Graphical Method

1. An organization is about to conduct a series of experiments with satellites carrying animals. The animal in each satellite is to be fed two types of meals L and M, in the form of bars that drop into a box from automatic dispensers. The minimum daily nutritional requirements for the animal in terms of nutrients A, B and C, can be supplied with different combinations of quantities for foods. The content of each food type and the weight of a single bar are shown below.

Nutrient	*Nutrient content, units*		*Daily requirements in units*
	L	*M*	
A	*2*	*3*	*90*
B	*8*	*2*	*160*
C	*4*	*2*	*120*
Weight per bar, gm	*3*	*2*	

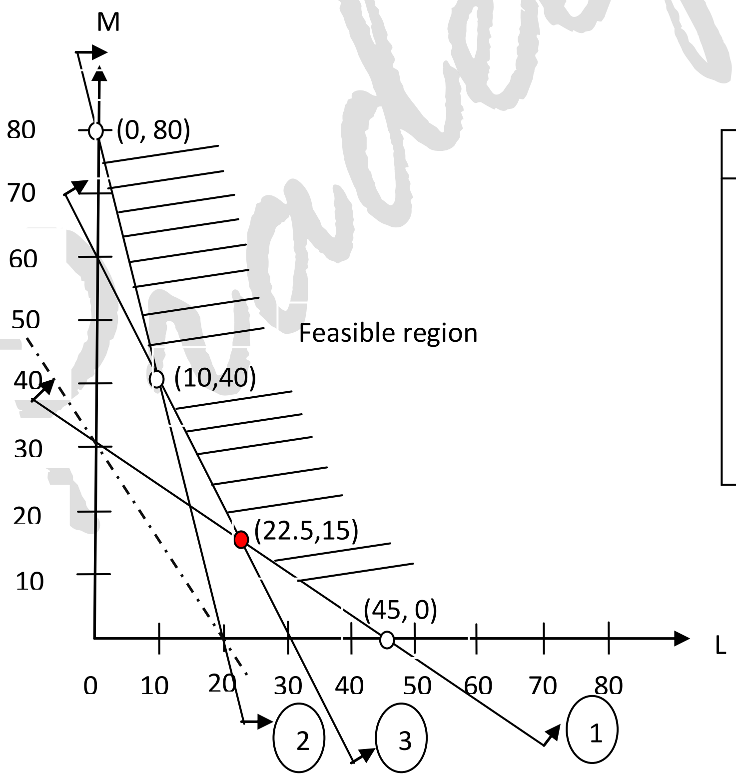
Using graphical method, find the combination of food types to be supplied so that daily requirements can be met while minimizing the total daily weight of the food needed.

Solution:

$$Minimize\ z = 3L + 2M$$

Subject to the constraints

$$2L + 3M \geq 90 \ldots (1)$$
$$8L + 2M \geq 160 \ldots (2)$$
$$4L + 2M \geq 120 \ldots (3)$$
$$L, M \geq 0$$



L	M	z
0	80	160
10	40	110
22.5	**15**	**97.5**
45	0	135

$$L = 22.5, M = 15$$
$$Minimize\ z = 3(22.5) + 2(15) = 97.5$$

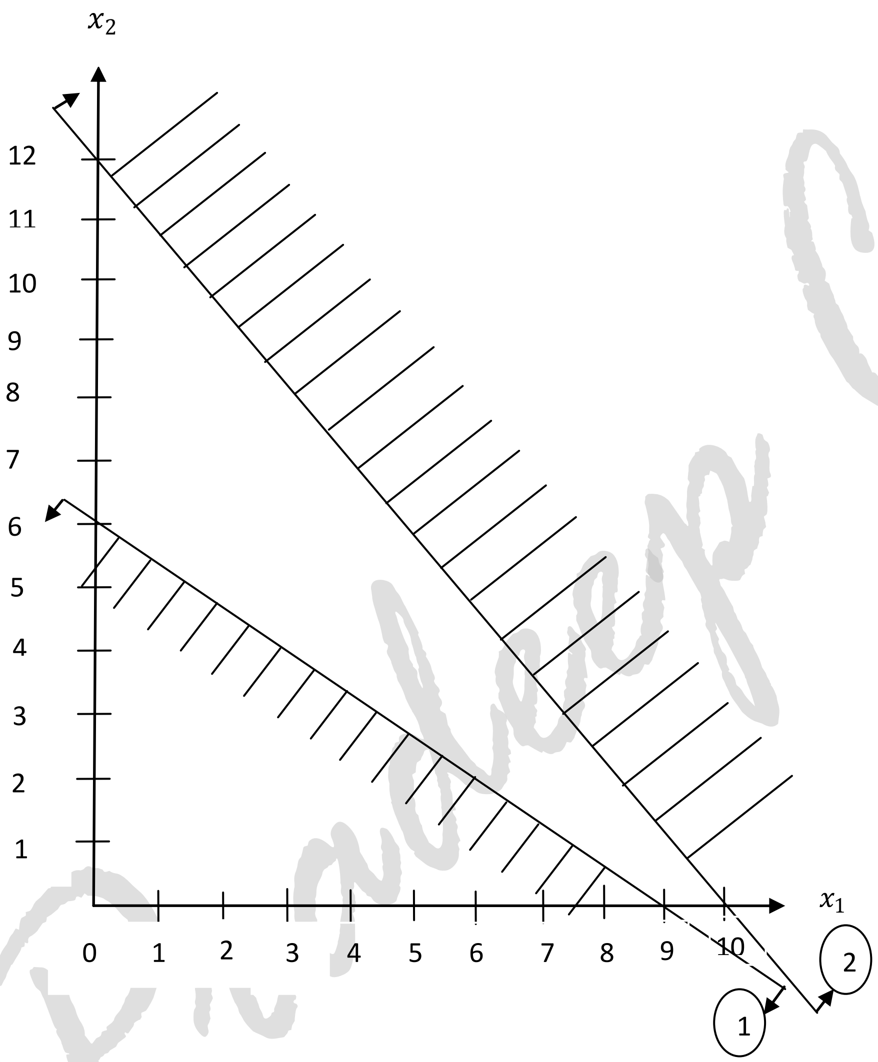
2. Solve the following LP problem graphically:

$$Maximize\ z = 10x_1 + 3x_2$$

Subject to

$$2x_1 + 3x_2 \leq 18$$
$$6x_1 + 5x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Solution:



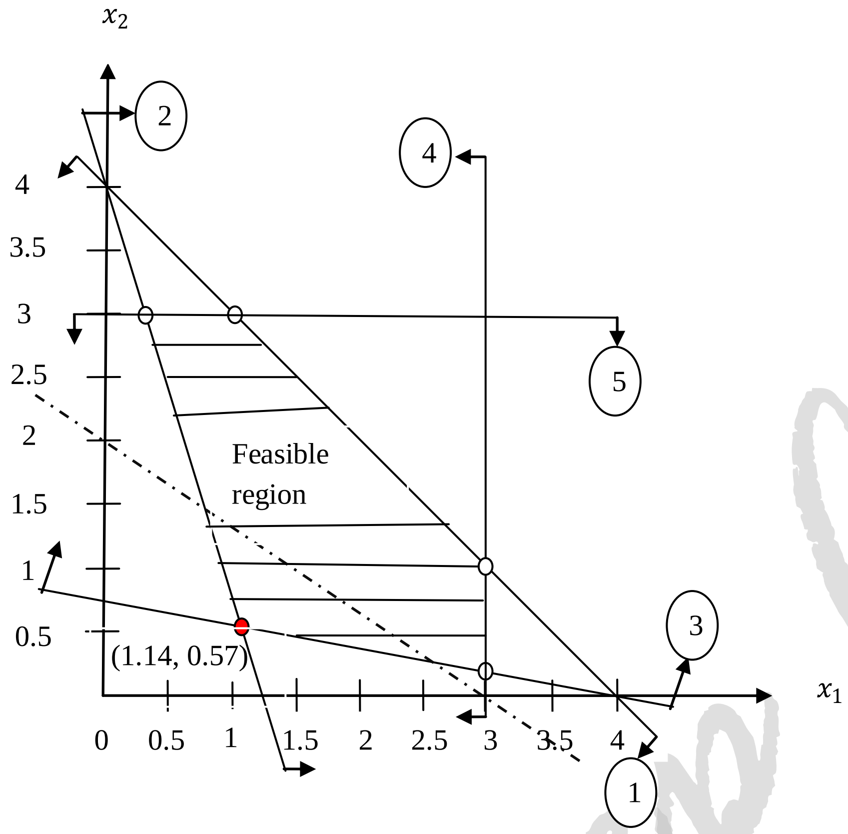
Since there is no feasible region, the solution is infeasible.

3. $Minimize\ z = 2x_1 + 3x_2$

Subject to the constraints

$$x_1 + x_2 \leq 4 \ldots (1)$$
$$6x_1 + 2x_2 \geq 8 \ldots (2)$$
$$x_1 + 5x_2 \geq 4 \ldots (3)$$
$$0 \leq x_1 \leq 3 \ldots (4)$$
$$0 \leq x_2 \leq 3 \ldots (5)$$

Solving the problem by graphical method,



$$x_1 = 1.14, x_2 = 0.57$$
$$Minimize\ z = 2(1.14) + 3(0.57) = 3.99 \approx 4$$

Simplex method

Standard Form

Before the simplex algorithm can be used to solve an LP, the LP must be converted into a problem where all the constraints are equations and all variables are nonnegative. An LP in this form is said to be in standard form.

Conversion of an LP to Standard Form

- A constraint i of an LP is a $\leq$ constraint: adding a slack variable (S_i) to the i th constraint and adding the sign restriction $S_i \geq 0$. A slack variable is the amount of the resource unused in the i th constraint.
- A constraint i of an LP is a $\geq$ constraint : subtracting the Surplus variable (S_i) from the i^{th} constraint and adding the sign restriction $S_i \geq 0$. We define S_i to be the amount by which i th constraint is over satisfied.

1. Solve the following LPP by simplex method

$$Maxmize\ z = 8x_1 - 2x_2$$
$$Subject\ to$$
$$-4x_1 + 2x_2 \leq 1,$$
$$5x_1 - 4x_2 \leq 3,$$
$$x_1, x_2 \geq 0.$$

Solution:

By introducing the slack variables to the $\leq$ constraints the given equation can be rewritten as

$$Maxmize\ z - 8x_1 + 2x_2 - 0s_1 - 0s_2 = 0$$

Subject to

$$-4x_1 + 2x_2 + s_1 = 1,$$
$$5x_1 - 4x_2 + s_2 = 3,$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

Basis	z	x_1	x_2	s_1	s_2	RATIO	SOL
s_1	0	-4	2	1	0		1
s_2	0	5	-4	0	1	3/5=0.6	3
$z_j - c_{ij}$	1	-8	2	0	0		0
s_1	0	0	-1.2	1	0.8		3.4
x_1	0	1	-0.8	0	0.2		0.6
$z_j - c_{ij}$	1	0	-4.4	0	1.6		4.8

Since all the elements in the pivot column are non positive, the solution is unbounded.

2. Solve by simplex method

$$Maximize\ z = 15R + 10P$$

Subject to the restrictions

$$3R + 0P \leq 180$$
$$0R + 5P \leq 200$$
$$4R + 6P \leq 360$$
$$R, P \geq 0$$

Solution:

The problem is rearranged as

$$Maximize\ z - 15R - 10P + 0s_1 + 0s_2 + 0s_3 = 0$$

Subject to the restrictions

$$3R + 0P + s_1 = 180$$
$$0R + 5P + s_2 = 200$$
$$4R + 6P + s_3 = 360$$
$$R, P, s_1, s_2, s_3 \geq 0$$

Basic	z	R	P	s_1	s_2	s_3	Solution	Ratio
s_1	0	3	0	1	0	0	180	60
s_2	0	0	5	0	1	0	200	
s_3	0	4	6	0	0	1	360	90
$z_j - c_j$	1	-15	-10	0	0	0	0	
R	0	1	0	0.333333	0	0	60	
s_2	0	0	5	0	1	0	200	40
s_3	0	0	6	-1.33333	0	1	120	20
$z_j - c_j$	1	0	-10	5	0	0	900	
R	0	1	0	0.333333	0	0	60	
s_2	0	0	0	1.111111	1	-0.83333	100	
P	0	0	1	-0.22222	0	0.166667	20	
$z_j - c_j$	1	0	0	2.777778	0	1.666667	1100	

Since all the values in the row $z_j - c_j \geq 0$, condition for optimality is satisfied.
∴ The optimum solution is $R = 60, P = 20$ *and Maximize* $z = 1100$

3. Solve the following LPP by simplex method

$$Maxmize\ z = 2x_1 + x_2$$

Subject to

$$-x_1 + x_2 \leq 1,$$
$$x_1 - 2x_2 \leq 2,$$
$$x_1, x_2 \geq 0.$$

Solution:

By introducing the slack variables to the ≤ constraints the given equation can be rewritten as

$$Maxmize\ z - 2x_1 - x_2 - 0s_1 - 0s_2 = 0$$

Subject to

$$-x_1 + x_2 + s_1 = 1,$$
$$x_1 - 2x_2 + s_2 = 2,$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

Basis	z	x_1	x_2	s_1	s_2	RATIO	SOL
s_1	0	-1	1	1	0		1
s_2	0	1	-2	0	1	2	2
$z_j - c_{ij}$	1	-2	-1	0	0		0
s_1	0	0	-1	1	1		3
x_1	0	1	-2	0	1		2
$z_j - c_{ij}$	1	0	-5	0	2		4

Since all the elements in the pivot column are non positive, the solution is unbounded.

4. Solve the following LPP by simplex method

$$Maxmize\ z = 4x_1 - x_2 + 2x_3$$

Subject to

$$2x_1 + x_2 + 2x_3 \leq 6,$$
$$x_1 - 4x_2 + 2x_3 \leq 0,$$
$$5x_1 - 2x_2 - 2x_3 \leq 4,$$
$$x_1, x_2, x_3 \geq 0.$$

Solution:

By introducing the slack variables to the ≤ constraints the given equation can be rewritten as

$$Maxmize\ z - 4x_1 + x_2 - 2x_3 - 0s_1 - 0s_2 - 0s_3 = 0$$

Subject to

$$2x_1 + x_2 + 2x_3 + s_1 = 6,$$
$$x_1 - 4x_2 + 2x_3 + s_2 = 0,$$
$$5x_1 - 2x_2 - 2x_3 + s_3 = 4,$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	Ratio	Solution
s_1	0	2	1	2	1	0	0	6/2=3	6
s_2	0	1	-4	2	0	1	0	0/1=0	0
s_3	0	5	-2	-2	0	0	1	4/5=0.8	4
$z_j - c_{ij}$	1	-4	1	-2	0	0	0		0
s_1	0	0	9	-2	1	-2	0	0.666667	6
x_1	0	1	-4	2	0	1	0		0
s_3	0	0	18	-12	0	-5	1	0.222222	4
$z_j - c_{ij}$	1	0	-15	6	0	4	0		0
s_1	0	0	0	4	1	0.5	-0.5	4/4=1	4
x_1	0	1	0	-0.66667	0	-0.11111	0.222222		0.888889
x_2	0	0	1	-0.66667	0	-0.27778	0.055556		0.222222
$z_j - c_{ij}$	1	0	0	-4	0	-0.16667	0.833333		3.333333
x_3	0	0	0	1	0.25	0.125	-0.125		1
x_1	0	1	0	0	0.166667	-0.02778	0.138889		1.555556
x_2	0	0	1	0	0.166667	-0.19444	-0.02778		0.888889
$z_j - c_{ij}$	1	0	0	0	1	0.333333	0.333333		7.333333

Since all the values in the $z_j - c_{ij} \geq 0$, we reached the optimal solution.
The optimal solution $x_1 = 1.555556, x_2 = 0.888889, x_3 = 1$ and $z = 7.333333$

5. Solve the following LPP by simplex method

$$Maxmize\ z = 2x_1 + x_2$$

Subject to

$$3x_1 + x_2 \leq 6,$$
$$x_1 - x_2 \leq 2,$$
$$x_2 \leq 3,$$
$$x_1, x_2 \geq 0.$$

Solution:

By introducing the slack variables to the $\leq$ constraints the given equation can be rewritten as

$$Maxmize\ z - 2x_1 - x_2 - 0s_1 - 0s_2 - 0s_3 = 0$$

Subject to

$$3x_1 + x_2 + s_1 = 6,$$
$$x_1 - x_2 + s_2 = 2,$$
$$x_2 + s_3 = 3,$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Basis	z	x_1	x_2	s_1	s_2	s_3	Ratio	Solution
s_1	0	3	1	1	0	0	2	6
s_2	0	1	-1	0	1	0	2	2
s_3	0	0	1	0	0	1		3
$z_j - c_{ij}$	1	-2	-1	0	0	0		0
s_1	0	0	4	1	-3	0	0	0
x_1	0	1	-1	0	1	0		2
s_3	0	0	1	0	0	1	3	3
$z_j - c_{ij}$	1	0	-3	0	2	0		4
x_2	0	0	1	0.25	-0.75	0		0
x_1	0	1	0	0.25	0.25	0	8	2
s_3	0	0	0	-0.25	0.75	1	4	3
$z_j - c_{ij}$	1	0	0	0.75	-0.25	0		4
x_2	0	0	1	0	0	1		3
x_1	0	1	0	0.333333	0	-0.33333		1
s_2	0	0	0	-0.33333	1	1.333333		4
$z_j - c_{ij}$	1	0	0	0.666667	0	0.333333		5

Since all the values in the $z_j - c_{ij}$ row is ≥ 0, we reached the optimal solution.
The optimal solution is $x_1 = 1, x_2 = 3$, $Max\ z = 5$

6. $Max\ z = 10\,x_1 + 15\,x_2 + 20\,x_3$
Subject to
$2x_1 + 4x_2 + 6x_3 \leq 24$
$3x_1 + 9x_2 + 6x_3 \leq 30$
$x_1, x_2\ and\ x_3 \geq 0$
Solution:
The problem is rearranged as
$Max\ z - 10\,x_1 - 15\,x_2 - 20\,x_3 + 0s_1 + 0s_2$
Subject to
$2x_1 + 4x_2 + 6x_3 + s_1 \leq 24$
$3x_1 + 9x_2 + 6x_3 + s_2 \leq 30$
$x_1, x_2, x_3, s_1\ and\ s_2 \geq 0$

Basis	z	x_1	x_2	x_3	s_1	s_2	Solution	Ratio
s_1	0	2	4	6	1	0	24	4
s_2	0	3	9	6	0	1	30	5
$z_j - c_j$	1	-10	-15	-20	0	0	0	
x_3	0	0.333333	0.666667	1	0.166667	0	4	12
s_2	0	1	5	0	-1	1	6	6
$z_j - c_j$	1	-3.33333	-1.66667	0	3.333333	0	80	
x_3	0	0	-1	1	0.5	-0.3333	2	
x_1	0	1	5	0	-1	1	6	
$z_j - c_j$	1	0	15	0	0	3.3333	100	

Since all the values in the row $z_j - c_j \geq 0$, we reach the optimum solution.

$x_1 = 6, x_2 = 0 \ and \ x_3 = 2$
$Max \ z = 100.$

7. $Max \ z = 16 \ x_1 + 17 \ x_2 + 10 \ x_3$
Subject to
$x_1 + 2x_2 + 4x_3 \leq 2000$
$2x_1 + x_2 + x_3 \leq 3600$
$x_1 + 2x_2 + 2x_3 \leq 2400$
$x_1 \leq 30$
$x_1, x_2 \ and \ x_3 \geq 0$
Solution:
The problem is rearranged as
$Max \ z - 16 \ x_1 - 17 \ x_2 - 10 \ x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$
Subject to
$x_1 + 2x_2 + 4x_3 + s_1 = 2000$
$2x_1 + x_2 + x_3 + s_2 = 3600$
$x_1 + 2x_2 + 2x_3 + s_3 = 2400$
$x_1 + s_4 = 30$
$x_1, x_2, x_3, s_1, s_2, s_3 \ and \ s_4 \geq 0$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Solution	Ratio
s_1	0	1	2	4	1	0	0	0	2000	1000
s_2	0	2	1	1	0	1	0	0	3600	3600
s_3	0	1	2	2	0	0	1	0	2400	1200
s_4	0	1	0	0	0	0	0	1	30	
$z_j - c_j$	1	-16	-17	-10	0	0	0	0	0	
x_2	0	0.5	1	2	0.5	0	0	0	1000	2000
s_2	0	1.5	0	-1	-0.5	1	0	0	2600	1733.333
s_3	0	0	0	-2	-1	0	1	0	400	
s_4	0	1	0	0	0	0	0	1	30	30
$z_j - c_j$	1	-7.5	0	24	8.5	0	0	0	17000	
x_2	0	0	1	2	0.5	0	0	-0.5	985	
s_2	0	0	0	-1	-0.5	1	0	-1.5	2555	
s_3	0	0	0	-2	-1	0	1	0	400	
x_1	0	1	0	0	0	0	0	1	30	
$z_j - c_j$	1	0	0	24	8.5	0	0	7.5	17225	

Since all the values in the row $z_j - c_j \geq 0$, we reach the optimum solution.
$x_1 = 30, x_2 = 985 \ and \ x_3 = 0$
$Max \ z = 17225.$

8. $Max \ z = 3 \ x + 2 \ y$
Subject to
$2x + y \leq 6$
$x + 2y \leq 6$

$x \text{ and } y \geq 0$

Solution:

The problem is rearranged as

$Max\ z - 3\,x - 2\,y + 0s_1 + 0s_2 = 0$

Subject to

$2x + y + s_1 = 6$

$x + 2y + s_2 = 6$

$x, y, s_1 \text{ and } s_2 \geq 0$

Basis	z	x	y	s_1	s_2	Solution	Ratio
s_1	0	2	1	1	0	6	3
s_2	0	1	2	0	1	6	6
$z_j - c_j$	1	-3	-2	0	0	0	
x	0	1	0.5	0.5	0	3	6
s_2	0	0	1.5	-0.5	1	3	2
$z_j - c_j$	1	0	-0.5	1.5	0	9	
x	0	1	0	0.666667	-0.33333	2	
y	0	0	1	-0.33333	0.666667	2	
$z_j - c_j$	1	0	0	1.333333	0.333333	10	

Since all the values in the row $z_j - c_j \geq 0$, we reached the optimum solution.

$x = 2, y = 2$

$Max\ z = 10.$

9. *A company produces 2 types of hats A and B. Every hat B requires twice as much labour time as hat A. The company can produce a total of 500 hats a day. The market limits daily sales of the A and B to 150 and 250 hats respectively. The profits on hats A and B are Rs. 8 and Rs. 5 respectively. Solve for an optimal solution.*

Solution:

Let x_1 and $x_2$represents the number of hat A and B respectively.

$$Maximize\ z = 8x_1 + 5x_2$$

Subject to

$$x_1 + 2x_2 \leq 500$$
$$x_1 \leq 150$$
$$x_2 \leq 250$$
$$x_1, x_2 \geq 0$$

The problem is rearranged as

$Max\ z - 8\,x_1 - 5\,x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

Subject to

$x_1 + 2x_2 + s_1 = 500$

$x_1 + s_2 = 150$

$2x_2 + s_3 = 250$

$x_1, x_2, , s_1\ , s_2, s_3\ \geq 0$

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	1	2	1	0	0	500	500
s_2	0	1	0	0	1	0	150	150
s_3	0	0	1	0	0	1	250	
$z_j - c_j$	1	-8	-5	0	0	0	0	
s_1	0	0	2	1	-1	0	350	175
x_1	0	1	0	0	1	0	150	
s_3	0	0	1	0	0	1	250	250
$z_j - c_j$	1	0	-5	0	8	0	1200	
x_2	0	0	1	0.5	-0.5	0	175	
x_1	0	1	0	0	1	0	150	
s_3	0	0	0	-0.5	0.5	1	75	
$z_j - c_j$	1	0	0	2.5	5.5	0	2075	

Since all the values in the row $z_j - c_j \geq 0$, we reach the optimum solution.
$x_1 = 150, x_2 = 175$
$Max\ z = 2075.$

Dual Simplex method

1. Solve the following LPP by dual simplex method

$$Min\ 5x_1 + 6x_2$$

Subject to:

$$x_1 + x_2 \geq 2$$
$$4x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

Solution:

$$Max - z = -5x_1 - 6x_2$$
$$Max\ z^* = -5x_1 - 6x_2 \text{ where } z^* = -z$$

Subject to:

$$-x_1 - x_2 \leq -2$$
$$-4x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

The problem is rearranged as

$$Max\ z^* + 5x_1 + 6x_2 + 0s_1 + 0s_2 = 0$$

Subject to:

$$-x_1 - x_2 + s_1 = -2$$
$$-4x_1 - x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Basis	z^*	x_1	x_2	s_1	s_2	solution
s_1	0	-1	-1	1	0	-2
s_2	0	-4	-1	0	1	-4
$z_j^* - c_j$	1	5	6	0	0	0
Ratio		1.25	6			

Basis	z^*	x_1	x_2	s_1	s_2	solution
s_1	0	0	-0.75	1	-0.25	-1
x_1	0	1	0.25	0	-0.25	1
$z_j^* - c_j$	1	0	4.75	0	1.25	-5
Ratio			6.333333		5	

Basis	z^*	x_1	x_2	s_1	s_2	solution
s_2	0	0	3	-4	1	4
x_1	0	1	1	-1	0	2
$z_j^* - c_j$	1	0	1	5	0	-10

Since all the basic variables are ≥ 0, Optimum solution is reached.

$$x_1 = 2, x_2 = 0$$

$$Max\ z^* = -10 \Rightarrow Min\ z = 10$$

2. Use dual simplex method to solve the following:

$$Min\ z = 5x + 2y + 4z$$

Subject to:

$$3x + y + 2z \geq 4$$

$$6x + 3y + 5z \geq 10$$

$$x, y, z \geq 0$$

Solution:

$$Max - z = -5x - 2y - 4z$$

$$Max\ z^* = -5x - 2y - 4z \ \ where\ z^* = -z$$

Subject to:

$$-3x - y - 2z \leq -4$$

$$-6x - 3y - 5z \leq -10$$

$$x, y, z \geq 0$$

The problem is rearranged as

$$Max\ z^* + 5x + 2y + 4z + 0s_1 + 0s_2 = 0$$

Subject to:

$$-3x - y - 2z + s_1 = -4$$

$$-6x - 3y - 5z + s_2 = -10$$

$$x, y, z, , s_1, s_2 \geq 0$$

Basis	z^*	x	y	z	s_1	s_2	solution
s_1	0	-3	-1	-2	1	0	-4
s_2	0	-6	-3	-5	0	1	-10
$z_j^* - c_j$	1	5	2	4	0	0	0
Ratio		0.833333	0.666667	0.8			

Basis	z^*	x	y	z	s_1	s_2	solution
s_1	0	-1	0	-0.33333	1	-0.33333	-0.66667
y	0	2	1	1.666667	0	-0.33333	3.333333
$z_j^* - c_j$	1	1	0	0.666667	0	0.666667	-6.66667
Ratio		1		2		2	

Basis	z^*	x	y	z	s_1	s_2	solution
x	0	1	0	0.333333	-1	0.333333	0.666667
y	0	0	1	1	2	-1	2
$z_j^* - c_j$	1	0	0	0.333333	1	0.333333	-7.33333

Since all the basic variables are ≥ 0, Optimum solution is reached.

$$x = 0.666667, y = 2, z = 0$$

$$Max\ z^* = -7.33333 \Rightarrow Min\ z = 7.33333$$

3. A Person requires 10, 12 and 12 units of a dry and liquid combination of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per jar. A dry product contains 1, 2 and 4 units of A, B and C per carbon. If the liquid product sells for Rs. 3 per jar and the dry product sells for Rs. 2 per carbon. How many of each should he purchase in order to minimize the cost and meet the requirement?

Solution:

Let x_1, x_2 represents the number of dry product and liquid product of chemicals A, B and C respectively.

$$Minimize\ z = 2x_1 + 3x_2$$

Subject to

$$x_1 + 5x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$4x_1 + x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Converting minimization problem in to maximization problem and solving by dual simplex method as follows

$$Max - z = -2x_1 - 3x_2$$
$$Max\ z^* = -2x_1 - 3x_2 \ \ where\ z^* = -z$$

Subject to:

$$-x_1 - 5x_2 \leq -10$$
$$-2x_1 - 2x_2 \leq -12$$
$$-4x_1 - x_2 \leq -12$$
$$x_1, x_2 \geq 0$$

The problem is rearranged as

$$Max\ z^* + 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to:

$$-x_1 - 5x_2 + s_1 = -10$$
$$-2x_1 - 2x_2 + s_2 = -12$$
$$-4x_1 - x_2 + s_3 = -12$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Basic	z^*	x_1	x_2	s_1	s_2	s_3	solution
s_1	0	-1	-5	1	0	0	-10
s_2	0	-2	-2	0	1	0	-12
s3	0	-4	-1	0	0	1	-12
$Z_j^* - c_j$	1	2	3	0	0	0	0
Ratio		1	1.5				

Basic	z^*	x_1	x_2	s_1	s_2	s_3	solution
s_1	0	0	-4	1	-0.5	0	-4
x_1	0	1	1	0	-0.5	0	6
s_3	0	0	3	0	-2	1	12
$Z_j^* - c_j$	1	0	1	0	1	0	-12
Ratio			0.25		2		

Basic	z^*	x_1	x_2	s_1	s_2	s_3	solution
x_2	0	0	1	-0.25	0.125	0	1
x_1	0	1	0	0.25	-0.625	0	5
s_3	0	0	0	0.75	-2.375	1	9
$Z_j^* - c_j$	1	0	0	0.25	0.875	0	-13

Since all the basic variables are ≥ 0, Optimum solution is reached.

$$x_1 = 5, x_2 = 1$$
$$Max\ z^* = -13 \Rightarrow Min\ z = 13$$

4. Use dually to solve the following linear programming problem.

$$Maximize\ z = 2x_1 + x_2$$

Subject to the constraints

$$x_1 + 2x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$
$$x_1 - 2x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Solution:

Dual of the primal problem is

$$Minimize\ z = 10y_1 + 6y_2 + 2y_3 + y_4$$

Subject to

$$y_1 + y_2 + y_3 + y_4 \geq 2$$

$$2y_1 + y_2 - y_3 - 2y_4 \geq 1$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Solving by dual simplex method, we get

$$Maximize\ z^* = -10y_1 - 6y_2 - 2y_3 - y_4 \qquad where\ z^* = -z$$

Subject to

$$-y_1 - y_2 - y_3 - y_4 \leq -2$$

$$-2y_1 - y_2 + y_3 + 2y_4 \leq -1$$

$$y_1, y_2, y_3, y_4 \geq 0$$

The problem is rearranged as

$$Maximize\ z^* + 10y_1 + 6y_2 + 2y_3 + y_4 + 0s_1 + 0s_2 = 0$$

Subject to

$$-y_1 - y_2 - y_3 - y_4 + s_1 = -2$$
$$-2y_1 - y_2 + y_3 + 2y_4 + s_2 = -1$$
$$y_1, y_2, y_3, y_4, s_1, s_2 \geq 0$$

Basic	z	y_1	y_2	y_3	y_4	s_1	s_2	Solution
s_1	0	-1	-1	-1	-1	1	0	-2
s_2	0	-2	-1	1	2	0	1	-1
$z_j - c_j$	1	10	6	2	1	0	0	0
Ratio		10	6	2	1			

Basic	z	y_1	y_2	y_3	y_4	s_1	s_2	Solution
y_4	0	1	1	1	1	-1	0	2
s_2	0	-4	-3	-1	0	2	1	-5
$z_j - c_j$	1	9	5	1	0	1	0	-2
Ratio		2.25	1.666667	1				

Basic	z	y_1	y_2	y_3	y_4	s_1	s_2	Solution
y_4	0	-3	-2	0	1	1	1	-3
y_3	0	4	3	1	0	-2	-1	5
$z_j - c_j$	1	5	2	0	0	3	1	-7
Ratio		1.666667	1					

Basic	z	y_1	y_2	y_3	y_4	s_1	s_2	Solution
y_2	0	1.5	1	0	-0.5	-0.5	-0.5	1.5
y_3	0	-0.5	0	1	1.5	-0.5	0.5	0.5
$z_j - c_j$	1	2	0	0	1	4	2	-10

Since all the basic variables are ≥ 0, Optimum solution is reached.

$$y_1 = 0, y_2 = 1.5, y_3 = 0.5, y_4 = 0$$

$$Max\ z^* = -10 \Rightarrow Min\ z = 10$$

Big M method

1. Modify the constraints so that the rhs of each constraint is nonnegative. Identify each constraint that is now an $=$ or $\geq$ const.
2. Convert each inequality constraints to standard form (add a slack variable for $\leq$ constraints, add an excess variable or surplus variable for $\geq$ consts).
3. For each $\geq$ or $=$ constraint, add artificial variables. Add sign restriction $A_i \geq 0$.
4. Let M denote a very large positive number. Add (for each artificial variable) MA_i to min problem objective functions or $-MA_i$ to max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row $z_j - c_{ij}$ before beginning the simplex. Remembering M represents a very large number, solve the transformed problem by the simplex.

1. $Maximize\ Z = 5x - 2y + 3z$
Subject to:
$2x + 2y - z \geq 2$
$3x - 4y \leq 3$
$y + 3z \leq 5$
$x, y, z \geq 0$
Solution:
The problem is rearranged as
$Maximize\ Z - 5x + 2y - 3z + 0s_1 + 0s_2 + 0s_3 + MA_1 = 0$
Subject to:
$2x + 2y - z - s_1 + A_1 = 2$
$3x - 4y + s_2 = 3$
$y + 3z + s_3 = 5$

$x, y, z, s_1, s_2, s_3, A_1 \geq 0$

Basic	z	x	y	z	s_1	s_2	s_3	A_1	solution	Ratio
A_1	0	2	2	-1	-1	0	0	1	2	1
s_2	0	3	-4	0	0	1	0	0	3	1
s_3	0	0	1	3	0	0	1	0	5	
$z_j - c_j$	1	-5	2	-3	0	0	0	M	0	
$z_j - c_j$	1	-5-2M	2-2M	-3+M	M	0	0	0	-2M	
x	0	1	1	-0.5	-0.5	0	0	0.5	1	
s_2	0	0	-7	1.5	1.5	1	0	-1.5	0	0
s_3	0	0	1	3	0	0	1	0	5	1.666
$z_j - c_j$	1	0	7	-5.5	-2.5	0	0	2.5+M	5	
x	0	1	-1.333	0	0	0.333	0	0	1	
z	0	0	-4.667	1	1	0.667	0	-1	0	
s_3	0	0	15	0	-3	-2	1	3	5	0.333
$z_j - c_j$	1	0	-18.667	0	3	3.667	0	-3+M	5	
x	0	1	0	0	-0.267	0.156	0.089	0.267	1.444	
z	0	0	0	1	0.067	0.044	0.311	-0.067	1.556	
y	0	0	1	0	-0.2	-0.133	0.067	0.2	0.333	
$z_j - c_j$	1	0	0	0	-0.733	1.178	1.244	0.733+M	11.222	
x	0	1	0	4	0	0.333	1.333	0	7.667	
s_1	0	0	0	15	1	0.667	4.667	-1	23.333	
y	0	0	1	3	0	0	1	0	5	
$z_j - c_j$	1	0	0	11	0	1.667	4.667	M	28.333	

Since all the values in the row $z_j - c_j$ are ≥ 0 and the artificial variable is a non basic variable, therefore Optimum solution is reached.

$$x = 7.667, y = 5, z = 0$$
$$Max\ z = 28.333$$

2. The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per Kg and B_2 costs Rs. 3 per Kg. Strength considerations dictate that the brick should contain not more than 4 Kg of B_1 and a minimum of 2 Kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out the minimum cost of the brick.

Solution:

Let x_1 and x_2 represents the number of Kilograms of ingredients B_1 and B_2 respectively.

$$Minimize\ z = 5x_1 + 3x_2$$

Subject to

$$x_1 + x_2 = 5$$

$$x_1 \leq 4$$

$$x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

The above problem is rearranged as follows

$$Minimize\ z - 5x_1 - 3x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

Subject to

$$x_1 + x_2 + A_1 = 5$$

$$x_1 + s_1 = 4$$

$$x_2 - s_2 + A_2 \geq 2$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Basic	z	x_1	x_2	s_1	s_2	A_1	A_2	**Solution**	**Ratio**
A_1	0	1	1	0	0	1	0	5	5
s_1	0	1	0	1	0	0	0	4	
A_2	0	0	1	0	-1	0	1	2	2
$z_j - c_j$	1	-5	-3	0	0	-M	-M	0	
$z_j - c_j$	1	-5+M	-3+2M	0	-M	0	0	7M	
A_1	0	1	0	0	1	1	-1	3	3
s_1	0	1	0	1	0	0	0	4	
x_2	0	0	1	0	-1	0	1	2	
$z_j - c_j$	1	-5+M	0	0	-3+M	0	3-2M	6+3M	
s_2	0	1	0	0	1	1	-1	3	
s_1	0	1	0	1	0	0	0	4	
x_2	0	1	1	0	0	1	0	5	
$z_j - c_j$	1	-2	0	0	0	3-M	-M	15	

Since all the values in the row $z_j - c_j \leq 0$ and artificial variable is not in the basic, optimum solution is reached.

$$x_1 = 0, x_2 = 5$$

$$Minimize\ z = 15$$

3. Write down the dual of the following LPP and solve.

$$Maximize\ z = 4x_1 + 2x_2$$

Subject to the constraints

$$x_1 + x_2 \geq 3$$

$$x_1 - x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Convert all $\geq$ constraints to $\leq$ constraints

$$x_1 + x_2 \geq 3 \Rightarrow -x_1 - x_2 \leq -3$$

$$x_1 - x_2 \geq 2 \Rightarrow -x_1 + x_2 \leq -2$$

Dual of the primal problem is

$$Minimize\ z = -3y_1 - 2y_2$$

Subject to

$$-y_1 - y_2 \geq 4$$

$$-y_1 + y_2 \geq 2$$

$$y_1, y_2 \geq 0$$

Rearranging the problem and solving by Big M method,

$$Minimize\ z + 3y_1 + 2y_2 + 0s_1 + 0s_2 - MA_1 - MA_2 = 0$$

Subject to

$$-y_1 - y_2 - s_1 + A_1 = 4$$

$$-y_1 + y_2 - s_2 + A_2 = 2$$

$$y_1, y_2 \geq 0$$

Basic	z	y_1	y_2	S_1	S_2	A_1	A_2	**Solution**	**Ratio**
A_1	0	-1	-1	-1	0	1	0	4	
A_2	0	-1	1	0	-1	0	1	2	2
$z_j - c_j$	1	3	2	0	0	-M	-M	0	
$z_j - c_j$	1	3-2M	2	-M	-M	0	0	6M	

Basic	z	y_1	y_2	S_1	S_2	A_1	A_2	**Solution**
A_1	0	-2	0	-1	-1	1	1	6
y_2	0	-1	1	0	-1	0	1	2
$z_j - c_j$	1	5-2M	0	-M	2-M	0	-2	6M-4

Since all the values in the row $z_j - c_j \leq 0$, condition for optimality is satisfied. But artificial variable is still in the basic. Therefore the solution is unbounded.

Two phase method

Two phase simplex method is used for problems involving '≥' or '=' type constraints.
Algorithm for two-Phase method id discussed as follows.
Step 0: Obtain the canonical form of the given problem.
Phase 1
Step 1: Form a modified problem for phase I from the canonical form of the original problem by replacing the objective function with the sum of only the artificial variables along with the same set of constraints of the canonical form of the original problem.
Step 2: Prepare the initial table for phase 1.
Step 3: Apply the usual simplex method till the optimality is reached.
Step 4: Check whether the objective function value is zero in the optimal table of phase 1. If yes, go to phase 2; otherwise, conclude that the original problem has no feasible solution and stop.
Phase 2
Step 5: Obtain a modified table using the following steps:
Drop the columns in the optimum table of phase 1 corresponding to the artificial variables which are currently non basic.
If some artificial variables are present at zero level in the basic solution of the optimal table of phase 1, substitute its objective function coefficients with zero.
Substitute the coefficients of the original objective function in the optimal table of the phase 1 for the remaining variables.
Step 6: Carry out further iterations till the optimality is reached and then stop.

1. $Minimize\ z = 2x_1 + 3x_2$
Subject to

$$x_1 + 5x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$4x_1 + x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution:
Phase 1

$$Minimize\ z = A_1 + A_2 + A_3$$

Subject to

$$x_1 + 5x_2 - s_1 + A_1 = 10$$
$$2x_1 + 2x_2 - s_2 + A_2 = 12$$
$$4x_1 + x_2 - s_3 + A_3 = 12$$
$$x_1, x_2, s_1, s_2, s_3, A_1, A_2, A_3 \geq 0$$

Basic	z	x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	Solution	Ratio
A_1	0	1	5	-1	0	0	1	0	0	10	2
A_2	0	2	2	0	-1	0	0	1	0	12	6
A_3	0	4	1	0	0	-1	0	0	1	12	12
$z_j - c_j$	1	0	0	0	0	0	-1	-1	-1	0	
$z_j - c_j$	1	7	8	-1	-1	-1	0	0	0	34	
x_2	0	0.2	1	-0.2	0	0	0.2	0	0	2	10
A_2	0	1.6	0	0.4	-1	0	-0.4	1	0	8	5
A_3	0	3.8	0	0.2	0	-1	-0.2	0	1	10	2.631
$z_j - c_j$	1	5.4	0	0.6	-1	-1	-1.6	0	0	18	
x_2	0	0	1	-0.211	0	0.053	0.211	0	-0.053	1.474	28
A_2	0	0	0	0.316	-1	0.421	-0.316	1	-0.421	3.789	9
x_1	0	1	0	0.053	0	-0.263	-0.053	0	0.263	2.632	
$z_j - c_j$	1	0	0	0.315	-1	0.421	-1.316	0	-1.421	3.789	
x_2	0	0	1	-0.25	0.125	0	0.25	-0.125	0	1	
s_3	0	0	0	0.75	-2.375	1	-0.75	2.375	-1	9	
x_1	0	1	0	0.25	-0.625	0	-0.25	0.625	0	5	
$z_j - c_j$	1	0	0	0	0	0	-1	-1	-1	0	

Criterion for optimality is reached and all the artificial variables are non Basic variables. Therefore we move to Phase 2.

Phase 2:

$$Minimize\ z - 2x_1 - 3x_2 = 0$$

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
x_2	0	0	1	-0.25	0.125	0	1
s_3	0	0	0	0.75	-2.375	1	9
x_1	0	1	0	0.25	-0.625	0	5
$z_j - c_j$	1	-2	-3	0	0	0	0
$z_j - c_j$	1	0	0	-0.25	-0.875	0	13

Since all the values in the $z_j - c_j$ rows are ≤ 0, condition for optimality is satisfied.

$$x_1 = 5, x_2 = 1, Minimize\ z = 13$$

2. $Minimize\ z = 10x_1 + 6x_2 + 2x_3$

Subject to the constraints

$$-x_1 + x_2 + x_3 \geq 1$$
$$3x_1 + x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

Phase 1

$$Minimize\ z = A_1 + A_2$$

Subject to the constraints

$$-x_1 + x_2 + x_3 - s_1 + A_1 = 1$$
$$3x_1 + x_2 - x_3 - s_2 + A_2 = 2$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

Basic	z	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Solution	Ratio
A_1	0	-1	1	1	-1	0	1	0	1	
A_2	0	3	1	-1	0	-1	0	1	2	0.666667
$z_j - c_j$	1	0	0	0	0	0	-1	-1	0	
$z_j - c_j$	1	2	2	0	-1	-1	0	0	3	
A_1	0	0	1.333333	0.666667	-1	-0.33333	1	0.333333	1.666667	1.25
x_1	0	1	0.333333	-0.33333	0	-0.33333	0	0.333333	0.666667	2
$z_j - c_j$	1	0	1.333333	0.666667	-1	-0.33333	0	-0.66667	1.666667	
x_2	0	0	1	0.5	-0.75	-0.25	0.75	0.25	1.25	
x_1	0	1	0	-0.5	0.25	-0.25	-0.25	0.25	0.25	
$z_j - c_j$	1	0	0	0	0	0	-1	-1	0	

Criterion for optimality is reached and all the artificial variables are non Basic variables. Therefore we move to Phase 2.

Phase 2:

Basic	z	x_1	x_2	x_3	s_1	s_2	Solution
x_2	0	0	1	0.5	-0.75	-0.25	1.25
x_1	0	1	0	-0.5	0.25	-0.25	0.25
$z_j - c_j$	1	-10	-6	-2	0	0	0
$z_j - c_j$	1	0	0	-4	-2	-4	10

Since all the values in the $z_j - c_j$ rows are ≤ 0, condition for optimality is satisfied.

$$x_1 = 0.25, x_2 = 1.25, x_3 = 0, Minimize\ z = 10$$