**SRI RAMAKRISHNA INSTITUTE OF TECHNOLOGY, COIMBATORE- 10**
**DEPARTMENT OF SCIENCE AND HUMANITIES**
**SUBJECT: NUMERICAL METHODS ( SEMESTER – VI )**
**UNIT V**

## BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQAUTIONS

1. State the conditions for the equation.

$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$ where $A, B, C, D, E, F, G$ are function of $x$ $and$ $y$ to be (i) . Elliptic (ii). Parabolic (iii). Hyperbolic.

Solution :

The given equation is said to be

(i). Elliptic at appoint $(x, y)$ in the plane if $B^2 - 4\,A\,C < 0$

(ii). Parabolic if $B^2 - 4\,A\,C = 0$

(iii). Hyperbolic if $B^2 - 4\,A\,C > 0$.

2. State the conditions for the equation.

$A\,u_{xx} + 2\,B\,u_{xy} + C\,u_{yy} = f(u_x, u_y, x, y)$ to be

(i) . Elliptic (ii). Parabolic (iii). Hyperbolic when $A, B, C$ are functions of $x$ $and$ $y$.

Solution :

The given equation is elliptic if $(2\,\mathrm{B}^2) - 4\,A\,C < 0$

$$i,e\,, \quad B^2 - A\,C < 0.$$

The given equation is parabolic if $\mathrm{B}^2 - A\,C = 0$.

The given equation is hyperbolic if $\mathrm{B}^2 - A\,C > 0$.

3. Fill up the blank.

The equation $y\,u_{xx} + u_{yy} = 0$ is hyperbolic in the region ………………………

Solution :

Here $A = y, B = 0, C = 1$

$\mathrm{B}^2 - 4\,A\,C = 0 - 4\,y = -4\,y.$

The equation is hyperbolic in the region $(x, y)$ where $\mathrm{B}^2 - 4A\,C > 0$.

$$i,e\,,\ B^2 - A\,C > -4\,y > 0 \ \ or \ \ y > 0.$$

It is a parabolic in the region $y > 0$.

4. What is the classification of $f_x - f_{yy} = 0$?

Solution :

Here $A = 0,\ B = 0,\ C = -1$

$$B^2 - 4AC = 0 - 4 \times 0 \times (-1)$$

So the equation is parabolic.

5. Give an example of parabolic equation.

Solution :

The one dimensional heat equation $\frac{\partial u}{\partial t} = \propto^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

6. State Schmidt's explicit formula for solving heat flow equation.

Solution :

$$u_{i,j+1} = \lambda\, u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

$$\text{If } \lambda = \frac{1}{2}, \quad u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$$

7. Fill in the blank.

Bender Schmidt's recurrence formula is useful to solve ………………… equation.

Solution : One dimensional heat equation.

8. Write an explicit formula to solve numerically the heat equation ( parabolic equation ) $u_{xx} - a\, u_t = 0$.

Solution :

$$u_{i,j+1} = \lambda\, u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

Where $\lambda = \frac{k}{h^2\, a}$ ( h is the space for the variable $x$ and $k$ is the space in the time direction ).

9. What is the value of $k$ to solve $\frac{\partial u}{\partial t} = \frac{1}{2} u_{xx}$ by Bender Schmidt's method with $h = 1$ if $h$ *and* $t$ respectively ?

Solution :

Given $u_{xx} = 2 \frac{\partial u}{\partial t}$

Here $\alpha^2 = 2,\ h = 1$

$\lambda = \frac{k\,\alpha^2}{h^2} = \frac{k\,(2)}{1} = 2\,k$

$\lambda = 2\,k = \frac{1}{2}$

$k = \frac{1}{4}$.

10. What is the classification of one dimensional flow equation.

Solution :

Here $A = 1,\ B = 0,\ C = 0$

$\therefore B^2 - 4\,A\,C = 0.$

Hence the one dimensional heat flow equation is parabolic.

11. Write down the Crank – Nicolson formula to solve $u_t = u_{xx}$.

Solution :

$$\frac{1}{2}\,\lambda\, u_{i+1,j+1} + \frac{1}{2}\,\lambda\; u_{i-1,j+1} - (\lambda+1)\, u_{i,j+1}$$
$$= -\,\frac{1}{2}\,\lambda\, u_{i+1,j} - \frac{1}{2}\,\lambda\, u_{i-1,j} + (\lambda-1)\, u_{i,j}$$
$$(or) \quad \lambda\left(u_{i+1,j+1} + u_{i-1,j+1}\right) - 2(\lambda+1)\, u_{i,j+1}$$
$$= 2(\lambda-1)\, u_{i,j} - \lambda\left(u_{i+1,j} + u_{i-1,j}\right)$$

12. Write down the implicit formula to solve one dimensional heat flow equation.

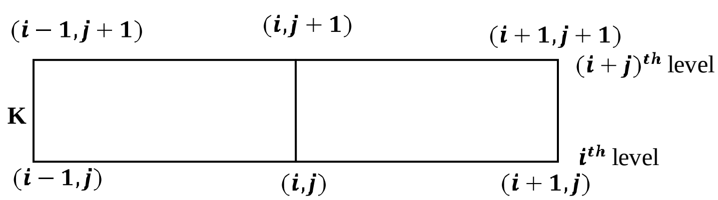
Solution :

$$u_{xx} = \frac{1}{c^2}\, u_t$$

13. Why is Crank Nicholson's scheme called an implicit scheme ?

Solution :

The schematic representation of Crank Nicholson's method is shown below.




The solution value at any point $(i, j+1)$ on the $(j+1)^{th}$ level is dependent on the solution values at the neighboring points on the same level and three values on the $j^{th}$ level. Hence it is implicit.

14. Fill up blanks.

In the parabolic equation $u_t = \alpha^2\, u_{xx}$ $\;if\;\; \lambda = \frac{k\,\alpha^2}{h^2}$ where $k = \Delta t \;\; and \;\; h = \Delta x$, then

(a). explicit method is stable only if $\lambda = \cdots$ … … … … ….

(b). implicit method is convergent when $\lambda = \cdots$ … … …

Solution :

(a). explicit method is stable only if $\lambda < \frac{1}{2}$.

(b). implicit method is convergent when $\lambda = \frac{1}{2}$.

15. what type of equations can be solved by using Crank Nicholson's difference formula ?

Solution :

Crank Nicholson's difference formula is used to solve parabolic equations is of the form

$$u_{xx} a\ u_t\ .$$

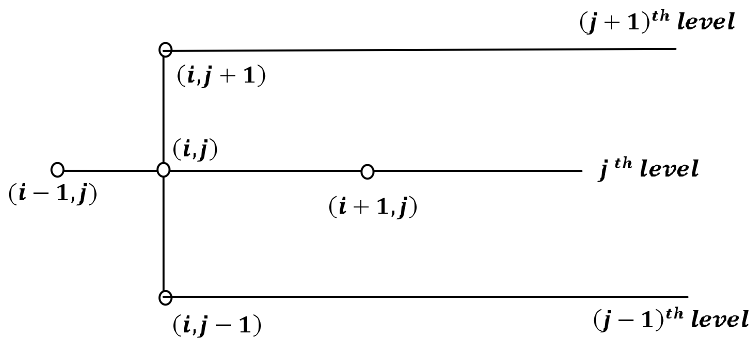
16. State the explicit formula for the solution of the wave equation.

Solution :

The formula to solve numerically the wave equation

$$a^2\ u_{xx} = 2\ (1 - \lambda^2\ a^2)\ u_{i,j} + \lambda^2\ a^2\ \left(u_{i+1,j} + u_{i-1,j}\right) - u_{i,j-1}$$

The schematic representation is shown below.




17. Write down the general and simplest forms of the difference equation corresponding to the hyperbolic equation $u_{tt} = a^2 u_{xx}$.

Solution :

The general form of the difference equation to solve the equation $u_{tt} = a^2 u_{xx}$ is

$$u_{i,j+1} = 2(1 - \lambda^2\ a^2)u_{i,j}\ + \lambda^2\ a^2\ \left(u_{i+1,j} + u_{i-1,j}\right)\ - u_{i,j-1}$$

If $\lambda^2 a^2 = 1,\ coefficient\ of\ \ u_{i,j}\ \ in\ \ (1)\ is\ = 0$

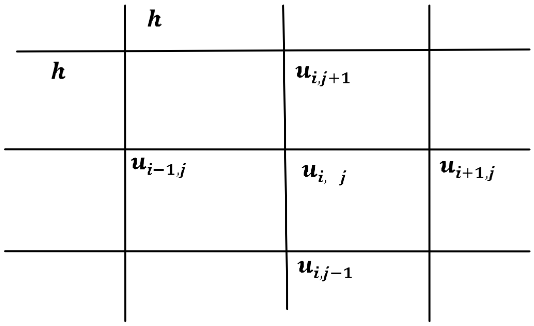
The recurrence equation (1) takes the simplified form

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

18. Write the diagonal five point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$.

Solution :

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$




19. Write down the standard five point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$.

Solution :

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

20. Write the difference scheme for solving the Laplace equation.

Solution :

The five point difference formula for $\nabla^2\varphi = 0$ is

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

21. What is the purpose of Leibmann's process?

Solution :

The purpose of Leibmann's process is to find the solution of the Laplace equation $u_{xx} + u_{yy} = 0$ by iteration.

22. If $u$ satisfies the Laplace equation and $u = 100$ on the boundary of a square what will be the value of $u$ at an interior gird point.

Solution :

Since $u$ satisfies Laplace equation and $u = 100$ on the boundary square.

$$u_{i,j} = \frac{1}{4}[100 + 100 + 100 + 100]$$

$$u_{i,j} = 100.$$

23. Write the Laplace equations $u_{xx} + u_{yy} = 0$ in difference quotients.

Solution :

$$\frac{u_{i-1,j} - 2\,u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2\,u_{i,j} + u_{i,j+1}}{k^2} = 0$$

24. Define a difference quotient.

Solution :

A difference quotient is the quotient obtained by dividing the difference between two values of a function by the difference between two corresponding values of the independent variable.

25. State Leibmann's iterative formula.

Solution :

$$u_{i,j}^{n+1} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} n + u_{i,j-1}^{n} + u_{i,j+1}^{n+1}\right]$$

26. The number of conditions required to solve the Laplace equation is …………… ?

Solution :

Four.

27. Write down the finite difference form of the equation $\nabla^2 u = f(x, y)$.

Solution :

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\, u_{i,j} = h^2\, f(ih, jh)$$

28. Write the difference scheme for $\nabla^2 u = f(x, y)$.

Solution :

Consider a square mesh with interval of differencing as $h$.

Taking $x = ih$ *and* $y = jh$ the difference equation reduces to

$$\frac{u_{i-1,j} - 2\, u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2\, u_{i,j} + u_{i,j+1}}{k^2} = f(ih, jh)$$

29. State the five point formula to solve the Poisson equation $u_{xx} + u_{yy} = 100$.

Solution :

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\, u_{i,j} = h^2\, f(ih, jh) = 100. \ \ (Since\ \ h = 1)$$

30. State the general form of Poisson's equation in partial derivatives.

Solution :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

**PART – B**

1. Using finite difference method, find $y(0.25), y(0.5)$ *and* $y(0.75)$ satisfying the differential equation $\frac{d^2y}{dx^2} + y = x$ subject to the boundary condition $y(0) = 0,\ y(1) = 1$.

2. Using finite difference method, solve $\frac{d^2y}{dx^2} = y$ in $(0,2)$ given $y(0) = 0,$

$y(2) = 3.63$ subdividing the range of $x$ into 4 equal parts.

3. Solve $y'' - y = x,\ x \epsilon (0,1)\ given\ y(0) = y(1) = 0$ using finite difference method dividing the intervals into 4 equal parts.

4. Solve the equation $y''(x) - x\,y(x) = 0\ \ for\ \ y(x_i), x_i 0,\ \frac{1}{3},\ \frac{2}{3}$, given that $y(0) + y'(0) = 1,\ \&\ \ y(1) = 1.$

5. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ t \ge 0$ with $u(x,0) = x(1-x),$ $0 < x < 1\ and\ u(0,t) = u(1,t) = 0,\ \forall\ t > 0$ using explicit method with $\Delta x = 0.2$ for 3 time steps.

6. Solve $u_{xx} = 32\,u_t$, taking $h = 0.25$ for $t > 0\,, 0 < x < 1\ \ and\ \ u(x,0) = 0,\ u(0,t) = 0,$ $u(1,t) = t.$

7. Solve $u_t = u_{xx},$ subject $u(0,t) = 0,\ u(1,t) = 0\ \ \&\ \ u(x,0) = \sin \pi\, x\,,\ \ 0 < x < 1.$

8. Solve $\frac{\partial u}{\partial t} = \frac{1}{2}\ \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 12,\ 0 \le t \le 12$ with boundary conditions and initial Conditions

$u(x,0) = \frac{1}{4}\ x\,(15 - x);\ \ 0 \le x \le 12$

$= u(0,t) = 0,\quad u(12,t) = 9,\quad 0 \le t \le 12.$

9. Solve $\frac{\partial^2 u}{\partial x^2} - 2\ \frac{\partial u}{\partial t} = 0$ given $u(0,t) = 0,\ u(4,t) = 0,\ u(x,0) = x(4-x)$. Assume $h = 1.$ Find the values of $u$ up to $t = 5.$

10. Solve $\frac{\partial u}{\partial t} = \ \frac{\partial^2 u}{\partial x^2}\ \ in\ \ 0 < x < 5, t \ge 0$ given that $u(x,0) = 20,\ u(0,t) = 0,\ u(5,t) = 100,$ compute $u$ for the time-step with $h = 1$ by Crank-Nicholson's method.

11. Solve by Crank-Nicholson's method the equation $u_{xx} = u_t$ subject to $u(x,0) = 0, u(0,t) = 0\ \ and\ \ u(1,t) = t$, for two-time steps.

12. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < 2,\ \ t > 0\ ,\ u(0,t) = u(2,t) = 0, t > 0\ \ and\ \ u(x,0) = \sin\frac{\pi\, x}{2},$ $0 \le x \le 2$ using $\Delta x = 0.5\ \ and\ \ \Delta t = 0.25$ for two-time steps by Crank-Nicholson's implicit finite difference method.

13. Solve $y_{tt} = y_{xx}\ \ upto\ t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0, y(1,t) = 0, y_t(x,0) = 0\ \ and\ \ y(x,0) = 10 + x(1-x).$

14. Approximate the solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ , $0 < x < 1, t > 0, u(0,t) = u(1,t) = 0,\ t > 0, u(x,0) = \sin 2\pi x,\ 0 \le x \le 1\ and\ \frac{\partial u}{\partial t}(x,0) = 0,\ 0 \le x \le 1\ with\ \Delta t = 0.25$ for three time steps.

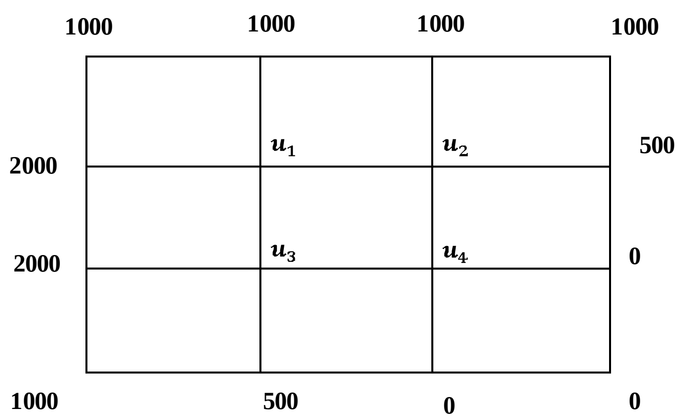
15.Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0, given\ u(x,0) = u_t(x,0) = u(0,t) = 0\ and\ u(1,t) = 100 \sin \pi t$. Compute $u$ for four time steps with $h = 0.25$.

16. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0 < x < 1,\ t > 0,\ given\ u(x,0) = 100\,(x - x^2),\ \frac{\partial u}{\partial t}(x,0) = 0,$ $u(0,t) = u(1,t) = 0,\ t > 0$ by finite difference method for one time step method with $h = 0.25$.

17. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ t > 0,\ given\ u(0,t) = 0, u(1,t) = 0,\ u(x,0) = (x - x^2),$ $\frac{\partial u}{\partial t}(x,0) = 0,$ taking $h = 0.2$ up to one-half of the period of vibration by taking appropriate time step.

18. Obtain a finite difference scheme to solve the Laplace equation. Solve $\Delta^2 u = 0$ at the pivotal points in the square shown fitted with square mesh. Use Leibmann's iteration procedure.
(5 iteration only).

Figure




19. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8\,x^2\,y^2$ in the square mesh given $u = 0$ on the four boundaries dividing the square into $16$ sub squares of length $1$ unit.

20. Solve the Poisson equation $u_{xx} + u_{yy} = -81\,x\,y,\ 0 < x < 1,\ 0 < y < 1$ given that $u(0,y) = 0,\ u(x,0) = 0,\ u(1,y) = 100,\ u(x,1) = 100\ \ and\ \ h = \frac{1}{3}$.