UNIT - I

VECTOR CALCULUS

Definition:

The operator ∇ is denoted by $\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$.

Also $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Definition: Gradient of a scalar function

Let $\phi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector

$$\nabla\phi = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

is called gradient of the function ∅ and is denoted as $grad\ \emptyset = \nabla\emptyset$.

Example:

If $\phi = 4x^2y - y^3z^2$, find $grad\ \phi$ at $(1, -1, 2)$.

Answer:

Given $\phi = 4x^2y - y^3z^2$

$$\frac{\partial\phi}{\partial x} = 8xy, \quad \frac{\partial\phi}{\partial y} = 4x^2 - 3y^2z^2, \quad \frac{\partial\phi}{\partial z} = -2y^3z$$

$$\nabla\phi = \vec{i}\,(8xy) + \vec{j}\,(4x^2 - 3y^2z^2) + \vec{k}\,(-2y^3z)$$

$$\nabla\phi_{(1,-1,2)} = -8\vec{i} + \vec{j}\,(4 - 12) + \vec{k}\,(-2 * -1 * 2)$$

$$\nabla\phi_{(1,-1,2)} = -8\vec{i} - 8\vec{j} + 4\vec{k}$$

Example:

If $\phi = xyz$, find $\nabla\phi$.

Answer:

Given $\phi = xyz$

$$\frac{\partial\phi}{\partial \mathrm{x}} = yz, \quad \frac{\partial\phi}{\partial \mathrm{y}} = xz, \quad \frac{\partial\phi}{\partial \mathrm{z}} = xy$$

$$\nabla\phi = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$

Example:

If $\phi = 2xz^4 - x^3y$. Find $|\nabla\phi|$ at $(2,-2,-1)$.

Answer:

Given $\phi = 2xz^4 - x^3y$

$$\frac{\partial\phi}{\partial \mathrm{x}} = 2z^4 - 3x^2y, \quad \frac{\partial\phi}{\partial \mathrm{y}} = -\mathrm{x}^3, \quad \frac{\partial\phi}{\partial \mathrm{z}} = 8\mathrm{xz}^3$$

$$\nabla\phi = \vec{\mathrm{i}}\,(2z^4 - 3x^2y) + \vec{\mathrm{j}}\,(-\mathrm{x}^3) + \vec{\mathrm{k}}\,(8\mathrm{xz}^3)$$

$$\nabla\phi_{(2,-2,-1)} = 26\,\vec{\mathrm{i}} - 8\,\vec{\mathrm{j}} - 16\,\vec{\mathrm{k}}$$

$$|\nabla\phi| = \sqrt{26^2 + 8^2 + 16^2} = \sqrt{996}$$

Example:

Find the gradient of the function $\log(x^2 + y^2 + z^2)$ *or* $\log r^2$.

Answer:

Given $\phi = \log(x^2 + y^2 + z^2)$

$$\frac{\partial\phi}{\partial \mathrm{x}} = \frac{1}{x^2 + y^2 + z^2} * 2x = \frac{2x}{x^2 + y^2 + z^2}$$

$$\frac{\partial\phi}{\partial \mathrm{y}} = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial\phi}{\partial \mathrm{z}} = \frac{2z}{x^2 + y^2 + z^2}$$

$$\nabla\phi = \frac{2}{x^2 + y^2 + z^2}\left[\vec{i}\,(x) + \vec{j}\,(y) + \vec{k}\,(z)\right]$$

$$\nabla\phi = \frac{2\vec{r}}{r^2}$$

Example:

Find ∇r^n where $\vec{r} = x\,\vec{\imath} + \mathrm{y}\,\vec{\jmath} + \mathrm{z}\,\vec{\mathrm{k}}$.

Answer:

$$\mathrm{r}^2 = x^2 + y^2 + z^2$$

Differentiating partially with respect to x, y and z respectively

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, 2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}, 2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = \vec{\imath}\,\frac{\partial r^n}{\partial \mathrm{x}} + \vec{\jmath}\,\frac{\partial r^n}{\partial \mathrm{y}} + \vec{\mathrm{k}}\,\frac{\partial r^n}{\partial \mathrm{z}}$$

$$\nabla r^n = nr^{n-1}\frac{\partial r}{\partial x}\vec{\imath} + nr^{n-1}\frac{\partial r}{\partial y}\vec{\jmath} + nr^{n-1}\frac{\partial r}{\partial z}\vec{\mathrm{k}}$$

$$= nr^{n-1}\frac{x}{r}\vec{\imath} + nr^{n-1}\frac{y}{r}\vec{\jmath} + nr^{n-1}\frac{z}{r}\vec{\mathrm{k}}$$

$$= nr^{n-2}\left[x\vec{\imath} + y\vec{\jmath} + z\vec{\mathrm{k}}\right] = nr^{n-2}\,\vec{r}$$

Example:

Find ∇r where $r = x\,\vec{\imath} + \mathrm{y}\,\vec{\jmath} + \mathrm{z}\,\vec{\mathrm{k}}$.

Answer:

$$\mathrm{r} = |\vec{\mathrm{r}}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial \phi}{\partial \mathrm{x}} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} * 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial \phi}{\partial \mathrm{y}} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial \phi}{\partial \mathrm{z}} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla \phi = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left[\vec{\imath}\,(x) + \vec{\jmath}\,(\mathrm{y}) + \vec{\mathrm{k}}\,(\mathrm{z})\right]$$

$$\nabla \phi = \frac{\vec{r}}{\sqrt{r^2}} = \frac{\vec{r}}{r}$$

Definition: Unit normal to the surface

Unit normal to the surface ϕ is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

Example:

Find the unit vector normal to the surface $x^2 + y^2 + z^2 = 1 \ \ at\ (1,1,1)$.

Answer:

$$\phi = x^2 + y^2 + z^2 - 1$$

$$\nabla\phi = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal to the surface ϕ is given by $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}}{2\sqrt{3}} = \dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

Example:

Find the unit vector normal to the surface $xy + 2xz^2 = 8$ at (1,0,2).

Answer:

$$\phi = xy + 2xz^2 - 8$$

$$\nabla\phi = (y + 2z^2)\vec{i} + (x)\vec{j} + (4xz)\,\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\,\vec{i} + \vec{j} + 8\,\vec{k}$$

$$|\nabla\phi| = \sqrt{64+1+64} = \sqrt{129}$$

Unit normal to the surface ϕ is given by $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{8\,\vec{i} + \vec{j} + 8\,\vec{k}}{\sqrt{129}}$.

Definition: Angle between the surfaces

The angle between the surfaces $\phi_1 \ and \ \phi_2$ is given by

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Example:

Find the angle between the surfaces $x^2 + y^2 - z = 3$ and $x^2 + y^2 + z^2 = 9$ at $(2, -1,2)$.

Answer:

$$\phi_1 = x^2 + y^2 - z - 3$$

$$\nabla\phi_1 = 2x\,\vec{\imath} + 2y\,\vec{\jmath} - \vec{k}$$

$$\nabla\phi_{1(2,-1,2)} = 4\,\vec{\imath} - 2\,\vec{\jmath} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\nabla\phi_2 = 2x\,\vec{\imath} + 2y\,\vec{\jmath} + 2z\,\vec{k}$$

$$\nabla\phi_{2(2,-1,2)} = 4\,\vec{\imath} - 2\,\vec{\jmath} + 4\,\vec{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36}$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(4\,\vec{\imath} - 2\,\vec{\jmath} - \vec{k}).(4\,\vec{\imath} - 2\,\vec{\jmath} + 4\,\vec{k})}{\sqrt{21}\sqrt{36}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{36}}$$

Example:

Find the angle between the surfaces $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 - 5$ at $(3,0,4)$.

Answer:

$$\phi_1 = x^2 + y^2 + z^2 - 25$$

$$\nabla\phi_1 = 2x\,\vec{\imath} + 2y\,\vec{\jmath} + 2z\,\vec{k}$$

$$\nabla\phi_{1(3,0,4)} = 6\,\vec{\imath} + 0\,\vec{\jmath} + 8\,\vec{k}$$

$$|\nabla\phi_1| = \sqrt{36+64} = \sqrt{100} = 10$$

$$\phi_2 = x^2 + y^2 - 5$$

$$\nabla\phi_2 = 2x\,\vec{\imath} + 2y\,\vec{\jmath} + 0\,\vec{k}$$

$$\nabla\phi_{2(3,0,4)} = 6\,\vec{\imath} + 0\,\vec{\jmath}$$

$$|\nabla\phi_2| = \sqrt{36} = 6$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(6\,\vec{i} + 0\,\vec{j} + 8\,\vec{k}).(6\,\vec{i})}{10 * 6}$$

$$\cos\theta = \frac{36}{60}$$

Definition: Divergence of a scalar point function

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function in a region of space then the divergence $\vec{F}$ is defined by

$$div\,\vec{F} = \nabla.\vec{F} = \vec{\imath}\,\frac{\partial\vec{F}}{\partial x} + \vec{\jmath}\,\frac{\partial\vec{F}}{\partial y} + \vec{k}\,\frac{\partial\vec{F}}{\partial z}$$

If $\quad \vec{F} = F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k},\quad$ then

$$div\,\vec{F} = \nabla.\left(F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k}\right) = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k}\right)$$

$$div\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Definition: Curl of a vector function

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function defined in each point (x, y, z) then the $curl\ of\ \vec{F}$ is defined by

$$curl\,\vec{F} = \nabla\times\vec{F} = \vec{\imath}\times\frac{\partial\vec{F}}{\partial x} + \vec{\jmath}\times\frac{\partial\vec{F}}{\partial y} + \vec{k}\times\frac{\partial\vec{F}}{\partial z}$$

If $\quad \vec{F} = F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k},\quad$ then

$$curl\,\vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F_1} & \mathrm{F_2} & \mathrm{F_3} \end{vmatrix} = \vec{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Example:

If $\vec{F} = x^2\,\vec{\imath} + y^2\,\vec{\jmath} + z^2\,\vec{k}$, then find $\nabla.\vec{F}\ and\ \nabla\times\vec{F}$.

Answer:

$$\vec{F} = x^2\,\vec{\imath} + y^2\,\vec{\jmath} + z^2\,\vec{k}$$

$$div\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= (2x) + (2y) + (2z)$$

$$div\,\vec{F} = 2(x + y + z)$$

$$curl\,\vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F_1} & \mathrm{F_2} & \mathrm{F_3} \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{x^2} & \mathrm{y^2} & \mathrm{z^2} \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}\,z^2 - \frac{\partial}{\partial z}\,y^2\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}\,z^2 - \frac{\partial}{\partial z}\,x^2\right) + \vec{k}\left(\frac{\partial}{\partial x}\,y^2 - \frac{\partial}{\partial y}\,x^2\right)$$

$$curl\,\vec{F} = 0\,\vec{\imath} + 0\,\vec{\jmath} + 0\,\vec{k}$$

Example:

If $\vec{F} = xyz\,\vec{\imath}\, + 3x^2y\,\,\vec{\jmath} + (xz^2 - y^2z)\,\,\vec{k},$ then find $\nabla.\vec{F}\ and\ \nabla \times \vec{F}$ at $(1,2,-1)$.

Answer:

$$\vec{F} = xyz\,\vec{\imath}\, + 3x^2y\,\,\vec{\jmath} + (xz^2 - y^2z)$$

$$div\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= (yz) + (3x^2) + (2xz - y^2)$$

$$\left[div\,\vec{F}\right]_{(1,2,-1)} = -2 + 3 + (-2 - 4) = -5$$

$$curl\,\vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F_1} & \mathrm{F_2} & \mathrm{F_3} \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{xyz} & \mathrm{3x^2y} & \mathrm{xz^2 - y^2z} \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}\,(\mathrm{xz^2 - y^2z}) - \frac{\partial}{\partial z}\,3x^2y\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(\mathrm{xz^2 - y^2z}) - \frac{\partial}{\partial z}xyz\right) + \vec{k}\left(\frac{\partial}{\partial x}\,3x^2y - \frac{\partial}{\partial y}xyz\right)$$

$$= \vec{\imath}(-2yz) - \vec{\jmath}(z^2 - xy) + \vec{k}(6xy - xz)$$

$$\left[curl\,\vec{F}\right]_{(1,2,-1)} = 4\,\vec{\imath} + \vec{\jmath} + 13\vec{k}$$

Definition: Laplace Equation

If ϕ is a scalar point function then $\Delta^2\phi = \frac{\partial^2\phi}{\partial \mathrm{x}^2} + \frac{\partial^2\phi}{\partial \mathrm{y}^2} + \frac{\partial^2\phi}{\partial \mathrm{z}^2} = 0$ is called the Laplace equation.

Example:

Find the value of $\nabla^2\left(\frac{1}{x+y+z}\right)$.

Answer:

$$\nabla^2\vec{F} = \frac{\partial^2\vec{F}}{\partial x^2} + \frac{\partial^2\vec{F}}{\partial y^2} + \frac{\partial^2\vec{F}}{\partial z^2} = \sum \frac{\partial^2\vec{F}}{\partial x^2}$$

$$= \sum \frac{\partial}{\partial x}\left(-\frac{1}{(x+y+z)^2}\right)$$

$$= \sum \frac{2}{(x+y+z)^3}$$

$$= \frac{2}{(x+y+z)^3} + \frac{2}{(x+y+z)^3} + \frac{2}{(x+y+z)^3}$$

$$\nabla^2\vec{F} = \frac{6}{(x+y+z)^3}$$

Definition: Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $\mathbf{div}\,\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla.\vec{\mathbf{F}} = \mathbf{0}$.

Definition: Irrotational vector

A vector $\vec{F}$ is said to be Irrotational if $\mathbf{curl}\,\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$.

Definition: Scalar Potential

If $\vec{F}$ is irrotational vector, then there exists a scalar function ϕ such that $\vec{\mathbf{F}} = \nabla\boldsymbol{\phi}$. Such a scalar function is called scalar potential of $\vec{F}$.

Example:

Prove that $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ is Solenoidal.

Answer:

A vector $\vec{F}$ is said to be solenoidal if $\mathbf{div}\,\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla.\vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$

$$\nabla.\vec{\mathrm{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}y + \frac{\partial}{\partial y}z + \frac{\partial}{\partial z}x = 0$$

Hence $\vec{\mathrm{F}}$ is Solenoidal.

Example:

Prove that $\vec{F} = 3y^4z^2\,\vec{\imath} + 4x^3z^2\,\vec{\jmath} - 3x^2y^2\,\vec{k}$ is Solenoidal.

Answer:

A vector $\vec{\mathrm{F}}$ is said to be solenoidal if $\mathbf{div}\,\vec{\mathbf{F}} = \mathbf{0}$. That is if $\boldsymbol{\nabla}.\vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{\mathrm{F}} = 3y^4z^2\,\vec{\imath} + 4x^3z^2\,\vec{\jmath} - 3x^2y^2\,\vec{k}$

$$\nabla.\vec{\mathrm{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) = 0$$

$$= 0 + 0 + 0 = 0$$

Hence $\vec{\mathrm{F}}$ is Solenoidal.

Example:

If $\vec{F} = (x + 3y)\,\vec{\imath} + (y - 2z)\,\vec{\jmath} + (x + \lambda z)\,\vec{k}$ is Solenoidal find the value of λ.

Answer:

A vector $\vec{\mathrm{F}}$ is said to be solenoidal if $\mathbf{div}\,\vec{\mathbf{F}} = \mathbf{0}$. That is if $\boldsymbol{\nabla}.\vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{\mathrm{F}} = (x + 3y)\,\vec{\imath} + (y - 2z)\,\vec{\jmath} + (x + \lambda z)\,\vec{k}$

$$\nabla.\vec{\mathrm{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$1 + 1 + \lambda = 0 \Longrightarrow \quad \lambda = -2$$

Example:

Prove that $\vec{F} = yz\,\vec{\imath} + zx\,\vec{\jmath} + xy\,\vec{k}$ is Irrotational.

Answer:

A vector $\vec{F}$ is said to be Irrotational if **curl** $\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}\,xy - \frac{\partial}{\partial z}\,zx\right) - \vec{j}\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}\,yz\right) + \vec{k}\left(\frac{\partial}{\partial x}\,zx - \frac{\partial}{\partial y}yz\right)$$

$$= \vec{i}(x - x) - \vec{j}(y - y) + \vec{k}(z - z)$$

$$\nabla \times \vec{F} = 0$$

Hence $\vec{F}$ is Irrotational.

Example:

Find the constants a, b, c so that $\vec{F} = (x + 2y + az)\,\vec{i} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\vec{k}$ is Irrotational.

Answer:

A vector $\vec{F}$ is said to be Irrotational if **curl** $\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{F} = (x + 2y + az)\,\vec{i} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\vec{i}\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}\,(bx - 3y - z)\right] = 0$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right]$$

$$\vec{i}(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$

$$\therefore \quad c + 1 = 0 \;\; \Rightarrow c = -1, \;\; 4 - a = 0 \quad \Rightarrow a = 4 \;\; and \;\; b = 2$$

Example:

Prove that $\vec{F} = (2x + yz)\,\vec{\imath} + (4y + zx)\,\vec{\jmath} - (6z - xy)\,\vec{k}$ is Solenoidal as well as Irrotational. Also find the scalar potential of $\vec{F}$.

Answer:

A vector $\vec{\mathrm{F}}$ is said to be solenoidal if **div** $\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla.\vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{\mathrm{F}} = (2x + yz)\,\vec{\imath} + (4y + zx)\,\vec{\jmath} - (6z - xy)\,\vec{k}$

$$\nabla.\vec{\mathrm{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\frac{\partial}{\partial x}(2x + yz) + \frac{\partial}{\partial y}(4y + zx) - \frac{\partial}{\partial z}(6z - xy) = 0$$

$$2 + 4 - 6 = 0$$

Hence $\vec{\mathbf{F}}$ is Solenoidal.

A vector $\vec{\mathrm{F}}$ is said to be Irrotational if **curl** $\vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{\mathrm{F}} = (2x + yz)\,\vec{\imath} + (4y + zx)\,\vec{\jmath} - (6z - xy)\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + yz & 4y + zx & -6z + xy \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(-6z + xy) - \frac{\partial}{\partial z}(4y + zx)\right)$$

$$-\vec{\jmath}\left(\frac{\partial}{\partial x}(-6z + xy) - \frac{\partial}{\partial z}(2x + yz)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(4y + zx) - \frac{\partial}{\partial y}(2x + yz)\right)$$

$$= \vec{\imath}(x - x) - \vec{\jmath}(y - y) + \vec{k}(z - z)$$

$$\nabla \times \vec{F} = 0$$

Hence $\vec{\mathrm{F}}$ is Irrotational.

To find scalar potential:

To find ϕ such that $\vec{F} = \nabla\phi$.

$$(2x + yz)\,\vec{\imath} + (4y + zx)\,\vec{\jmath} - (6z - xy)\,\vec{k} = \frac{\partial\phi}{\partial x}\,\vec{\imath} + \frac{\partial\phi}{\partial y}\,\vec{\jmath} + \frac{\partial\phi}{\partial z}\,\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2x + yz$$

$$\frac{\partial\phi}{\partial y} = 4y + zx$$

$$\frac{\partial\phi}{\partial x} = -6z + xy$$

Integrating with respect to x, y, z respectively, we get

$$\phi(x, y, z) = x^2 + xyz + f(y, z)$$

$$\phi(x, y, z) = 2y^2 + xyz + f(x, z)$$

$$\phi(x, y, z) = -3z^2 + xyz + f(x, y)$$

Combining, we get $\phi(x, y, z) = x^2 + xyz + 2y^2 + xyz - 3z^2 + xyz + k$

where k is a constant.

Therefore ϕ is a scalar potential.

Example:

Prove that $\vec{F} = (6xy + z^3)\,\vec{\imath} + (3x^2 - z)\,\vec{\jmath} + (3xz^2 - y)\,\vec{k}$ is Irrotational. Also find the scalar potential of $\vec{F}$.

Answer:

A vector $\vec{\mathrm{F}}$ is said to be Irrotational if $\mathbf{curl}\ \vec{\mathbf{F}} = \mathbf{0}$. That is if $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$.

Given $\vec{\mathrm{F}} = (6xy + z^3)\,\vec{\imath} + (3x^2 - z)\,\vec{\jmath} + (3xz^2 - y)\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}\,(3xz^2 - y) - \frac{\partial}{\partial z}\,(3x^2 - z)\right)$$

$$-\vec{\jmath}\left(\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}\,(6xy + z^3)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right)$$

$$= \vec{\imath}(-1+1) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(6x - 6x)$$

$$\nabla \times \vec{F} = 0$$

Hence $\vec{F}$ is Irrotational.

To find scalar potential:

To find ϕ such that $\vec{F} = \nabla\phi$.

$$(6xy + z^3)\,\vec{\imath} + (3x^2 - z)\,\vec{\jmath} + (3xz^2 - y)\,\vec{k} = \frac{\partial\phi}{\partial x}\,\vec{\imath} + \frac{\partial\phi}{\partial y}\,\vec{\jmath} + \frac{\partial\phi}{\partial z}\,\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z$$

$$\frac{\partial\phi}{\partial x} = 3xz^2 - y$$

Integrating with respect to x, y, z respectively, we get

$$\phi(x,y,z) = 3x^2y + xz^3 + f(y,z)$$

$$\phi(x,y,z) = 3x^2y - yz + f(x,z)$$

$$\phi(x,y,z) = xz^3 - yz + f(x,y)$$

Combining, we get $\phi(x,y,z) = 3x^2y + xz^3 + 3x^2y - yz + xz^3 - yz + k$

where k is a constant. Therefore ϕ is a scalar potential.

Vector integration

Line integral

Let $\vec{F}$ be a vector field in space and let AB be a curve described in the sense $A\ to\ B$.

$$\lim_{n\to\infty}\sum_{A}^{B}\overrightarrow{F_n}.\,d\overrightarrow{r_n} = \int_{A}^{B}\vec{F}.\,d\vec{r} \quad \text{is called line integral.}$$

If the line integral along the curve C then it is denoted by

$$\int_C \vec{F}.d\vec{r} \quad \text{or} \quad \oint_C \vec{F}.d\vec{r} \quad \text{if C is closed curve.}$$

Example:

If $\vec{F} = (3x^2 + 6y)\vec{\imath} - 14yz\,\vec{\jmath} + 20\,xz^2\vec{k}$. Evaluate $\oint_C \vec{F}.d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve.

Solution:

The end points are $(0,0,0)$ and $(1,1,1)$

These points correspond to $t = 0$ and $t = 1$

$$dx = dt, \quad dy = 2t\ dt, \quad dz = 3t^2\ dt$$

$$\oint_C \vec{F}.d\vec{r} = \int_0^1 [(3x^2 + 6y)dx - 14yz\,dy + 20\,xz^2\,dz]$$

$$= \int_0^1 [(3t^2 + 6t)dt - 14tt\ 2t\,dt + 20\,tt^2\,3t^2dt]$$

$$= \int_0^1 [3t^2 - 28\,t^6 + 60\,t^9]\,dt = \left[\frac{3t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$$

$$= 1 - \frac{28}{7} + \frac{60}{10} = 5$$

Example:

Find the work done by the moving particle in the force field $\vec{F} = 3x^2\,\vec{\imath} + (2xz - y)\,\vec{\jmath} - z\,\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, $y = t$ and $z = 4t^3$.

Solution:

$$\text{Work done} = \oint_C \vec{F}.d\vec{r} = \int_0^1 [3x^2dx + (2xz - y)\,dy - z\,dz]$$

$$= \int_0^1 [3 \times 4t^4\ 4t\,dt + [2 \times 2t^2 \times 4t^3 - t]dt - 4t^3 * 12\,t^2dt]$$

$$= \int_0^1 [48\, t^5 + 16t^5 - t - 48\, t^5]\, \mathrm{dt} = \left[16\frac{\mathrm{t}^6}{6} - \frac{\mathrm{t}^2}{2}\right]_0^1$$

$$= \frac{16}{6} - \frac{1}{2} = \frac{13}{6}$$

Greens theorem in plane

If $M(x, y)$ and $N(x, y)$ are continuous functions with continuous, partial derivatives in a region R of the xy plane bounded by a simple closed curve C then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

where C is the curve described in the positive direction.

Example:

Evaluate using Green's theorem in the plane for $\int (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Answer:

By Green's theorem $\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Given $M = xy + y^2$; $N = x^2$

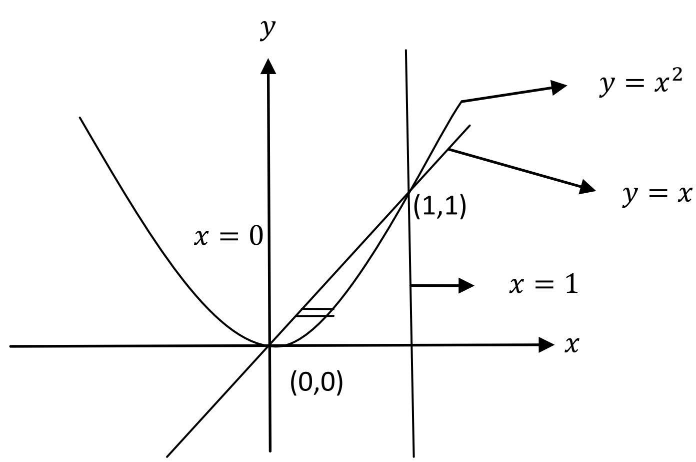
$$\frac{\partial \mathrm{M}}{\partial \mathrm{y}} = x + 2y; \quad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \int_C (xy + y^2)dx + x^2 dy = \iint_R (2x - x - 2y)dx\, dy$$

$$I = \iint_R (x - 2y)dx\, dy$$

$$I = \int_0^1 \int_y^{\sqrt{y}} (x - 2y)dx\ dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$



$$= \int_0^1 \left[\left(\frac{y}{2} - 2\sqrt{y}y\right) - \left(\frac{y^2}{2} - 2y^2\right)\right] dy$$

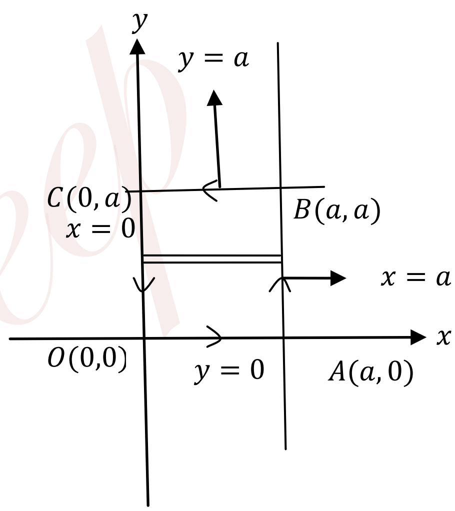
$$= \int_0^1 \left[\left(\frac{y}{2} - 2y^{\frac{3}{2}} + \frac{3}{2}y^2\right)\right] dy$$

$$= \left(\frac{y^2}{2*2} - \frac{2y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{\frac{3}{2}y^3}{3}\right)_0^1 = \left(\frac{1}{4} - \frac{4}{5} + \frac{3}{6}\right) = -\frac{1}{20}$$

Example:

Verify Green's theorem in the plane for $\int_C (x^2 dx + xy\, dy)$ where C is the curve in the plane given by $x = 0,\ y = 0,\ x = a,\ y = a\ (a > 0)$.

Answer:



By Green's theorem

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Given $M = x^2$; $N = xy$

$$\frac{\partial M}{\partial y} = 0; \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \int_C x^2 dx + xy\, dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along $OA, y = 0 \Rightarrow dy = 0$

$$\int_C x^2 dx + xy\, dy = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB, $x = a \Rightarrow dx = 0$

$$\int_C x^2 dx + xy\, dy = \int_0^a ay dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

Along BC, $y = a \Rightarrow dy = 0$

$$\int_C x^2 dx + xy\, dy = \int_a^0 x^2 dx = \left[\frac{y^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

Along $CO, x = 0 \Rightarrow dx = 0$

$$\int_C x^2dx + xy\,dy = \int_a^0 0 = 0$$

$$\therefore \int_C x^2dx + xy\,dy = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

Also $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^a \int_0^a y\,dx\,dy$$

$$= \int_0^a y\,[x]_0^a\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

Hence $$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Example:

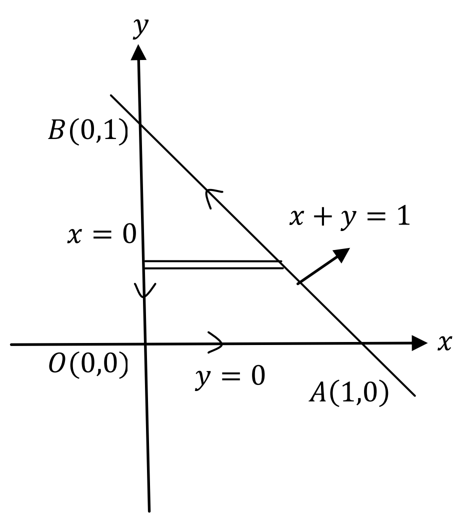
Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region bounded by $x = 0,\ y = 0,\ x + y = 1.$

Answer:

By Green's theorem $$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \int_0^1 \int_0^{1-y} 10y\,dx\,dy = 10\int_0^1 y\,[x]_0^{1-y}\,dy$$



$$= 10\int_0^1 y(1-y)\,dy = 10\int_0^1 [y - y^2]\,dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = 10\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{5}{3}$$

Given $M = 3x^2 - 8y^2; \quad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y; \quad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_{OA} + \int_{AB} + \int_{BO}$$

Along OA, $y = 0 \Rightarrow dy = 0$

$$\int_C (3x^2)\, dx = \int_0^1 (3x^2)\, dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1 = 1$$

Along $AB, y = 1 - x \Rightarrow dy = -dx$

$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 (3x^2 - 8(1-x)^2)\, dx + \big(4(1-x) - 6x(1-x)\big)(-dx\,)$$

$$= \int_1^0 (-11x^2 + 26x - 12)\, dx$$

$$= \left[-11\frac{x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0$

$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 y dy = \int_1^0 \frac{y^2}{2} = -2$$

$$\therefore \int_C M dx + N dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$$\text{Hence} \quad \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Example:

Verify Green's theorem in the plane for $\int_C x^2(1+y)dx + (y^3 + x^3)\, dy$ where C is the square bounded by $x = \pm a,\ y = \pm a$.

Answer:

$$\text{By Green's theorem} \quad \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\therefore \int_C x^2(1+y)dx + (y^3 + x^3)\, dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DO}$$

Along $AB, y = -a \Rightarrow dy = 0$

$$\int_{-a}^{a} x^2(1-a)dx = (1-a)\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= (1-a)2\frac{a^3}{3} = 2\frac{a^3}{3} - 2\frac{a^4}{3}$$

Along $BC, x = a \Rightarrow dx = 0$

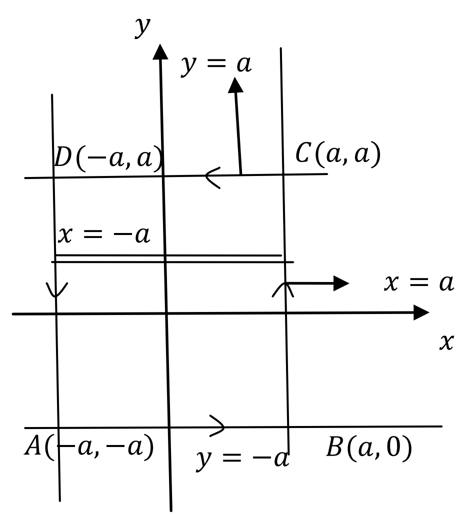
$$\int_{-a}^{a} (y^3 + a^3)\, dy = \left[\frac{y^4}{4} + a^3 y\right]_{-a}^{a}$$

$$= \frac{a^4}{4} + a^4 - \frac{a^4}{4} + a^4 = 2a^4$$

Along CD , $y = a \Rightarrow dy = 0$

$$\int_{a}^{-a} x^2(1+a)dx = (1+a)\left[\frac{x^3}{3}\right]_{a}^{-a}$$

$$= (1+a)\left(-\frac{a^3}{3} - \frac{a^3}{3}\right) = -\frac{2a^3}{3} - \frac{2a^4}{3}$$



Along DA , $x = -a \Rightarrow dx = 0$

$$\int_{a}^{-a} (-a^3 + y^3)\, dy = \left[\frac{y^4}{4} - a^3 y\right]_{a}^{-a} = 2a^4$$

$$\therefore \int_C x^2 dx + xy\, dy = 2\frac{a^3}{3} - 2\frac{a^4}{3} + 2a^4 - \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4 = \frac{8}{3}a^4$$

Given $M = x^2(1+y); \quad N = y^3 + x^3$

$$\frac{\partial M}{\partial y} = x^2; \quad \frac{\partial N}{\partial x} = 3x^2$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{-a}^{a}\int_{-a}^{a} (3x^2 - x^2)\, dx\, dy$$

$$= 2\int_{0}^{a} \left[\frac{x^3}{3}\right]_{-a}^{a} dy = 2\int_{-a}^{a} \frac{2a^3}{3} dy$$

$$= 4\frac{a^3}{3}[y]_a^a = \frac{4a^3}{3}(2a) = \frac{8a^4}{3}$$

Hence $\displaystyle\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Example:

Find the area bounded between the curves $y^2 = 4x$ and $x^2 = 4y$ using Green's theorem.

Answer:

By Green's theorem $\displaystyle\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

The point of intersection of $x^2 = 4y$ *and* $y^2 = 4x$ are $(0,0)$ and $(4,4)$.

The area enclosed by a single curve C is $\frac{1}{2}\int_C (xdy - ydx)$.

$$\text{Hence the area} = \frac{1}{2}\int_{C_1} (xdy - ydx) + \frac{1}{2}\int_{C_2} (xdy - ydx)$$

Along C_1 $(x^2 = 4y)$

$$4dy = 2xdx$$

$$\int_C (xdy - ydx) = \int_0^4 \left(\frac{x^2}{2}dx - \frac{x^2}{4}dx\right)$$

$$= \int_0^4 \frac{x^2}{4}\, dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{16}{3}$$

Along C_2 $(y^2 = 4x)$

$$2ydy = 4dx$$

$$\int_C (xdy - ydx) = \int_4^0 \left(\frac{y^2}{4}dy - \frac{y^2}{2}dy\right)$$

$$= -\int_4^0 \frac{y^2}{4}\, dy = -\frac{1}{4}\left[\frac{y^3}{3}\right]_4^0 = \frac{16}{3}$$

$$\therefore \text{ Area} = \frac{1}{2}\left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}.$$

Stoke's theorem

If S is a open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in S and so on C, then

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}\ .\hat{n}\,ds$$

where $\hat{n}$ is the unit vector normal to the surface. That is, the surface integral of the normal component of $curl\,\vec{F}$ is equal to the line integral of the tangential component of $\vec{F}$ taken around C.

Example:

Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the XOY plane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b$.

Answer:

$$\text{By Stoke's theorem } \int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$$

$$\text{Given } \vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j} + 0\vec{k}$$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\vec{F}.d\vec{r} = (x^2 - y^2)dx + 2xy\,dy$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2\mathrm{xy} & 0 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right) - \vec{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 - y^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y)$$

$$\nabla \times \vec{F} = 4y\,\vec{k}$$

Here the surface S denotes the rectangle $OABC$ and the unit outward normal to the vector is $\vec{k}$.

That is $\vec{n} = \vec{k}$

$$\text{Now} \quad \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds = \iint_S 4y\,\vec{k}\ .\vec{n}\,ds = \iint_S 4y\,\vec{k}\ .\vec{k}\,ds$$

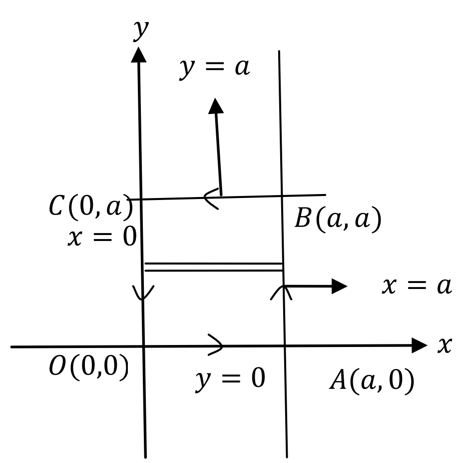
$$= \iint_S 4y\,dx\,dy = \int_0^b \int_0^a 4y\ dx\,dy$$

$$= 4\int_0^b y[x]_0^a\ dy = 4a\left[\frac{y^2}{2}\right]_0^b = 2ab^2$$

$$\text{Now} \quad \int_S \vec{F}.d\vec{r} = \int_C (x^2 - y^2)dx + 2xy\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along $OA, y = 0 \Rightarrow dy = 0$

$$\int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$



Along AB , $x = a \Rightarrow dx = 0$

$$\int_0^b 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = ab^2$$

Along BC , $y = a \Rightarrow dy = 0$

$$\int_a^0 (x^2 - b^2)dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along $CO, x = 0 \Rightarrow dx = 0$

$$\int_{CO} (x^2 - y^2)dx + 2xy\, dy = 0$$

$$\therefore \int_S \vec{F}.d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2$$

$$\text{Hence } \int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\, ds$$

Example:

Verify Stoke's theorem when $\vec{F} = (2xy - x^2)\vec{\imath} - (x^2 - y^2)\,\vec{\jmath}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x\ \ and\ \ x^2 = y$.

Answer:

$$\text{By Stoke'stheorem } \int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\, ds$$

$$\text{Given } \vec{F} = (2xy - x^2)\vec{\imath} - (-x^2 - y^2)\,\vec{\jmath} + 0\vec{k}$$

$$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

$$\vec{F}.d\vec{r} = (2xy - x^2)dx - (-x^2 - y^2)\,dy$$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F_1} & \mathrm{F_2} & \mathrm{F_3} \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - x^2 & -x^2 - y^2 & 0 \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x^2 - y^2)\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy - x^2)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(-x^2 - y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right)$$

$$= \vec{\imath}(0) - \vec{\jmath}(0) + \vec{k}(-2x - 2x)$$

$$\nabla \times \vec{F} = -4x\,\vec{k}$$

Here the surface S denotes the surface of the XOY plane. The unit outward normal to the vector is $\vec{k}$.

That is $\vec{n} = \vec{k}$

$$\text{Now} \iint_S \nabla \times \vec{F}\,.\vec{n}\,ds = \iint_S -4x\,\vec{k}\,.\vec{n}\,ds = \iint_S -4x\,\vec{k}\,.\vec{k}\,ds$$

$$= -\iint_S 4x\,dx\,dy = -4\int_0^1 \int_{\sqrt{y}}^{y^2} x\,dx\,dy$$

$$= -4\int_0^1 \left[\frac{x^2}{2}\right]_{\sqrt{y}}^{y^2} dy = -2\int_0^1 (y^4 - y)dy$$

$$= -2\left[\frac{y^5}{5} - \frac{y^2}{2}\right]_0^1 = -2\left(\frac{1}{5} - \frac{1}{2}\right) = 2 * \frac{-3}{10} = -\frac{6}{10} = -\frac{3}{5}$$

Now $\int_S \vec{F}.d\vec{r}$ can be taken OA and AO.

$$\text{i.e.,} \int_S \vec{F}.d\vec{r} = \int_S (2xy - x^2)dx - (x^2 - y^2)\,dy = \int_{OA} + \int_{AO}$$

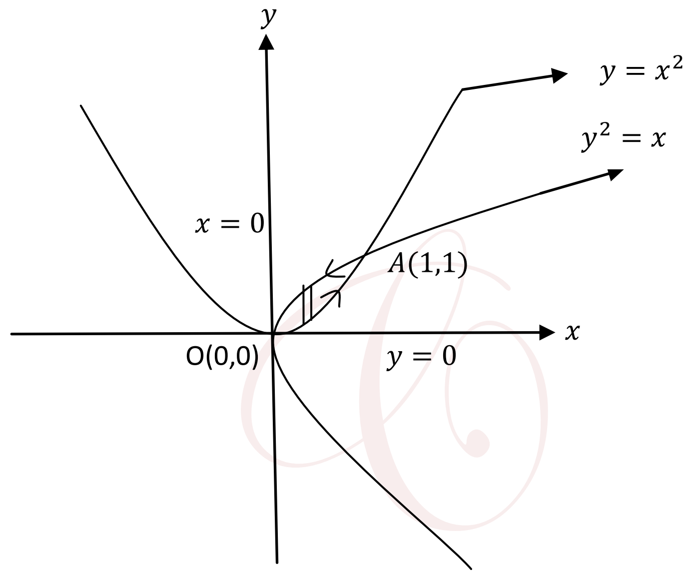
Along OA, $y = x^2 \Rightarrow dy = 2xdx$

$$\int_0^1 (2xx^2 - x^2)dx - (x^2 - x^4)2xdx$$

$$= \int_0^1 (2x^3 - x^2 - 2x^3 + 2x^5)dx$$

$$= \int_0^1 (2x^5 - x^2)dx = \left[2\frac{x^6}{6} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{3}\right) = 0$$



Along AO, $x = y^2 \Rightarrow dx = 2y\,dy$

$$\int_S (2y^2y - y^4)2ydy - (y^4 - y^2)\,dy$$

$$= \int_1^0 (4y^4 - 2y^5 - y^4 + y^2)dy$$

$$= \int_1^0 (3y^4 - 2y^5 + y^2)dy = \left[3\frac{y^5}{5} - 2\frac{y^6}{6} + \frac{y^3}{3}\right]_1^0$$

$$= -\left(\frac{3}{5} - \frac{2}{6} + \frac{1}{3}\right) = -\frac{3}{5}$$

$$\therefore \int_S \vec{F}.d\vec{r} = -\frac{3}{5}$$

$$\text{Hence } \int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$$

Example:

Verify Stoke's theorem for $\vec{F} = y^2\vec{\imath} + y\,\vec{\jmath} - xz\,\vec{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2, z \geq 0$.

Answer:

$$\text{By Stoke's theorem } \int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$$

Given $\vec{F} = y^2\vec{\imath} + y\,\vec{\jmath} - xz\,\vec{k}$

$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$

$\vec{F}.d\vec{r} = y^2dx + y\,dy - xz\,dz$

$\vec{F}.d\vec{r} = y^2dx + y\,dy \quad [\because z = 0]$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & \mathrm{y} & -\mathrm{xz} \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(y)\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(y^2)\right)$$

$$= \vec{\imath}(0) - \vec{\jmath}(-\mathrm{z}) - \vec{k}(2y)$$

$$\nabla \times \vec{F} = z\,\vec{\jmath} - 2y\,\vec{k}$$

Let $\phi = x^2 + y^2 + z^2 - a^2$

$\nabla\phi = 2x\,\vec{\imath} + 2y\,\vec{\jmath} + 2z\,\vec{k}$

$|\nabla\phi| = 2\sqrt{x^2 + y^2 + z^2} = 2a$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\vec{\imath} + 2y\,\vec{\jmath} + 2z\,\vec{k}}{2a} = \frac{x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}}{a}$$

Here the surface S denotes the rectangle $OABC$ and the unit outward normal to the vector is $\vec{k}$.

Thus $\operatorname{curl} \vec{\mathrm{F}}.\vec{\mathrm{n}} = \left(z\,\vec{\jmath} - 2y\,\vec{k}\right).\left(\frac{x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}}{a}\right)$

$$= \frac{1}{\mathrm{a}}(\mathrm{zy} - 2\mathrm{yz}) = -\frac{\mathrm{yz}}{\mathrm{a}}$$

Now $\displaystyle\iint_S \nabla \times \vec{F}\,.\vec{n}\,ds = \iint_R -\frac{\mathrm{yz}}{\mathrm{a}}\,\frac{dx\,dy}{|\vec{n}.\vec{k}|}$

$$= \iint_R \left(-\frac{yz}{a}\right)\frac{dx\,dy}{z/a}$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (-y)\ dy\,dx$$

$$= \int_{-a}^{a} \left[-\frac{y^2}{2}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx = -\frac{1}{2}\int_{-a}^{a} [a^2 - x^2 - a^2 + x^2]\ dx$$

$$\iint_S \nabla \times \vec{F}\ .\vec{n}\,ds = 0$$

Now $\int_S \vec{F}.d\vec{r} = \int_C y^2 dx + y\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Since $x^2 + y^2 = a^2$, is a circle.

$Put\ \ x = a\cos\theta$ and $y = a\sin\theta$

$dx = -a\sin\theta$ and $dy = a\cos\theta$

θ varies from $0\ \ to\ 2\pi$

$$\int_S \vec{F}.d\vec{r} = \int_0^{2\pi} [a^2 \sin^2\theta\,(-a\sin\theta) + a\sin\theta\,(a\cos\theta)]\,d\theta$$

$$= \int_0^{2\pi} -a^3 \sin^3\theta\ \ d\theta + \frac{a^2}{2}\int_0^{2\pi} sin\,2\theta\ \ d\theta$$

$$= \frac{a^3}{4}\int_0^{2\pi} (\sin 3\theta - 3\sin\theta)d\theta + \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^3}{4}\left[-\frac{\cos 3\theta}{3} + 3\cos\theta\right]_0^{2\pi} + \frac{a^2}{2}\left[-\frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{a^3}{4}\left[-\frac{1}{3} + 3 + \frac{1}{3} - 3\right] + \frac{a^2}{2}\left[-\frac{1}{2} + \frac{1}{2}\right]$$

$$= 0$$

Hence $\int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$

Example:

Verify Stoke's theorem for the function $\vec{F} = x^2\,\vec{\imath} + xy\,\vec{\jmath}$ integrated round the square in the $z = 0$ plane whose sides are along the line $x = 0,\ y = 0,\ x = a,\ y = a$.

Answer:

$$\text{By Stoke's theorem} \quad \int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$$

$$\text{Given} \quad \vec{F} = x^2\vec{\imath} + xy\,\vec{\jmath} + 0\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{vmatrix} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & \mathrm{xy} & 0 \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy)\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right)$$

$$= \vec{\imath}(0) - \vec{\jmath}(0) + \vec{k}(y)$$

$$\nabla \times \vec{F} = y\,\vec{k}$$

Here the surface S denotes the rectangle $OABC$ and the unit outward normal to the vector is $\vec{k}$.

That is $\vec{n} = \vec{k}$

$$\text{Now} \quad \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds = \iint_S y\,\vec{k}\ .\vec{n}\,ds = \iint_S y\,\vec{k}\ .\vec{k}\,ds$$

$$= \iint_S y\,dx\,dy = \int_0^a \int_0^a y\ dx\,dy$$

$$= \frac{a^3}{2}$$

$$\text{Given} \quad \vec{F} = x^2\vec{\imath} + xy\,\vec{\jmath} + 0\vec{k}$$

$$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

$$\vec{F}.d\vec{r} = x^2dx + xy\,dy$$

$$\text{Now} \quad \int_C \vec{F}.d\vec{r} = \int_C x^2dx + xy\,dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along $OA, y = 0 \Rightarrow dy = 0$

$$\int_C x^2dx + xy\,dy = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Along AB , $x = a \Rightarrow dx = 0$

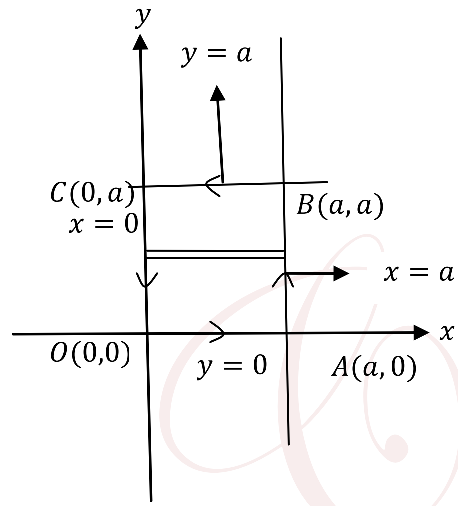
$$\int_C x^2dx + xy\,dy = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

Along BC , $y = a \Rightarrow dy = 0$

$$\int_C x^2dx + xy\,dy = \int_a^0 x^2dx = \left[\frac{y^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

Along $CO, x = 0 \Rightarrow dx = 0$

$$\int_C x^2dx + xy\,dy = \int_a^0 0 = 0$$



$$\therefore \int_C x^2dx + xy\,dy = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

$$\therefore \int_S \vec{F}.d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$$

Hence $\int_S \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds$

Example:

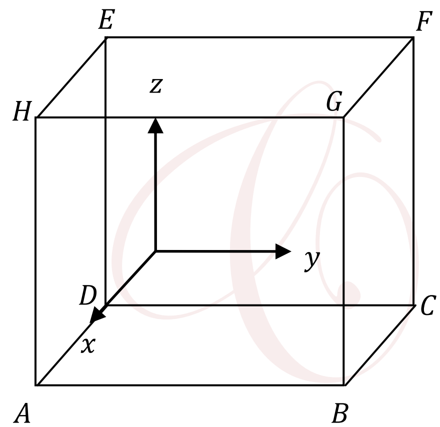
Verify Stoke's theorem for the function $\vec{F} = y^2z\,\vec{i} + z^2x\,\vec{j} + x^2y\,\vec{k}$ where S is the open surface of the cube formed by the planes $x = \pm a,\ y = \pm a\ \ and\ \ z = \pm a$ in which the plane $z = -a$ is cut.

Answer:

By Stoke'stheorem $\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \times \vec{F})\ .\vec{n}\,ds$

Given $\vec{F} = y^2z\,\vec{i} + z^2x\,\vec{j} + x^2y\,\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$



$$= \vec{\imath}\left(\frac{\partial}{\partial y}\,(x^2 y) - \frac{\partial}{\partial z}\,(z^2 x)\right)$$

$$-\vec{\jmath}\left(\frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial z}\,(y^2 z)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(z^2 x) - \frac{\partial}{\partial y}(y^2 z)\right)$$

$$\nabla \times \vec{F} = \vec{\imath}(x^2 - 2zx) - \vec{\jmath}(y^2 - 2xy) + \vec{k}(z^2 - 2yz)$$

Since the surface $z = -a$ is cut.

$$\text{Now} \quad \iint_S (\nabla \times \vec{F})\,.\vec{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

Surface	**Equation**	$\vec{n}$	$(\nabla \times \vec{F}).\vec{n}$	ds
$ABGH\ (S_1)$	$x = a$	$\vec{\imath}$	$a^2 - 2za$	$dy\,dz$
$DCFE\ (S_2)$	$x = -a$	$\overrightarrow{-\imath}$	$-(a^2 + 2za)$	$dy\,dz$
$BCFG\ (S_3)$	$y = a$	$\vec{\jmath}$	$a^2 - 2ax$	$dx\,dz$
$ADEH\ (S_4)$	$y = -a$	$\overrightarrow{-\jmath}$	$-(a^2 + 2ax)$	$dx\,dz$
$EFGH\ (S_5)$	$z = a$	$\vec{k}$	$a^2 - 2ay$	$dx\,dy$

$$\therefore \iint_S \nabla \times \vec{F}\,.\vec{n}\,ds = \iint (a^2 - 4za)\,dy\,dz - \iint (a^2 - 4za)\,dy\,dz +$$

$$\iint (a^2 - 4ax)\,dx\,dz - \iint (a^2 - 4ax)\,dx\,dz + \iint (a^2 - 2ay)\,dx\,dy$$

$$= \iint -4za\,dy\,dz + \iint -4ax\,dx\,dz + \iint (a^2 - 2ay)\,dx\,dy$$

$$= -4a \int_{-a}^{a}\int_{-a}^{a} z\ dx\,dy - 4a \int_{-a}^{a}\int_{-a}^{a} x\ dx\,dz + \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ay)\ dx\,dy$$

$$= 0 + 0 + \int_{-a}^{a} [a^2x - 2ayx]_{-a}^{a}\,dy = \int_{-a}^{a} [a^3 - 2a^2y + a^3 - 2a^2y]\,dy$$

$$= \int_{-a}^{a} [2a^3 - 4a^2y]\,dy = \left[2a^3y - \frac{4a^2y^2}{2}\right]_{-a}^{a} = 2a^4 - 2a^4 + 2a^4 + 2a^4$$

$$\therefore \iint_S \nabla \times \vec{F}\ .\vec{n}\,ds = 4a^4$$

Now $\int_C \vec{F}.d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

The boundary bounded by the square $ABCD$ that lies in the plane $z = -a$.

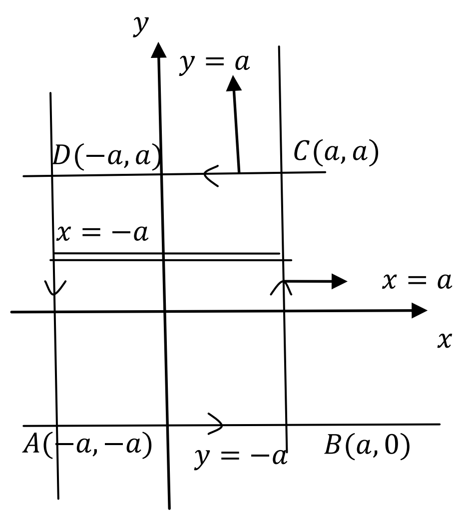
$\therefore dz = 0$

Now $\vec{F}.d\vec{r} = -ay^2dx + a^2x\,dy$

Along AB , $x = a \Rightarrow dx = 0$

$$\int_{-a}^{a} a^3dy = a^3[y]_{-a}^{a} = 2a^4$$

Along $BC, y = a \Rightarrow dy = 0$



$$\int_{a}^{-a} -a^3\ dx = -0a^3[x]_{a}^{-a} = 2a^4$$

Along $CD, x = -a \Rightarrow dx = 0$

$$\int_{a}^{-a} -a^3dy = 2a^4$$

Along $DA, y = -a \Rightarrow dy = 0$

$$\int_{-a}^{a} -a^3 dx = -2a^4$$

$$\therefore \int_S \vec{F}.d\vec{r} = 4a^4$$

$$\text{Hence } \int_S \vec{F}.d\vec{r} = \iint_S \nabla\times\vec{F}\ .\vec{n}\,ds$$

Gauss Divergence theorem

Statement:

The surface integral of the normal component of a vector function F over a closed surface S enclosing the volume V is equal to the volume of integral of the divergence of F taken throughout the volume V.

$$\iint_S \vec{F}.\hat{n}\,\mathrm{ds} = \iiint_V \nabla\times\vec{F}\ \mathrm{dv}$$

Example:

Verify Gauss divergence theorem for $\vec{F} = 4xz\,\vec{\imath} - y^2\,\vec{\jmath} + y\,z\,\vec{k}$ over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Answer:

$$\text{By Gauss stheorem } \iint_S \vec{F}.\hat{n}\,\mathrm{ds} = \iiint_V \nabla\times\vec{F}\ \mathrm{dv}$$

$$\text{Given } \nabla.\vec{F} = 4z - 2y + y = 4z - y$$

$$\text{Now } \iiint_V \nabla\times\vec{F}\ \mathrm{dv} = \int_0^1\int_0^1\int_0^1 (4\mathrm{z}-\mathrm{y})\mathrm{dx\,dy\,dz} = \int_0^1\int_0^1 [4\mathrm{zx}-\mathrm{yx}]_0^1\ dy\,dx$$

$$= \int_0^1\int_0^1 [4\mathrm{z}-\mathrm{y}]\ dy\,dx = \int_0^1 \left[4\mathrm{zy}-\frac{\mathrm{y}^2}{2}\right]_0^1 dx = \int_0^1 \left(4\mathrm{z}-\frac{1}{2}\right) dx$$

$$= \left(\frac{4\mathrm{z}^2}{2}-\frac{1}{2}\mathrm{z}\right)_0^1 = \frac{4}{2}-\frac{1}{2} = \frac{3}{2}$$

$$\text{Now} \quad \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	**Equation**	$\hat{n}$	$\vec{F}.\hat{n}$	ds
$AEGD\ (S_1)$	$x = 1$	$\vec{i}$	$4xz \Rightarrow 4z$	$dy\, dz$
$OBFC\ (S_2)$	$x = 0$	$-\vec{i}$	$-4xz \Rightarrow 0$	$dy\, dz$
$EBFG\ (S_3)$	$y = 1$	$\vec{j}$	$-y^2 \Rightarrow -1$	$dx\, dz$
$OADC\ (S_4)$	$y = 0$	$-\vec{j}$	$y^2 \Rightarrow 0$	$dx\, dz$
$DGFC\ (S_5)$	$z = 1$	$\vec{k}$	$yz \Rightarrow y$	$dx\, dy$
$OAEB\ (S_6)$	$z = 0$	$-\vec{k}$	$-yz \Rightarrow 0$	$dx\, dy$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, ds = 4\int_0^1\int_0^1 z\ dy\, dz - \int_0^1\int_0^1 dx\, dz + \int_0^1\int_0^1 y\ dx\, dy$$

$$= 2 - 1 + \frac{1}{2}$$

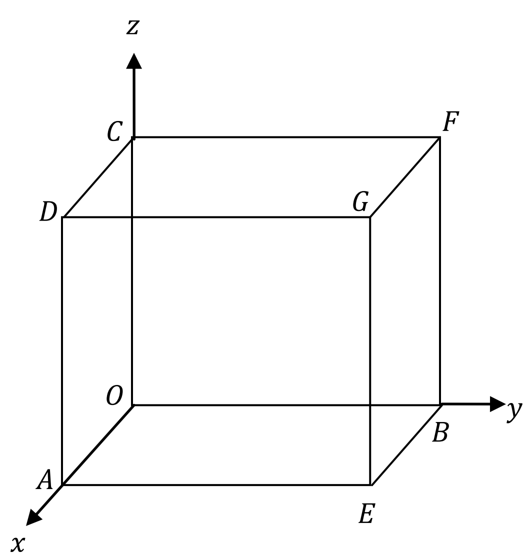
$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, ds = \frac{3}{2}$$

Gauss divergence theorem is verified.

Example:

Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\,\vec{i} + (y^2 - zx)\,\vec{j} + (z^2 - xy)\ \vec{k}$ over the rectangular parallelepiped $0 \le x \le a,\ 0 \le y \le b,$ $0 \le z \le c.$

Answer:



$$\text{By Gauss stheorem} \quad \iint_S \vec{F}.\hat{n}\,\text{ds}$$

$$= \iiint_V \nabla \times \vec{F}\ \text{dv}$$

$$\text{Given} \quad \nabla.\vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

$$\text{Now} \quad \iiint_V \nabla \times \vec{F}\ \text{dv}$$

$$= \int_0^a\int_0^b\int_0^c 2(x+y+z)dx\,dy\,dz = 2\int_0^a\int_0^b\left[\frac{x^2}{2}+yx+zx\right]_0^c dy\,dz$$

$$= 2\int_0^a\int_0^b\left[\frac{c^2}{2}+yc+zc\right] dy\,dz$$

$$= 2\int_0^b\left[\frac{c^2}{2}y+\frac{y^2}{2}c+zyc\right]_0^b dz$$

$$= 2\int_0^a\left(\frac{c^2}{2}b+\frac{b^2}{2}c+zbc\right) dz = 2\left[\frac{c^2}{2}bz+\frac{b^2}{2}cz+\frac{z^2}{2}bc\right]_0^a$$

$$= 2\ \frac{abc^2+b^2ca+a^2bc}{2}$$

$$\iiint_V \nabla\times\vec{F}\ dv = abc(a+b+c)$$

Now $\iint_S \vec{F}\ .\hat{n}\ ds = \iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}+\iint_{S_5}+\iint_{S_6}$

Surface	**Equation**	$\hat{n}$	$\vec{F}.\hat{n}$	ds
$AEGD\ (S_1)$	$x=a$	$\vec{i}$	x^2-yz	$dy\,dz$
$OBFC\ (S_2)$	$x=0$	$\overrightarrow{-i}$	$-(x^2-yz)$	$dy\,dz$
$EBFG\ (S_3)$	$y=b$	$\vec{j}$	y^2-zx	$dx\,dz$
$OADC\ (S_4)$	$y=0$	$\overrightarrow{-j}$	$-(y^2-zx)$	$dx\,dz$
$DGFC\ (S_5)$	$z=c$	$\vec{k}$	z^2-xy	$dx\,dy$
$OAEB\ (S_6)$	$z=0$	$-\vec{k}$	$-(z^2-xy)$	$dx\,dy$

$$\iint_{S_1} \vec{F}\ .\hat{n}\ ds = \int_0^c\int_0^b (a^2-yz)\ dy\,dz = \int_0^c\left(a^2y-\frac{y^2}{2}z\right)_0^b dz = \int_0^c\left(a^2b-\frac{b^2}{2}z\right)dz$$

$$= \left(a^2bz-\frac{b^2}{2}\frac{z^2}{2}\right)_0^c = a^2bc-\frac{b^2c^2}{4}$$

$$\iint_{S_2} \vec{F}\ .\hat{n}\ ds = \int_0^c\int_0^b -(0-yz)\ dy\,dz = \int_0^c\left(\frac{y^2}{2}z\right)_0^b dz = \int_0^c\left(\frac{b^2}{2}z\right)dz$$

$$= \left(\frac{b^2}{2}\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

$$\iint_{S_3} \vec{F}\ .\hat{n}\ ds = \int_0^c \int_0^a (b^2 - zx)\ dx\ dz = \int_0^c \left(b^2x - \frac{x^2}{2}z\right)_0^a\ dz = \int_0^c \left(b^2a - \frac{a^2}{2}z\right)dz$$

$$= \left(b^2az - \frac{a^2}{2}\frac{z^2}{2}\right)_0^c = b^2ac - \frac{a^2c^2}{4}$$

$$\iint_{S_4} \vec{F}\ .\hat{n}\ ds = \int_0^c \int_0^a -(0 - zx)\ dx\ dz = \int_0^c \left(\frac{x^2}{2}z\right)_0^a\ dz = \int_0^c \left(\frac{a^2}{2}z\right)dz$$

$$= \left(\frac{a^2}{2}\frac{z^2}{2}\right)_0^c = \frac{a^2c^2}{4}$$

$$\iint_{S_5} \vec{F}\ .\hat{n}\ ds = \int_0^b \int_0^a (c^2 - xy)\ dx\ dy = \int_0^b \left(c^2x - \frac{x^2}{2}y\right)_0^a\ dy = \int_0^b \left(c^2a - \frac{a^2}{2}y\right)dy$$

$$= \left(c^2ay - \frac{a^2}{2}\frac{y^2}{2}\right)_0^b = c^2ab - \frac{a^2b^2}{4}$$

$$\iint_{S_6} \vec{F}\ .\hat{n}\ ds = \int_0^b \int_0^a -(0 - xy)\ dx\ dy = \int_0^b \left(\frac{x^2}{2}y\right)_0^a\ dy = \int_0^b \left(\frac{a^2}{2}y\right)dy$$

$$= \left(\frac{a^2}{2}\frac{y^2}{2}\right)_0^b = \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\ .\hat{n}\ ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + b^2ac - \frac{a^2c^2}{4} + \frac{a^2c^2}{4} + c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + b^2ac + c^2ab$$

$$= abc(a + b + c)$$

Gauss divergence theorem is verified.