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Limits and Continuity – Partial derivatives – Total derivative – Differentiation of implicit functions – Jacobian and properties – Taylor's series for functions of two variables – Maxima and minima of functions of two variables – Lagrange's method of undetermined multipliers.

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Limits and Continuity

Limit:

The function $f(x,y)$ is said to tend to the limit 'ℓ' as $x \to a$ and $y \to b$ if and only if the limit 'ℓ' is independent of the path followed by the point (x,y) as $x \to a$ and $y \to b$. Then $\lim_{\substack{x\to a \\ y\to b}} f(x,y) = \ell$.

(or)

The function $f(x,y)$ in R is said to tend to the limit 'ℓ' as $x \to a$ and $y \to b$ if and only if corresponding to a positive number 'ε' in (a,b) there exist another $+ve$ no. δ such that $|f(x,y) - \ell| < \varepsilon$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$ for every point (a,b) in R.

Working rule for evaluation of limits:

Type I : Non – zero values of (a,b)

1. Find the value of $f(x,y)$ along $y \to b$ and $x \to a$ (say f_1).
2. Find the value of $f(x,y)$ along $x \to a$ and $y \to b$ (say f_2).
3. If $f_1 = f_2$, then the limit exists otherwise not.

Type II : $a = 0$, $b = 0$

1. Find the value of $f(x,y)$ along $y \to 0$ and $x \to 0$ (say f_1).
2. Find the value of $f(x,y)$ along $x \to 0$ and $y \to 0$ (say f_2).
3. Find the value of $f(x,y)$ along $y \to mx$ and $x \to 0$ (say f_3).
4. Find the value of $f(x,y)$ along $y \to mx^2$ and $x \to 0$ (say f_4).
5. If $f_1 = f_2 = f_3 = f_4$ then the limit exists otherwise not.

Note:

If the value of the limit does not contain m then limit exists. If it contains m the limit does not exists.

Continuity:

A function $f(x,y)$ is said to be continuous at a point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. A function is said to be continuous in a domain if it is continuous at every point of the domain.

Note:

A function which is not continuous at (a,b) is said to be discontinuous at (a,b).

Working rule for continuity at a point $(\boldsymbol{a},\boldsymbol{b})$.

Step 1. $f(a,b)$ should be well defined.

Step 2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ should exist. (must be unique and same along any path)

Step 3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Problems:

Type I

1. Evaluate $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \dfrac{xy+5}{x^2+2y^2}$.

Solution:

$$f_1 = \lim_{x\to\infty}\left(\lim_{y\to 2}\frac{xy+5}{x^2+2y^2}\right) = \lim_{x\to\infty}\left(\frac{2x+5}{x^2+8}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x\left(2+\frac{5}{x}\right)}{x^2\left(1+\frac{8}{x^2}\right)}\right) = \lim_{x\to\infty}\left(\frac{1}{x}\frac{\left(2+\frac{5}{x}\right)}{\left(1+\frac{8}{x^2}\right)}\right) = 0\left(\frac{2+0}{1+0}\right) = 0$$

$$f_2 = \lim_{y\to 2}\left(\lim_{x\to\infty}\frac{xy+5}{x^2+2y^2}\right) = \lim_{y\to 2}\left(\frac{x\left(y+\frac{5}{x}\right)}{x^2\left(1+\frac{2y^2}{x^2}\right)}\right)$$

$$= \lim_{y\to 2}\left(\frac{1}{x}\frac{\left(y+\frac{5}{x}\right)}{\left(1+\frac{2y^2}{x^2}\right)}\right) = \lim_{y\to 2}\left(0\frac{(y+0)}{(1+0)}\right) = \lim_{y\to 2} 0 = 0$$

$\because f_1 = f_2$ the limit exists and the value is 0.

2. Evaluate $\lim\limits_{\substack{x\to 1 \\ y\to 2}} \dfrac{x^2+2y}{x+y^2}$.

Solution:

$$f_1 = \lim_{x\to 1}\left(\lim_{y\to 2}\frac{x^2+2y}{x+y^2}\right) = \lim_{x\to 1}\left(\frac{x^2+4}{x+4}\right) = \frac{5}{5} = 1$$

$$f_2 = \lim_{y\to 2}\left(\lim_{x\to 1}\frac{x^2+2y}{x+y^2}\right) = \lim_{y\to 2}\left(\frac{1+2y}{1+y^2}\right) = \frac{1+(2\times 2)}{1+2^2} = \frac{5}{5} = 1$$

$\because\ f_1 = f_2$ the limit exists with the value 1.

Type II

3. Evaluate $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \left(x^3 + y^3\right)$.

Solution:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} x^3 + y^3\right) = \lim_{x \to 0} x^3 = 0$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} x^3 + y^3\right) = \lim_{y \to 0} y^3 = 0$$

$$f_3 = \lim_{x \to 0}\left(\lim_{y \to mx} x^3 + y^3\right) = \lim_{x \to 0}\left(x^3 + m^3x^3\right) = 0$$

$$f_4 = \lim_{x \to 0}\left(\lim_{y \to mx^2} x^3 + y^3\right) = \lim_{x \to 0}\left(x^3 + m^3x^6\right) = 0$$

$\because\ f_1 = f_2 = f_3 = f_4$ the limit exists with the value 0.

4. Evaluate $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{y - x^2}$.

Solution:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{xy}{y - x^2}\right) = \lim_{x \to 0} 0 = 0$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{xy}{y - x^2}\right) = \lim_{y \to 0} 0 = 0$$

$$f_3 = \lim_{x \to 0}\left(\lim_{y \to mx} \frac{xy}{y - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^2}{mx - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^2}{x\left(m - x^2\right)}\right) = \lim_{x \to 0}\left(\frac{mx}{\left(m - x^2\right)}\right) = \frac{0}{m - 0} = 0$$

$$f_3 = \lim_{x \to 0}\left(\lim_{y \to mx^2} \frac{xy}{y - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^3}{mx^2 - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^3}{x^2\left(m - 1\right)}\right) = \lim_{x \to 0}\left(\frac{mx}{\left(m - 1\right)}\right) = \frac{0}{m - 1} = 0$$

$\because\ f_1 = f_2 = f_3 = f_4$ the limit exists with the value 0.

5. Find $\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{y^2 - x^2}$.

Solution:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{xy}{y^2 - x^2}\right) = \lim_{x \to 0} 0 = 0$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{xy}{y^2 - x^2}\right) = \lim_{y \to 0} 0 = 0$$

$$f_3 = \lim_{x \to 0}\left(\lim_{y \to mx} \frac{xy}{y^2 - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^2}{m^2x^2 - x^2}\right) = \lim_{x \to 0}\left(\frac{mx^2}{x^2\left(m^2 - 1\right)}\right) = \lim_{x \to 0}\left(\frac{m}{\left(m^2 - 1\right)}\right) = \frac{m}{\left(m^2 - 1\right)}$$

Since the limit depends up on m, (or $\because f_1 = f_2 \neq f_3$) the limit does not exist.

6. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - y}{x^2 + y^2}$.

Solution:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{x - y}{x^2 + y^2}\right) = \lim_{x \to 0}\left(\frac{x}{x^2}\right) = \lim_{x \to 0} \frac{1}{x} = \infty$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{x - y}{x^2 + y^2}\right) = \lim_{y \to 0}\left(\frac{-y}{y^2}\right) = \lim_{y \to 0} \frac{-1}{y} = \infty$$

$\because f_1$ *and* $f_2 = \infty$, limit does not exist.

7. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2 - x^2}{x^2 + y^2}$.

Solution:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{y^2 - x^2}{x^2 + y^2}\right) = \lim_{x \to 0}\left(\frac{-x^2}{x^2}\right) = \lim_{x \to 0}(-1) = -1$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{y^2 - x^2}{x^2 + y^2}\right) = \lim_{y \to 0}\left(\frac{y^2}{y^2}\right) = \lim_{x \to 0}(1) = 1$$

$\because f_1 \neq f_2$, the limit does not exist.

Problems: (continuity)

1. Test the function $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } x \neq 0,\ y \neq 0 \\ 0 & \text{if } x = 0,\ y = 0 \end{cases}$ for continuity at the origin.

Solution:

Step 1:

The function is well defined at the origin i.e., $f(0,0) = 0$.

Step 2:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{x^3 - y^3}{x^2 + y^2}\right) = \lim_{x \to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x \to 0} x = 0$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{x^3 - y^3}{x^2 + y^2}\right) = \lim_{y \to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y \to 0}(-y) = 0$$

$$f_3 = \lim_{x\to 0}\left(\lim_{y\to mx}\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^3-m^3x^3}{x^2+m^2x^2}\right) = \lim_{x\to 0}\left(\frac{x^3\left(1-m^3\right)}{x^2(1+m^2)}\right) = \lim_{x\to 0}\left(\frac{x\left(1-m^3\right)}{(1+m^2)}\right) = 0$$

$$f_4 = \lim_{x\to 0}\left(\lim_{y\to mx^2}\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^3-m^3x^6}{x^2+m^2x^4}\right) = \lim_{x\to 0}\left(\frac{x^3\left(1-m^3x^3\right)}{x^2(1+m^2x^2)}\right) = \lim_{x\to 0}\left(\frac{x\left(1-m^3x^3\right)}{(1+m^2x^2)}\right) = \frac{0(1-0)}{(1+0)} = 0$$

$\because f_1 = f_2 = f_3 = f_4$, the limit exist and the value is 0.

i.e. $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Step 3:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+y^2} = 0 \text{ and } f(0,0) = 0 \text{ (defined in problem)}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

$\therefore$ the function is continuous at the origin.

2. Discuss the continuity of $f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & x\neq 0,\ y\neq 0 \\ 2 & x=0,\ y=0 \end{cases}$ at the origin.

Solution:

Step 1:

The function is well defined at the origin. i.e. $f(0,0) = 2$.

Step 2:

$$f_1 = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0}\left(\frac{x}{\sqrt{x^2}}\right) = \lim_{x\to 0}\left(\frac{x}{x}\right) = \lim_{x\to 0} 1 = 1$$

$$f_2 = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0}\left(\frac{0}{\sqrt{y^2}}\right) = \lim_{x\to 0} 0 = 0$$

$\because f_1 \neq f_2$, the limit does not exist.

$\therefore f(x,y)$ is not continuous at the origin.

3. Examine for continuity at origin of the function defined by

$f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & x\neq 0,\ y\neq 0 \\ 3 & x=0,\ y=0 \end{cases}$. Redefine the function to make it as continuous.

Solution:

Step 1:

The function is well defined at the origin. i.e., $f(0,0) = 3$

Step 2:

$$f_1 = \lim_{x \to 0}\left(\lim_{y \to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right) = \lim_{x \to 0}\left(\frac{x^2}{\sqrt{x^2}}\right) = \lim_{x \to 0}\left(\frac{x^2}{x}\right) = \lim_{x \to 0} x = 0$$

$$f_2 = \lim_{y \to 0}\left(\lim_{x \to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right) = \lim_{y \to 0}\left(\frac{0}{\sqrt{y^2}}\right) = \lim_{x \to 0} 0 = 0$$

$$f_3 = \lim_{x \to 0}\left(\lim_{y \to mx} \frac{x^2}{\sqrt{x^2+y^2}}\right) = \lim_{x \to 0}\left(\frac{x^2}{\sqrt{x^2+m^2x^2}}\right) = \lim_{x \to 0}\left(\frac{x^2}{x\sqrt{1+m^2}}\right) = \lim_{x \to 0}\left(\frac{x}{\sqrt{1+m^2}}\right) = 0$$

$$f_4 = \lim_{x \to 0}\left(\lim_{y \to mx^2} \frac{x^2}{\sqrt{x^2+y^2}}\right) = \lim_{x \to 0}\left(\frac{x^2}{\sqrt{x^2+m^2x^4}}\right) = \lim_{x \to 0}\left(\frac{x^2}{x\sqrt{1+m^2x^2}}\right) = \lim_{x \to 0}\left(\frac{x}{\sqrt{1+m^2x^2}}\right) = 0$$

$\because f_1 = f_2 = f_3 = f_4$, the limit exists with the value 1. i.e., $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Step 3:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \text{and} \quad f(0,0) = 3 \;\; \text{(defined in problem)}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0).$$

$\therefore$ the function is discontinuous at the origin.

The function can be made continuous at origin by redefining as $f(0,0) = 0$. i.e.,

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & x \neq 0,\; y \neq 0 \\ 0 & x = 0,\; y = 0 \end{cases}$$

Partial derivatives

Problems:

1. Find $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *if* $u = y^x$

Solution:

$$\frac{\partial u}{\partial x} = y^x \log y \quad \text{and} \quad \frac{\partial u}{\partial y} = xy^{x-1}.$$

2. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solution:

$$\frac{\partial u}{\partial x} = 4(x-y)^3 - 4(z-x)^3 \quad \ldots(1)$$

$$\frac{\partial u}{\partial y} = -4(x-y)^3 + 4(y-z)^3 \quad \ldots(2)$$

$$\frac{\partial u}{\partial z} = -4(y-z)^3 + 4(z-x)^3 \quad ...(3)$$

(1)+(2)+(3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 4(x-y)^3 \; -4(z-x)^3 \; -4(x-y)^3 \; +4(y-z)^3 \; -4(y-z)^3 \; +4(z-x)^3 = 0$$

3. If $u = e^{xy}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$.

Solution:

$$\frac{\partial u}{\partial x} = y\,e^{xy} \quad ...(1) \quad \text{and} \quad \frac{\partial u}{\partial y} = x\,e^{xy} \quad ...(2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\,e^{xy}\right) = y^2\,e^{xy} \quad ...(3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\,e^{xy}\right) = x^2\,e^{xy} \quad ...(4)$$

$$(3) + (4) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y^2\; e^{xy} + x^2\; e^{xy} = \left(y^2 + x^2\right)e^{xy} \quad ...(5)$$

$$\frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{1}{u}\left[\left(y^2\; e^{2xy}\right) + x^2\; e^{2xy}\right] \quad \textit{from } (1) \textit{ and } (2)$$

$$= \frac{e^{2xy}}{e^{xy}}\left[y^2 + x^2\right]$$

$$= e^{xy}\left[y^2 + x^2\right]$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \textit{from } (5)$$

4. If $u = x^y$ then show that (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (ii) $u_{xxy} = u_{xyx}$.

Solution:

$$u = x^y \Rightarrow \frac{\partial u}{\partial x} = yx^{y-1} \quad \textit{and} \quad \frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^y \log x\right) = yx^{y-1}\log x + x^y\left(\frac{1}{x}\right) = yx^{y-1}\log x + x^{y-1} \quad ...(1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(yx^{y-1}\right) = yx^{y-1}\log x + x^{y-1}(1) = yx^{y-1}\log x + x^{y-1} \quad ...(2)$$

From (1) and (2) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

$$u_{xxy} = \frac{\partial^3 u}{\partial^2 x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(yx^{y-1}\log x + x^{y-1}\right)$$

$$= y\left((y-1)x^{y-2}\log x + x^{y-1}\left(\frac{1}{x}\right)\right) + (y-1)x^{y-2}$$

$$= y(y-1)x^{y-2}\log x + x^{y-2} + (y-1)x^{y-2}$$

$$= x^{y-2}\left(y(y-1)\log x + 1 + y - 1\right)$$

$$= x^{y-2}\left(y(y-1)\log x + y\right) \qquad ..(3)$$

$$u_{xyx} = \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\left(yx^{y-1}\log x + x^{y-1}\right)$$

$$= y\left((y-1)x^{y-2}\log x + x^{y-1}\left(\frac{1}{x}\right)\right) + (y-1)x^{y-2}$$

$$= y(y-1)x^{y-2}\log x + x^{y-2} + (y-1)x^{y-2}$$

$$= x^{y-2}\left(y(y-1)\log x + 1 + y - 1\right)$$

$$= x^{y-2}\left(y(y-1)\log x + y\right) \qquad ..(4)$$

From (3) and (4) $u_{xxy} = u_{xyx}$.

5. If $u = \log\left(x^2 + y^2\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $u_{xx} + u_{yy} = 0$.

Solution:

$$u = \log\left(x^2 + y^2\right) + \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

Diff. (1) w.r.t 'x' and 'y' partially,

$$\Rightarrow u_x = \frac{2x}{\left(x^2+y^2\right)} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow u_x = \frac{2x}{\left(x^2+y^2\right)} - \frac{x^2}{\left(x^2+y^2\right)}\left(\frac{y}{x^2}\right)$$

$$\Rightarrow u_x = \frac{2x}{\left(x^2+y^2\right)} - \frac{y}{\left(x^2+y^2\right)}$$

$$\Rightarrow u_x = \frac{2x-y}{\left(x^2+y^2\right)} \qquad ...(2)$$

$$\Rightarrow u_y = \frac{2y}{\left(x^2+y^2\right)} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$\Rightarrow u_y = \frac{2y}{\left(x^2+y^2\right)} + \frac{x^2}{\left(x^2+y^2\right)}\left(\frac{1}{x}\right)$$

$$\Rightarrow u_y = \frac{2y}{\left(x^2+y^2\right)} + \frac{x}{\left(x^2+y^2\right)}$$

$$\Rightarrow u_y = \frac{2y+x}{\left(x^2+y^2\right)} \qquad ...(3)$$

Diff. (2) and (3) w.r.t 'x' and 'y' partially,

$$\Rightarrow u_{xx} = \frac{\left(x^2+y^2\right)(2)-(2x-y)(2x)}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow u_{xx} = \frac{2x^2+2y^2-4x^2+2xy}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow u_{xx} = \frac{2y^2-2x^2+2xy}{\left(x^2+y^2\right)^2} \quad ...(4)$$

$$\Rightarrow u_{yy} = \frac{\left(x^2+y^2\right)(2)-(2y+x)(2y)}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow u_{yy} = \frac{2x^2+2y^2-4y^2-2xy}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow u_{yy} = \frac{2x^2-2y^2-2xy}{\left(x^2+y^2\right)^2} \quad ...(5)$$

(4) + (5) ⇒

$$u_{xx}+u_{yy} = \frac{2y^2-2x^2+2xy}{\left(x^2+y^2\right)^2} + \frac{2x^2-2y^2-2xy}{\left(x^2+y^2\right)^2} = 0$$

Homogeneous function

A function $u = f(x,y)$ is said to be a homogeneous function in x *and* y of degree n if $f(tx,ty) = t^n f(x,y)$.

Euler's theorem

If $u = f(x,y)$ is a homogeneous function of degree n then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf$.

Note:

(i) If $u = f(x,y)$ is a homogeneous function of degree n then $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

(ii) If If $u = f(x,y)$ is a homogeneous function of degree n and $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = f(u)$ then $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = f(u)\left(f'(u)-1\right)$.

Problems:

1. If $u = \frac{y}{z}+\frac{z}{x}+\frac{x}{y}$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 0$.

Solution:

Given $u = f(x,y,z) = \frac{y}{z}+\frac{z}{x}+\frac{x}{y}$.

Now, $f(tx,ty,tz) = \frac{ty}{tz}+\frac{tz}{tx}+\frac{tx}{ty} = t^0\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right) = t^0 f(x,y,z)$

⇒ $u = f(x,y,z)$ is a homogeneous function of degree $n = 0$.

By Euler's theorem, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf=0\times f=0$.

2. If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$.

Solution:

Given $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\Rightarrow \boxed{\cos u=\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=f(x,y)}$

Now, $f(tx,ty)=\left(\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right)=\frac{t}{\sqrt{t}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=t^{\frac{1}{2}}f(x,y)$

$\Rightarrow u=f(x,y)$ is a homogeneous function of degree $n=\frac{1}{2}$.

By Euler's theorem, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$.

$$\Rightarrow x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=\frac{1}{2}(\cos u)$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}=\frac{1}{2}(\cos u) \quad \div -\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\cos u}{\sin u}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

3. If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}$.

Solution:

Given $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\Rightarrow \boxed{\sin u=\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=f(x,y)}$

Now, $f(tx,ty)=\left(\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right)=\frac{t}{\sqrt{t}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=t^{\frac{1}{2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)=t^{\frac{1}{2}}f(x,y)$

$\Rightarrow u=f(x,y)$ is a homogeneous function of degree $n=\frac{1}{2}$.

By Euler's theorem, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$.

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y}=\frac{1}{2}(\sin u)$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}(\sin u) \quad \div\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$

By Euler's theorem on second derivatives, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=f(u)\left(f'(u)-1\right)$.

Here $f(u)=\frac{1}{2}\tan u \Rightarrow f'(u)=\frac{1}{2}\sec^2 u$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=f(u)\left(f'(u)-1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u-1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1}{2\cos^2 u}-1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\tan u\left(\frac{1-2\cos^2 u}{2\cos^2 u}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{1-2\cos^2 u}{2\cos^2 u}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\left(\frac{\sin u\cos 2u}{4\cos^3 u}\right)$$

4. If $u=(x-y)\,f\left(\frac{y}{x}\right)$ then find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

Solution:

Given $u=(x-y)\,f\left(\frac{y}{x}\right)=f(x,y)$

Now, $f(tx,ty)=(tx-ty)\,f\left(\frac{ty}{tx}\right)=t\left((x-y)\,f\left(\frac{y}{x}\right)\right)=t\;f(x,y)$

$\Rightarrow$ $u=f(x,y)$ is a homogeneous function of degree $n=1$.

By Euler's theorem on second derivatives, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1(1-1)u=0$$

5. If $u(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)$ then find $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

Solution:

Given $u(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)=f(x,y)$

Now, $f(tx,ty)=t^2x^2\tan^{-1}\left(\frac{ty}{tx}\right)=t^2\left(x^2\tan^{-1}\left(\frac{y}{x}\right)\right)=t^2\ f(x,y)$

$\Rightarrow u=f(x,y)$ is a homogeneous function of degree $n=2$.

By Euler's theorem on second derivatives, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2(2-1)u=2u=2x^2\tan^{-1}\left(\frac{y}{x}\right)$$

6. If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan^3 u$.

Solution:

Given $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)\Rightarrow \boxed{\sin u=\left(\frac{x^2+y^2}{x+y}\right)=f(x,y)}$

Now, $f(tx,ty)=\left(\frac{t^2x^2+t^2y^2}{tx+ty}\right)=\frac{t^2}{t}\left(\frac{x^2+y^2}{x+y}\right)=t\left(\frac{x^2+y^2}{x+y}\right)=t\ f(x,y)$

$\Rightarrow u=f(x,y)$ is a homogeneous function of degree $n=1$.

By Euler's theorem, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$.

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y}=\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u \quad \div\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$

By Euler's theorem on second derivatives, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=f(u)\left(f'(u)-1\right)$.

Here $f(u)=\tan u \Rightarrow f'(u)=\sec^2 u$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=f(u)\left(f'(u)-1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan u\left(\sec^2 u-1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan u\left(\tan^2 u\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan^3 u$$

Hence proved.

Total derivative

If $u=f(x,y)$ where $x=g_1(t)$, $y=g_2(t)$, then $\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$ is the total derivative of u.

Note:

- ✓ Extending the above result to a function $u=f(x_1,x_2,\dots,x_n)$ we get $\frac{du}{dt}=\frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt}+\dots+\frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}$.
- ✓ If $u=f(x,y)$ and y is a function of x then, $\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$.
- ✓ If $u=f(x,y)$ and x is a function of y then, $\frac{du}{dy}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dy}+\frac{\partial u}{\partial y}$.

Differentiation of implicit functions

If $f(x,y)=c$ is an implicit function then $\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=-\frac{f_x}{f_y}$.

Note:

Any function of the type $f(x,y)=c$ is called an implicit function. When x and y are implicitly related, it may not be possible in many cases to express y as a single valued function of x explicitly.

Partial derivatives of composite functions

If $z = f(x, y)$ where $x = g_1(u, v)$ and $y = g_2(u, v)$ then z is a function of u, v called the composite function of two variables and the partial derivatives of z w.r.t. u *and* v are given by $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$.

Problems

1. If $u = x^3y^2 + x^2y^3$ where $x = at^2$ and $y = 2at$ then find $\frac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = \left(3x^2y^2 + 2xy^3\right)(2at) + \left(2x^3y + 3x^2y^2\right)(2a)$$

$$\Rightarrow \frac{du}{dt} = \left(3(a^2t^4)(4a^2t^2) + 2(at^2)(8a^3t^3)\right)(2at) + \left(2(a^3t^6)(2at) + 3(a^2t^4)(4a^2t^2)\right)(2a)$$

$$\Rightarrow \frac{du}{dt} = \left(12a^4t^6 + 16a^4t^5\right)(2at) + \left(4a^4t^7 + 12a^4t^6\right)(2a)$$

$$\Rightarrow \frac{du}{dt} = 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$$

$$\Rightarrow \frac{du}{dt} = 32a^5t^7 + 56a^5t^6$$

2. Find $\frac{du}{dt}$ if $u = \sin\left(\frac{x}{y}\right)$ where $= e^t$, $y = t^2$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = \cos\left(\frac{x}{y}\right)\cdot\frac{1}{y}\cdot e^t + \cos\left(\frac{x}{y}\right)\cdot\frac{1}{y}\cdot e^t$$

3. Using the definition of total derivative, find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = (-4a)(2at) + (2y)(2a)$$

$$\Rightarrow \frac{du}{dt} = -8a^2t + (2(2at))(2a)$$

$$\Rightarrow \frac{du}{dt} = -8at^2 + 8at^2 = 0$$

4. If $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t}\cos 2t$, $z = e^{2t}\sin 2t$ find $\dfrac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = (2x)\left(2e^{2t}\right) + (2y)\left(2e^{2t}\cos 2t - 2e^{2t}\sin 2t\right) + (2z)\left(2e^{2t}\sin 2t + 2e^{2t}\cos 2t\right)$$

$$\Rightarrow \frac{du}{dt} = \left(2e^{2t}\right)\left(2e^{2t}\right) + \left(2e^{2t}\cos 2t\right)\left(2e^{2t}\cos 2t - 2e^{2t}\sin 2t\right) + \left(2e^{2t}\sin 2t\right)\left(2e^{2t}\sin 2t + 2e^{2t}\cos 2t\right)$$

$$\Rightarrow \frac{du}{dt} = 4e^{4t} + 4e^{4t}\cos^2 2t - \cancel{4e^{4t}\cos 2t\sin 2t} + 4e^{4t}\sin^2 2t + \cancel{4e^{4t}\sin 2t\cos 2t}$$

$$\Rightarrow \frac{du}{dt} = 4e^{4t} + 4e^{4t}\left(\cos^2 2t + \sin^2 2t\right)$$

$$\Rightarrow \frac{du}{dt} = 4e^{4t} + 4e^{4t}$$

$$\Rightarrow \frac{du}{dt} = 8e^{4t}$$

5. If $u = xy + yz + zx$ where $x = \dfrac{1}{t}$, $y = e^t$ *and* $z = e^{-t}$ find $\dfrac{du}{dt}$.

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)\left(e^t\right) + (y+x)\left(-e^{-t}\right)$$

$$\Rightarrow \frac{du}{dt} = \left(e^t + e^{-t}\right)\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)\left(e^t\right) + \left(e^t + \frac{1}{t}\right)\left(-e^{-t}\right)$$

$$\Rightarrow \frac{du}{dt} = -\frac{1}{t^2}\left(e^t + e^{-t}\right) + \frac{e^t}{t} + 1 - 1 - \frac{e^{-t}}{t}$$

$$\Rightarrow \frac{du}{dt} = -\frac{1}{t^2}\left(e^t + e^{-t}\right) + \frac{1}{t}\left(e^t - e^{-t}\right)$$

6. Find $\dfrac{du}{dx}$, if $u = \sin(x^2 + y^2)$ where $a^2x^2 + b^2y^2 = c^2$.

Solution:

$u = \sin(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$.

$\dfrac{\partial u}{\partial x} = 2x\ \cos(x^2 + y^2)$ *and* $\dfrac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$

$f(x, y) = a^2x^2 + b^2y^2 - c^2$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2a^2x}{2b^2y} = -\frac{a^2x}{b^2y}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\left(-\frac{a^2x}{b^2y}\right)$$

$$=2x\cos\left(x^2+y^2\right)\left(1-\frac{a^2}{b^2}\right)$$

7. If $u = x^2y$ *and* $x^2+xy+y^2=1$, then find $\frac{du}{dx}$.

Solution:

$u = x^2y$ *and* $x^2+xy+y^2=1$

$\frac{\partial u}{\partial x}=2xy$ *and* $\frac{\partial u}{\partial y}=x^2$

$f(x,y)=x^2+xy+y^2-1$

$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{2x+y}{x+2y}$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=2xy+x^2\left(-\frac{2x+y}{y+2x}\right)=x\left(2y-x\left(\frac{2x+y}{y+2x}\right)\right)$$

8. Find $\frac{du}{dx}$, if $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $y=\tan^2 x$.

Solution:

$u=\tan^{-1}\left(\frac{y}{x}\right)$ *and* $y=\tan^2 x$

$\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$ *and* $\frac{\partial u}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$

$f(x,y)=y-\tan^2 x$

$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{(-2\tan x\ .\sec^2 x)}{1}=2\tan x\sec^2 x$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=-\frac{y}{x^2+y^2}+\frac{x}{x^2+y^2}\left(2\tan x\sec^2 x\right)$$

$$=\frac{1}{x^2+y^2}\left(-y+2x\tan x\sec^2 x\right)$$

9. If $x^y+y^x=1$, then find $\frac{dy}{dx}$.

Solution:

$f(x,y)=x^y+y^x-1$

$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

10. If $u = f(x-y,\ y-z,\ z-x)$ show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Solution:

Let $r = x-y,\ s = y-z$ *and* $t = z-x$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot(1)+\frac{\partial u}{\partial s}\cdot(0)+\frac{\partial u}{\partial t}\cdot(-1)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot(-1)+\frac{\partial u}{\partial s}\cdot(1)+\frac{\partial u}{\partial t}\cdot(0)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot(0)+\frac{\partial u}{\partial s}\cdot(-1)+\frac{\partial u}{\partial t}\cdot(1)$$

$$\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

$$\therefore \sum\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0.$$

11. If $w = f(y-z,\ z-x,\ x-y)$, then show that $\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}=0$.

Solution:

Let $r = y-z,\ s = z-x$ *and* $t = x-y$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot(0)+\frac{\partial u}{\partial s}\cdot(-1)+\frac{\partial u}{\partial t}\cdot(1)$$

$$\frac{\partial u}{\partial x}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot(1)+\frac{\partial u}{\partial s}\cdot(0)+\frac{\partial u}{\partial t}\cdot(-1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot(-1)+\frac{\partial u}{\partial s}\cdot(1)+\frac{\partial u}{\partial t}\cdot(0)$$

$$\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

$$\therefore \sum\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}=0.$$

12. If $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Solution:

Let $r=\frac{x}{y}$, $s=\frac{y}{z}$ *and* $t=\frac{z}{x}$. *Then*

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\left(\frac{1}{y}\right)+\frac{\partial u}{\partial s}\cdot(0)+\frac{\partial u}{\partial t}\cdot\left(-\frac{z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x}=\frac{x}{y}\cdot\frac{\partial u}{\partial r}-\frac{z}{x}\cdot\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\cdot\left(\frac{1}{z}\right)+\frac{\partial u}{\partial t}\cdot(0)$$

$$y\frac{\partial u}{\partial y}=-\frac{x}{y}\cdot\frac{\partial u}{\partial r}+\frac{y}{z}\cdot\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot(0)+\frac{\partial u}{\partial s}\cdot\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\cdot\left(\frac{1}{x}\right)$$

$$z\frac{\partial u}{\partial z}=-\frac{y}{z}\cdot\frac{\partial u}{\partial s}+\frac{z}{x}\cdot\frac{\partial u}{\partial t}$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{x}{y}.\frac{\partial u}{\partial r}-\frac{z}{x}.\frac{\partial u}{\partial t}-\frac{x}{y}.\frac{\partial u}{\partial r}+\frac{y}{z}.\frac{\partial u}{\partial s}-\frac{y}{z}.\frac{\partial u}{\partial s}+\frac{z}{x}.\frac{\partial u}{\partial t}=0$$

13. If F is a function of x *and* y and if $x = e^u \sin v$ and $y = e^u \cos v$ prove that $\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=e^{-2u}\left[\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}\right]$.

Solution:

$$\frac{\partial F}{\partial u}=\frac{\partial F}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial F}{\partial y}.\frac{\partial F}{\partial u}$$

$$\frac{\partial F}{\partial u}=\frac{\partial F}{\partial x}(e^u \sin v)+\frac{\partial F}{\partial y}\,(e^u \cos v) \;\Rightarrow\; \frac{\partial}{\partial u}=\frac{\partial}{\partial x}(e^u \sin v)+\frac{\partial}{\partial y}\,(e^u \cos v)$$

$$\frac{\partial^2 F}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right)=\left(\frac{\partial}{\partial x}(e^u \sin v)+\frac{\partial}{\partial y}\,(e^u \cos v)\right)\left(\frac{\partial F}{\partial x}(e^u \sin v)+\frac{\partial F}{\partial y}\,(e^u \cos v)\right)$$

$$\frac{\partial^2 F}{\partial u^2}=e^{2u}\sin^2 v\,\frac{\partial^2 F}{\partial x^2}+e^{2u}\sin v\cos v\,\frac{\partial^2 F}{\partial x\partial y}+e^{2u}\cos v\,\sin v\frac{\partial^2 F}{\partial y\partial x}$$
$$+\,e^{2u}\cos^2 v\,\frac{\partial^2 F}{\partial y^2}\quad\ldots(1)$$

$$\frac{\partial F}{\partial v}=\frac{\partial F}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial F}{\partial y}.\frac{\partial F}{\partial v}$$

$$\frac{\partial F}{\partial v}=\frac{\partial F}{\partial x}(e^u \cos v)+\frac{\partial F}{\partial y}(-e^u \sin v) \;\Rightarrow\; \frac{\partial}{\partial v}=e^u\cos v\,\frac{\partial}{\partial x}-e^u \sin v\,\frac{\partial}{\partial y}$$

$$\frac{\partial^2 F}{\partial v^2}=\frac{\partial}{\partial v}\left(\frac{\partial F}{\partial v}\right)=\left(e^u\cos v\,\frac{\partial}{\partial x}-e^u \sin v\,\frac{\partial}{\partial y}\right)\left(\frac{\partial F}{\partial x}(e^u \cos v)+\frac{\partial F}{\partial y}(-e^u \sin v)\right)$$

$$\frac{\partial^2 F}{\partial v^2}=e^{2u}\cos^2 v\,\frac{\partial^2 F}{\partial x^2}-e^{2u}\sin v\cos v\,\frac{\partial^2 F}{\partial x\partial y}-e^{2u}\cos v\,\sin v\frac{\partial^2 F}{\partial y\partial x}$$
$$+\,e^{2u}\sin^2 v\,\frac{\partial^2 F}{\partial y^2}\quad\ldots(2)$$

(1) + (2) $\Rightarrow$

$$\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}=e^{2u}\sin^2 v\,\frac{\partial^2 F}{\partial x^2}+e^{2u}\cos^2 v\,\frac{\partial^2 F}{\partial y^2}+e^{2u}\cos^2 v\,\frac{\partial^2 F}{\partial x^2}+e^{2u}\sin^2 v\,\frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}=e^{2u}\,\frac{\partial^2 F}{\partial x^2}(\sin^2 v+\cos^2 v)+e^{2u}\frac{\partial^2 F}{\partial y^2}(\cos^2 v+\sin^2 v)$$

$$\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}=e^{2u}\,\frac{\partial^2 F}{\partial x^2}+e^{2u}\frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}=e^{2u}\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right) \;\Rightarrow\; \frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=e^{-2u}\left(\frac{\partial^2 F}{\partial u^2}+\frac{\partial^2 F}{\partial v^2}\right)$$

14. If $z = f(x,y)$ where $x = u^2 - v^2$, $y = 2uv$ prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4\left(u^2 + v^2\right)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$.

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)$$

$$\Rightarrow \frac{\partial}{\partial u} = \frac{\partial}{\partial x}(2u) + \frac{\partial}{\partial y}(2v)$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial x}(2u) + \frac{\partial}{\partial y}(2v)\right)\left(\frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4uv\frac{\partial^2 z}{\partial x \partial y} + 4uv\frac{\partial^2 z}{\partial y \partial x} + 4v^2\frac{\partial^2 z}{\partial y^2} \qquad ...(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\Rightarrow \frac{\partial}{\partial v} = \frac{\partial}{\partial x}(-2v) + \frac{\partial}{\partial y}(2u)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(\frac{\partial}{\partial x}(-2v) + \frac{\partial}{\partial y}(2u)\right)\left(\frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 4uv\frac{\partial^2 z}{\partial x \partial y} - 4uv\frac{\partial^2 z}{\partial y \partial x} + 4u^2\frac{\partial^2 z}{\partial y^2} \qquad ...(2)$$

$(1) + (2) \Rightarrow$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4v^2\frac{\partial^2 z}{\partial y^2} + 4v^2\frac{\partial^2 z}{\partial x^2} + 4u^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4\frac{\partial^2 z}{\partial x^2}\left(u^2 + v^2\right) + 4\frac{\partial^2 z}{\partial y^2}\left(u^2 + v^2\right)$$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4\left(u^2 + v^2\right)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

15. If $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$ and $V = f(x,y)$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}$.

Solution:

$$\frac{\partial V}{\partial u}=\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\Rightarrow\frac{\partial V}{\partial u}=\frac{\partial V}{\partial x}(\cos\alpha)+\frac{\partial V}{\partial y}(\sin\alpha)$$

$$\Rightarrow\frac{\partial}{\partial u}=\frac{\partial}{\partial x}(\cos\alpha)+\frac{\partial}{\partial y}(\sin\alpha)$$

$$\frac{\partial^2 V}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial V}{\partial u}\right)=\left(\frac{\partial}{\partial x}(\cos\alpha)+\frac{\partial}{\partial y}(\sin\alpha)\right)\left(\frac{\partial V}{\partial x}(\cos\alpha)+\frac{\partial V}{\partial y}(\sin\alpha)\right)$$

$$\Rightarrow\frac{\partial^2 V}{\partial u^2}=\cos^2\alpha\frac{\partial^2 V}{\partial x^2}+\cos\alpha\,\sin\alpha\frac{\partial^2 V}{\partial x\partial y}+\sin\alpha\,\cos\alpha\frac{\partial^2 V}{\partial y\partial x}+\sin^2\alpha\frac{\partial^2 V}{\partial y^2}\qquad\ldots(1)$$

$$\frac{\partial V}{\partial v}=\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\Rightarrow\frac{\partial V}{\partial v}=\frac{\partial V}{\partial x}(-\sin\alpha)+\frac{\partial V}{\partial y}(\cos\alpha)$$

$$\Rightarrow\frac{\partial}{\partial v}=\frac{\partial}{\partial x}(-\sin\alpha)+\frac{\partial}{\partial y}(\cos\alpha)$$

$$\frac{\partial^2 V}{\partial v^2}=\frac{\partial}{\partial v}\left(\frac{\partial V}{\partial v}\right)=\left(\frac{\partial}{\partial x}(-\sin\alpha)+\frac{\partial}{\partial y}(\cos\alpha)\right)\left(\frac{\partial V}{\partial x}(-\sin\alpha)+\frac{\partial V}{\partial y}(\cos\alpha)\right)$$

$$\Rightarrow\frac{\partial^2 V}{\partial v^2}=\sin^2\alpha\frac{\partial^2 V}{\partial x^2}-\sin\alpha\,\cos\alpha\frac{\partial^2 V}{\partial x\partial y}-\cos\alpha\,\sin\alpha\frac{\partial^2 V}{\partial y\partial x}+\cos^2\alpha\frac{\partial^2 V}{\partial y^2}\qquad\ldots(2)$$

$(1)+(2)\Rightarrow$

$$\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}=\cos^2\alpha\frac{\partial^2 V}{\partial x^2}+\sin^2\alpha\frac{\partial^2 V}{\partial y^2}+\sin^2\alpha\frac{\partial^2 V}{\partial x^2}+\cos^2\alpha\frac{\partial^2 V}{\partial y^2}$$

$$\Rightarrow\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}=\cos^2\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+\sin^2\alpha\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

$$\Rightarrow\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}=\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\left(\cos^2\alpha+\sin^2\alpha\right)$$

$$\Rightarrow\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}$$

16. If $u=f(x,y)$ where $x=r\cos\theta$, $y=r\sin\theta$ prove that $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$.

Solution:

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}(\cos\theta)+\frac{\partial u}{\partial y}(\sin\theta)$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2=\left(\frac{\partial u}{\partial x}(\cos\theta)+\frac{\partial u}{\partial y}(\sin\theta)\right)^2$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2=\cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2+2\cos\theta\sin\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2 \quad ...(1)$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow \left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)\right)^2$$

$$\Rightarrow \left(\frac{\partial u}{\partial \theta}\right)^2=r^2\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2-2r^2\cos\theta\sin\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+r^2\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$\Rightarrow \left(\frac{\partial u}{\partial \theta}\right)^2=r^2\left(\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2-2\cos\theta\sin\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$\Rightarrow \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2-2\cos\theta\sin\theta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2 \quad ...(2)$$

$(1)+(2)\Rightarrow$

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\cos^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial y}\right)^2+\sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2+\cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\cos^2\theta+\sin^2\theta\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

17. Find $\frac{dy}{dx}$ if $x^3+y^3=3ax^2y$.

Solution:

Let $f(x,y)=x^3+y^3-3ax^2y$.

$\frac{\partial f}{\partial x}=3x^2-6axy$ and $\frac{\partial f}{\partial y}=3y^2-3ax^2$.

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{3x^2-6axy}{3y^2-3ax^2}=\frac{2axy-x^2}{y^2-ax^2}.$$

18. If $e^y - e^x + xy = 0$ find $\frac{dy}{dx}$.

Solution:

Let $f(x,y) = e^y - e^x + xy$.

Differentiating $f(x,y)$ w.r.t x & y partially, $\frac{\partial f}{\partial x} = -e^x + y$ and $\frac{\partial f}{\partial y} = e^y + x$.

$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(y - e^x)}{(e^y + x)} = \frac{e^x - y}{e^y + x}$.

Jacobian and properties

Jacobian

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian or functional determinant of u, v with respect to x *and* y

and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$. Similarly, the Jacobian of u, v, w with respect to x, y, z

is defined as $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$.

Note:

The concept of Jacobians is used when we change the variables in multiple integrals.

Properties:

1. If u and v are functions of x *and* y, then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$.

2. If u and v are functions of r *and* s, where r *and* s are functions of x *and* y, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$.

Functionally dependent:

The functions u, v, w are said to be functionally dependent, if each can be expressed in terms of the others or equivalently $f(u,v,w) = 0$.

Condition for functionally dependent:

If u, v, w are functions of x, y, z such that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$ then u, v, w are functionally dependent. i.e., there exists a relation among them.

Problems:

1. If $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x *and* y with respect to u *and* v.

   Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2.$$

2. If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

   Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Now, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } 2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

3. If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\frac{\partial(r,\theta)}{\partial(x,y)}$.

   Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)}=\frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}}=\frac{1}{r} \qquad \because \frac{\partial(x,y)}{\partial(r,\theta)}\times\frac{\partial(r,\theta)}{\partial(x,y)}=1.$$

4. If $u=\frac{yz}{x}$, $v=\frac{zx}{y}$ and $w=\frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. (or) If $r=\frac{yz}{x}$, $s=\frac{zx}{y}$ and $t=\frac{xy}{z}$ find $\frac{\partial(r,s,t)}{\partial(x,y,z)}$.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}=\begin{vmatrix}-\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x}\\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y}\\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2}\end{vmatrix}$$

$$=\frac{1}{x^2}\frac{1}{y^2}\frac{1}{z^2}\begin{vmatrix}-yz & zx & yx\\ zy & -zx & xy\\ yz & xz & -xy\end{vmatrix}=\frac{(yz)(zx)(yx)}{x^2y^2z^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

(or)

$$\frac{\partial(r,s,t)}{\partial(x,y,z)}=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}=\begin{vmatrix}-\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x}\\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y}\\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2}\end{vmatrix}$$

$$=\frac{1}{x^2}\frac{1}{y^2}\frac{1}{z^2}\begin{vmatrix}-yz & zx & yx\\ zy & -zx & xy\\ yz & xz & -xy\end{vmatrix}=\frac{(yz)(zx)(yx)}{x^2y^2z^2}\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=4$$

5. Find the Jacobian of the transformation $x=r\cos\theta$ and $y=r\sin\theta$.

Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

6. If $u=\dfrac{y^2}{x}$ and $v=\dfrac{x^2}{y}$, find $\dfrac{\partial(x,y)}{\partial(u,v)}$.

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}-\dfrac{y^2}{x^2} & \dfrac{2y}{x}\\ \dfrac{2x}{y} & -\dfrac{x^2}{y^2}\end{vmatrix}=\frac{-y^2}{x^2}\times\frac{-x^2}{y^2}-\frac{2y}{x}\times\frac{2x}{y}=1-4=-3$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}=\frac{-1}{3}$$

7. If $u=\dfrac{x+y}{1-xy}$ and $v=\tan^{-1}x+\tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution:

$$u=\frac{x+y}{1-xy}$$

$$\therefore u_x=\frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2}=\frac{1-xy+xy+y^2}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2}\quad \textit{and}$$

$$u_y=\frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2}=\frac{1-xy+x^2+xy}{(1-xy)^2}=\frac{1+x^2}{(1-xy)^2}$$

$$v=\tan^{-1}x+\tan^{-1}y$$

$$\therefore v_x=\frac{1}{1+x^2}\quad \textit{and}\quad v_y=\frac{1}{1+y^2}.$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=\begin{vmatrix}\dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2}\\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2}\end{vmatrix}$$

$$=\frac{\cancel{1+y^2}}{(1-xy)^2}\times\frac{1}{\cancel{1+y^2}}-\frac{\cancel{1+x^2}}{(1-xy)^2}\times\frac{1}{\cancel{1+x^2}}=\frac{1}{(1-xy)^2}-\frac{1}{(1-xy)^2}=0$$

8. Find the Jacobian of y_1,y_2,y_3 with respect to x_1,x_2,x_3 if $y_1=\dfrac{x_2x_3}{x_1}$, $y_2=\dfrac{x_3x_1}{x_2}$ and $y_3=\dfrac{x_1x_2}{x_3}$.

Solution:

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2}\frac{1}{x_2^2}\frac{1}{x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_2x_1 \\ x_3x_2 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix} = \frac{(x_2x_3)(x_3x_1)(x_2x_1)}{x_1^2x_2^2x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$

9. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution:

Given $u = \frac{y^2}{2x}$ and $v = \frac{x^2+y^2}{2x} = \frac{x^2}{2x} + \frac{y^2}{2x} = \frac{x}{2} + \frac{y^2}{2x}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{2x^2} & \frac{\not{2}y}{\not{2}x} \\ \frac{1}{2} - \frac{y^2}{2x^2} & \frac{\not{2}y}{\not{2}x} \end{vmatrix} = \left(-\frac{y^2}{2x^2}\right)\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)\left(\frac{1}{2} - \frac{y^2}{2x^2}\right) = -\frac{\not{y^3}}{\not{2}x^3} - \frac{y}{2x} + \frac{\not{y^3}}{\not{2}x^3} = -\frac{y}{2x}$$

10. If $u = 2xy\,, = x^2 - y^2\,, x = r\cos\theta,\ y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Solution:

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \left(-4y^2 - 4x^2\right)\left(r\cos^2\theta + r\sin^2\theta\right)$$

$$= -4(y^2+x^2)r\left(\cos^2\theta + \sin^2\theta\right)$$

$$= -4r\left(r^2\cos^2\theta + r^2\sin^2\theta\right) = -4r^3\left(\cos^2\theta + \sin^2\theta\right) = -4r^3$$

11. If $x + y + z = u,\ y + z = uv,\ z = uvw$ prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Solution:

Given $\boxed{z = uvw}$, $\quad y + z = uv \Rightarrow y = uv - z \Rightarrow \boxed{y = uv - uvw}$ and

$x + y + z = u \Rightarrow x = u - (y + z) \Rightarrow \boxed{x = u - uv}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= uv \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -1 \\ vw & uw & 1 \end{vmatrix}$$

$$= uv\left[(1-v)(u-\cancel{uw}+\cancel{uw})+(u)(v-\cancel{vw}+\cancel{vw})+0\right]$$

$$= uv\left(u-\cancel{uv}+\cancel{uv}\right) = u^2v$$

12. Find the Jacobian $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Solution:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & \cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\sin\theta\left[\cos\theta\left(r\cos\theta\cos^2\phi + r\cos\theta\sin^2\phi\right) + r\sin\theta\left(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi\right) + 0\right]$$

$$= r\sin\theta\left[r\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$

$$= r\sin\theta\left[r\cos^2\theta + r\sin^2\theta\right]$$

$$= r^2\sin\theta\left[\cos^2\theta + \sin^2\theta\right] = r^2\sin\theta$$

(take $r\sin\theta$ outside and expand the determinant through the last column)

13. Show that the functions $u = \dfrac{x}{y}$, $v = \dfrac{x+y}{x-y}$ are functionally dependent and find the relation between them.

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & -\dfrac{x}{y^2} \\ -\dfrac{2y}{(x-y)^2} & \dfrac{2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{2x}{y(x-y)^2} - \frac{2xy}{y^2(x-y)^2} = \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$$

$\therefore$ u *and* v are functionally dependent.

Now, $v = \frac{x+y}{x-y} = \frac{\not{y}\left(\frac{x}{y}+1\right)}{\not{y}\left(\frac{x}{y}-1\right)} = \frac{u+1}{u-1}$.

14. Show that $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent. Find a relation between them.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= (2)(-1)\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \ \ (two\ cols\ are\ same)$$

$\therefore$ u, v $and\ w$ are functionally dependent.

Now, $w - v = 2z$ and $u - v = 4z$

$u - v = 4z \ \Rightarrow \ u - v = 2(w - v) \ \Rightarrow u - v - 2w + 2v = 0 \ \Rightarrow \ \boxed{u + v - 2w = 0}$

Taylor's series for functions of two variables

Statement

If $f(x,y)$ and all its partial derivatives are finite and continuous at all points (x,y) then,

$$f(x+h,y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots + \infty$$

Another form of Taylor's series of $f(x,y)$ at or near the point (a,b)

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)\right.$$
$$\left.+ (y-b)^3 f_{yyy}(a,b)\right] + \cdots + \infty$$

Note:

Taylor's series of $f(x,y)$ at or near the point $(0,0)$ is Maclaurins series of $f(x,y)$. The Maclaurins series of $f(x,y)$ in powers of x and y is given by,

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x\, f_x(0,0) + (y-b)f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots + \infty$$

Problems:

1. Expand $\sin xy$ at $\left(1,\frac{\pi}{2}\right)$ up to second degree terms using Taylor's series.

Solution:

$f(x,y) = \sin xy$ and $(a,b)=\left(1,\frac{\pi}{2}\right)$

$f(x,y)$ and its derivative	values at $\left(1,\frac{\pi}{2}\right)$
$f = \sin xy$	1
$f_x = y\cos xy$	0
$f_y = x\cos xy$	0
$f_{xx} = -y^2 \sin xy$	$-\frac{\pi^2}{4}$
$f_{xy} = \cos xy - xy\sin xy$	$-\frac{\pi}{2}$
$f_{yy} = -x^2 \sin xy$	-1

By Taylor's series,

$$f(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)\,f_x(a,b)+(y-b)\,f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2\,f_{xx}(a,b)+2(x-a)(y-b)\,f_{xy}(a,b)+(y-b)^2\,f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3\,f_{xxx}(a,b)+3(x-a)^2(y-b)\,f_{xxy}(a,b)+\\3(x-a)(y-b)^2\,f_{xyy}(a,b)+(y-b)^3\,f_{yyy}(a,b)\end{array}\right]+\ldots+\infty$$

$$\therefore \sin xy = 1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2+\ldots+\infty$$

2. Expand $e^x\log(1+y)$ in powers of x and y up to third degree terms using Taylor's theorem.

Solution:

$f(x,y) = e^x\log(1+y)$ and $(a,b)=(0,0)$

$f(x,y)$ and its derivative	values at $(0,0)$
$f = e^x\log(1+y)$	0
$f_x = e^x\log(1+y)$	0
$f_y = \frac{e^x}{1+y}$	1

$f_{xx} = e^{x}\log(1+y)$	0
$f_{xy} = \dfrac{e^{x}}{1+y}$	1
$f_{yy} = -\dfrac{e^{x}}{(1+y)^{2}}$	-1
$f_{xxx} = e^{x}\log(1+y)$	0
$f_{xxy} = \dfrac{e^{x}}{1+y}$	1
$f_{xyy} = -\dfrac{e^{x}}{(1+y)^{2}}$	-1
$f_{yyy} = \dfrac{2e^{x}}{(1+y)^{3}}$	2

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[\begin{matrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)\, f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{matrix}\right] + \ldots + \infty$$

$$\therefore e^{x}\log(1+y) = y + xy - \frac{y^2}{2} + \frac{1}{2}\left(x^2 y - xy^2\right) + \frac{y^3}{3} + \ldots\infty$$

3. Expand $e^{x}\cos y$ at $\left(0,\frac{\pi}{2}\right)$ up to the third term using Taylor's series.

Solution:

$f(x,y) = e^{x}\cos y$ and $(a,b) = \left(0,\frac{\pi}{2}\right)$

$f(x,y)$ and its derivative	values at $\left(0,\frac{\pi}{2}\right)$

$f = e^x \cos y$	0
$f_x = e^x \cos y$	0
$f_y = -e^x \sin y$	-1
$f_{xx} = e^x \cos y$	0
$f_{xy} = -e^x \sin y$	-1
$f_{yy} = -e^x \cos y$	0
$f_{xxx} = e^x \cos y$	0
$f_{xxy} = -e^x \sin y$	-1
$f_{xyy} = -e^x \cos y$	0
$f_{yyy} = e^x \sin y$	1

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\begin{bmatrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)\, f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{bmatrix} + \ldots + \infty$$

$$\therefore e^x \cos y = \frac{\pi}{2} - y - x\left(y - \frac{\pi}{2}\right) - \frac{1}{2}x^2\left(y - \frac{\pi}{2}\right) + \frac{1}{6}\left(y - \frac{\pi}{2}\right)^3 + \ldots + \infty$$

4. Obtain the Taylor series expansion up to the second degree of $e^x \cos y$ in powers of $(x+1)$ and $\left(y - \frac{\pi}{4}\right)$.

Solution:

$f(x,y) = e^x \cos y$ and $(a,b) = \left(-1, \frac{\pi}{4}\right)$

$f(x,y)$ and its derivative	values at $\left(-1, \frac{\pi}{4}\right)$
$f = e^x \cos y$	$\frac{1}{e\sqrt{2}}$
$f_x = e^x \cos y$	$\frac{1}{e\sqrt{2}}$
$f_y = -e^x \sin y$	$-\frac{1}{e\sqrt{2}}$

$f_{xx} = e^x \cos y$ $\qquad \frac{1}{e\sqrt{2}}$

$f_{xy} = -e^x \sin y$ $\qquad -\frac{1}{e\sqrt{2}}$

$f_{yy} = -e^x \cos y$ $\qquad -\frac{1}{e\sqrt{2}}$

$f_{xxx} = e^x \cos y$ $\qquad \frac{1}{e\sqrt{2}}$

$f_{xxy} = -e^x \sin y$ $\qquad -\frac{1}{e\sqrt{2}}$

$f_{xyy} = -e^x \cos y$ $\qquad -\frac{1}{e\sqrt{2}}$

$f_{yyy} = e^x \sin y$ $\qquad \frac{1}{e\sqrt{2}}$

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\begin{bmatrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2 (y-b) f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{bmatrix} + \ldots + \infty$$

$$e^x \cos y = \frac{1}{e\sqrt{2}}\begin{bmatrix}1 + (x+1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2}(x+1)^2 - (x+1)\left(y-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right)^2 + \\ \frac{1}{6}(x+1)^3 - \frac{1}{2}(x+1)^2\left(y-\frac{\pi}{4}\right) - \frac{1}{2}(x+1)\left(y-\frac{\pi}{4}\right)^2 + \frac{1}{6}\left(y-\frac{\pi}{4}\right)^3 + \ldots + \infty\end{bmatrix}$$

5. Find the Taylor series expansion of $e^x \cos y$ in the neighborhood of the point $\left(1, \frac{\pi}{4}\right)$ up to third degree terms.

Solution:

$f(x,y) = e^x \cos y$ and $(a,b) = \left(1, \frac{\pi}{4}\right)$

$f(x,y)$ and its derivative $\qquad$ values at $\left(1, \frac{\pi}{4}\right)$

$f = e^x \cos y$ $\qquad \frac{e}{\sqrt{2}}$

$f_x = e^x \cos y$ $\qquad \frac{e}{\sqrt{2}}$

$$f_y = -e^x \sin y \qquad -\frac{e}{\sqrt{2}}$$

$$f_{xx} = e^x \cos y \qquad \frac{e}{\sqrt{2}}$$

$$f_{xy} = -e^x \sin y \qquad -\frac{e}{\sqrt{2}}$$

$$f_{yy} = -e^x \cos y \qquad -\frac{e}{\sqrt{2}}$$

$$f_{xxx} = e^x \cos y \qquad \frac{e}{\sqrt{2}}$$

$$f_{xxy} = -e^x \sin y \qquad -\frac{e}{\sqrt{2}}$$

$$f_{xyy} = -e^x \cos y \qquad -\frac{e}{\sqrt{2}}$$

$$f_{yyy} = e^x \sin y \qquad \frac{e}{\sqrt{2}}$$

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[\begin{array}{l}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{array}\right] + \ldots + \infty$$

$$e^x \cos y = \frac{e}{\sqrt{2}}\left[\begin{array}{l}1 + (x-1) - \left(y-\frac{\pi}{4}\right) + \frac{1}{2}(x-1)^2 - (x-1)\left(y-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right)^2 + \\ \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)^2\left(y-\frac{\pi}{4}\right) - \frac{1}{2}(x-1)\left(y-\frac{\pi}{4}\right)^2 + \frac{1}{6}\left(y-\frac{\pi}{4}\right)^3 + \ldots + \infty\end{array}\right]$$

6. Expand $e^x \sin y$ by Taylor's theorem in powers of x and y as far as the terms of third degree.

Solution:

$f(x,y) = e^x \sin y$ and $(a,b) = (0,0)$.

$f(x,y)$ and its derivative — values at $(0,0)$

$$f = e^x \sin y \qquad 0$$

$$f_x = e^x \sin y \qquad 0$$

$$f_y = e^x \cos y \qquad 1$$

$f_{xx} = e^x \sin y$	0
$f_{xy} = e^x \cos y$	1
$f_{yy} = -e^x \sin y$	0
$f_{xxx} = e^x \sin y$	0
$f_{xxy} = e^x \cos y$	1
$f_{xyy} = -e^x \sin y$	0
$f_{yyy} = -e^x \cos y$	-1

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)\,f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{bmatrix} + \ldots + \infty$$

$$\therefore e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots + \infty$$

7. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem up to third degree terms.

Solution:

$f(x,y) = x^2y + 3y - 2$ and $(a,b) = (1,-2)$

$f(x,y)$ and its derivative	values at $(1,-2)$
$f = x^2y + 3y - 2$	-10
$f_x = 2xy$	-4
$f_y = x^2 + 3$	4
$f_{xx} = 2y$	-4
$f_{xy} = 2x$	2
$f_{yy} = 0$	0
$f_{xxx} = 0$	0
$f_{xxy} = 2$	2
$f_{xyy} = 0$	0
$f_{yyy} = 0$	0

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{matrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)\,f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{matrix}\right] + \ldots + \infty$$

$$\therefore x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + \ldots + \infty$$

8. Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ up to the third powers.

Solution:

$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$ and $(a,b) = (-2,1)$

$f(x,y)$ and its derivatives	values at $(-2,1)$
$f = x^2y^2 + 2x^2y + 3xy^2$	6
$f_x = 2xy^2 + 4xy + 3y^2$	-9
$f_y = 2x^2y + 2x^2 + 6xy$	4
$f_{xx} = 2y^2 + 4y$	6
$f_{xy} = 4xy + 4x + 6y$	-10
$f_{yy} = 2x^2 + 6x$	-4
$f_{xxx} = 0$	0
$f_{xxy} = 4y + 4$	8
$f_{xyy} = 4x + 6$	-2
$f_{yyy} = 0$	0

By Taylor's series,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\,f_x(a,b) + (y-b)\,f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{matrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)\,f_{xxy}(a,b) + \\ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{matrix}\right] + \ldots + \infty$$

$$\therefore x^2y^2+2x^2y+3xy^2=6+\left(-9(x+2)+4(y-1)\right)+\left(3(x+2)^2-10(x+2)(y-1)-2(y-1)^2\right)$$
$$+\left(4(x+2)^2(y-1)-(x+2)(y-1)^2\right)+\ldots.+\infty$$

9. Find the Taylor's series expansion of x^y near the point $(1,1)$ upto the second degree terms.

Solution:

$f(x,y)=x^y$ and $(a,b)=(1,1)$

$f(x,y)$ and its derivatives	values at $(1,1)$
$f=x^y$	1
$f_x=yx^{y-1}$	1
$f_y=x^y\log x$	0
$f_{xx}=y(y-1)x^{y-2}$	0
$f_{xy}=x^{y-1}+yx^{y-1}\log x$	1
$f_{yy}=x^y(\log x)^2$	0

By Taylor's series,

$$f(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+\\3(x-a)(y-b)^2f_{xyy}(a,b)+(y-b)^3f_{yyy}(a,b)\end{array}\right]+\ldots+\infty$$

$$\therefore x^y=1+(x-1)+(x-1)(y-1)+\ldots.+\infty$$

10. Expand $\dfrac{(x+h)(y+k)}{x+h+y+k}$ in series of powers of h and k up to the second degree terms.

Solution:

Let $f(x+h,y+k)=\dfrac{(x+h)(y+k)}{x+h+y+k}$. $\therefore f(x,y)=\dfrac{xy}{x+y}$.

$$f=\frac{xy}{x+y}$$

$$f_x=\frac{(x+y)y-xy(1)}{(x+y)^2}=\frac{y^2}{(x+y)^2}$$

$$f_y=\frac{(x+y)x-xy(1)}{(x+y)^2}=\frac{x^2}{(x+y)^2}$$

$$f_{xx}=\frac{(x+y)^2(0)-y^2 2(x+y)}{(x+y)^4}=\frac{-2y^2}{(x+y)^3}$$

$$f_{xy}=\frac{(x+y)2y-y^2 2(x+y)}{(x+y)^4}=\frac{2xy}{(x+y)^3}$$

$$f_{yy}=\frac{(x+y)^2(0)-x^2 2(x+y)}{(x+y)^4}=\frac{-2x^2}{(x+y)^3}$$

By Taylor's series,

$$f(x+h,y+k)=f(x,y)+\frac{1}{1!}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)+\cdots+\infty$$

$$\frac{(x+h)(y+k)}{x+h+y+k}=\frac{xy}{x+y}+\frac{hy^2}{(x+y)^2}+\frac{kx^2}{(x+y)^2}-\frac{h^2y^2}{(x+y)^3}+\frac{2hkxy}{(x+y)^3}-\frac{k^2x^2}{(x+y)^3}+\ldots+\infty$$

Maxima and minima of functions of two variables

A function $f(x,y)$ is said to have a relative maximum or simply maximum at $x=a$ and $y=b$ if $f(a,b)>f(a+h,b+k)$ for all small values of h *and* k.

A function $f(x,y)$ is said to have a relative minimum or simply minimum at $x=a$ and $y=b$ if $f(a,b)<f(a+h,b+k)$ for all small values of h *and* k.

A maximum or a minimum value of a function is called its extreme value.

Working rule to find the extreme values of a function $f(x,y)$

1. Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$.

2. Solve the equations $\frac{\partial f}{\partial x}=0$ *and* $\frac{\partial f}{\partial y}=0$ simultaneously. Let the solutions be (a,b), (c,d), ...

3. For each solution in step (2), find the values of $A=\frac{\partial^2 f}{\partial x^2}$, $B=\frac{\partial^2 f}{\partial x\partial y}$, $C=\frac{\partial^2 f}{\partial y^2}$ and $\Delta=AC-B^2$.

$\Delta>0$	A (or C) <0	$f(x,y)$ has a maximum value at (a,b)
$\Delta>0$	A (or C) >0	$f(x,y)$ has a minimum value at (a,b)
$\Delta<0$	----	$f(x,y)$ has neither a maximum nor a minimum value at (a,b)
$\Delta=0$	----	Doubtful case. Further investigations are required to decide the nature of

		extreme values of $f(x,y)$

Stationary points and Stationary values

The points at which $\frac{\partial f}{\partial x}=0$ *and* $\frac{\partial f}{\partial y}=0$ are called stationary points or critical points of the function $f(x,y)$. The values of $f(x,y)$ at the stationary points are called stationary values of $f(x,y)$.

Saddle point

If at a point $\Delta = AC - B^2 < 0$, then such points are called saddle points.

Necessary and sufficient conditions:

Necessary condition for $f(x,y)$ to have an extreme value at (a,b) is $\frac{\partial f}{\partial x}=0$ *and* $\frac{\partial f}{\partial y}=0$

Sufficient condition for $f(x,y)$ to have an extreme value at (a,b) is:

$\Delta > 0$	A (or C) < 0	$f(x,y)$ has a maximum value at (a,b)
$\Delta > 0$	A (or C) > 0	$f(x,y)$ has a minimum value at (a,b)

where $\Delta = AC - B^2$, $A=\frac{\partial^2 f}{\partial x^2}$, $B=\frac{\partial^2 f}{\partial x \partial y}$ and $C=\frac{\partial^2 f}{\partial y^2}$

Problems:

1. Test for maxima and minima of the function $f(x,y) = x^3 + y^3 - 12x - 3y + 20$.

Solution:

$f(x,y) = x^3 + y^3 - 12x - 3y + 20$

$\frac{\partial f}{\partial x} = 3x^2 - 12$ and $\frac{\partial f}{\partial y} = 3y^2 - 3$

$\frac{\partial f}{\partial x} = 0$

$\Rightarrow 3x^2 - 12 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x-2)(x+2) = 0$

$\Rightarrow x = 2,\ x = -2$

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3y^2 - 3 = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow (y-1)(y+1) = 0$

$\Rightarrow y = 1\ y = -1$

$\therefore$ The stationary points are $(2,1)$, $(2,-1)$, $(-2,1)$, $(-2,-1)$.

$A=\frac{\partial^2 f}{\partial x^2}=6x$, $B=\frac{\partial^2 f}{\partial x \partial y}=0$ and $C=\frac{\partial^2 f}{\partial y^2}=6y$.

Stationary points	$A=6x$	$B=0$	$C=6y$	$\Delta = AC - B^2$	Conclusion	Extreme value

$(2,1)$	$12>0$	0	6	$72>0$	Minimum	2
$(2,-1)$	12	0	-6	$-72<0$	Neither max. nor min.	Saddle points
$(-2,1)$	-12	0	6	$-72<0$		
$(-2,-1)$	$-12<0$	0	-6	$72>0$	Maximum	38

2. Test for an extrema of the function $x^4+y^4-x^2-y^2-1$.

Solution:

$$f(x,y)=x^4+y^4-x^2-y^2-1$$

$$\frac{\partial f}{\partial x}=4x^3-2x \quad \text{and} \quad \frac{\partial f}{\partial y}=4y^3-2y$$

$$\frac{\partial f}{\partial x}=0$$
$$\Rightarrow 4x^3-2x=0$$
$$\Rightarrow 2x(2x^2-1)=0$$
$$\Rightarrow x=0 \; and \; 2x^2-1=0$$
$$\Rightarrow x=0, \; and \; \left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)=0$$
$$\Rightarrow x=0, \; \left(\sqrt{2}x-1\right)=0 \; and \; \left(\sqrt{2}x+1\right)=0$$
$$\Rightarrow x=0, \; x=\frac{1}{\sqrt{2}}, \; x=-\frac{1}{\sqrt{2}}$$

$$\frac{\partial f}{\partial y}=0$$
$$\Rightarrow 4y^3-2y=0$$
$$\Rightarrow 2y(2y^2-1)=0$$
$$\Rightarrow y=0 \; and \; 2y^2-1=0$$
$$\Rightarrow y=0, \; and \; \left(\sqrt{2}y-1\right)\left(\sqrt{2}y+1\right)=0$$
$$\Rightarrow y=0, \; \left(\sqrt{2}y-1\right)=0 \; and \; \left(\sqrt{2}y+1\right)=0$$
$$\Rightarrow y=0, \; y=\frac{1}{\sqrt{2}}, \; y=-\frac{1}{\sqrt{2}}$$

$\therefore$ The stationary points are

$$(0,0), \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right), \left(0,\frac{\pm 1}{\sqrt{2}}\right), \left(\frac{\pm 1}{\sqrt{2}},0\right)$$

$$A=\frac{\partial^2 f}{\partial x^2}=12x^2-2, \quad B=\frac{\partial^2 f}{\partial x \partial y}=0 \quad \text{and} \quad C=\frac{\partial^2 f}{\partial y^2}=12y^2-2.$$

Stationary points	$A=12x^2-2$	$B=0$	$C=12y^2-2$	$\Delta=AC-B^2$	Conclusion	Extreme value
$(0,0)$	$-2<0$	0	-2	$4>0$	Maximum	-1
$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$4>0$	0	4	$16>0$	Minimum	$-\frac{3}{2}$
$\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$	$4>0$	0	4	$16>0$		

$\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$4>0$	0	4	$16>0$		
$\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$	$4>0$	0	4	$16>0$		
$\left(0,\frac{\pm1}{\sqrt{2}}\right)$	-2	0	4	$-8<0$	Neither max. nor min.	Saddle points
$\left(\frac{\pm1}{\sqrt{2}},0\right)$	4	0	-2	$-8<0$		

3. Discuss the maxima and minima of the function $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$.

Solution:

$f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

$\frac{\partial f}{\partial x}=4x^3-4x+4y$ and $\frac{\partial f}{\partial y}=4y^3+4x-4y$

$\frac{\partial f}{\partial x}=0$

$\Rightarrow 4(x^3-x+y)=0$

$\Rightarrow x^3-x+y=0$ .... (1)

$\frac{\partial f}{\partial y}=0$

$\Rightarrow 4(y^3+x-y)=0$

$\Rightarrow y^3+x-y=0$ .... (2)

Adding (1) and (2) we get $x^3+y^3=0 \Rightarrow y=-x$ ....(3)

Using (3) in (1), $x^3-x-x=0 \Rightarrow x^3-2x=0 \Rightarrow x(x^2-2)=0 \Rightarrow x=0,\ x=\sqrt{2}$, $x=-\sqrt{2}$

$\because y=-x$, we have $y=0,\ y=-\sqrt{2},\ y=\sqrt{2}$.

$\therefore$ The stationary points are $(0,0)$, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$

$A=\frac{\partial^2 f}{\partial x^2}=4(3x^2-1)$, $B=\frac{\partial^2 f}{\partial x\partial y}=4$, $C=\frac{\partial^2 f}{\partial y^2}=4(3y^2-1)$

Stationary points	$A=4(3x^2-1)$	$B=4$	$C=4(3y^2-1)$	$\Delta=AC-B^2$	Conclusion	Extreme value
$(0,0)$	-4	4	-4	$16-16$ $=0$	Doubtful case	---
$(\sqrt{2},-\sqrt{2})$	$20>0$	4	20	$384>0$	Minimum	8
$(-\sqrt{2},\sqrt{2})$	$20>0$	4	20	$384>0$		

4. Find the maximum and minimum values of x^2-xy+y^2-2x+y.

Solution:

$f(x,y) = x^2 - xy + y^2 - 2x + y$

$\frac{\partial f}{\partial x} = 2x - y - 2$ and $\frac{\partial f}{\partial y} = 2y - x + 1$

$\frac{\partial f}{\partial x} = 0$

$\Rightarrow 2x - y - 2 = 0$

$\Rightarrow 2x - y = 2$ .... (1)

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow 2y - x + 1 = 0$

$\Rightarrow 2y - x = -1$

$\Rightarrow x = 2y + 1$ .... (2)

Substitute (2) in (1) $2(2y+1) - y = 2 \Rightarrow 4y + 2 - y - 2 = 0 \Rightarrow 3y = 0 \Rightarrow y = 0$

Sub $y = 0$ in (2), $x = 2(0) + 1 \Rightarrow x = 1$

$\therefore$ The stationary point is $(1,0)$.

$A = \frac{\partial^2 f}{\partial x^2} = 2$, $B = \frac{\partial^2 f}{\partial x \partial y} = -1$, $C = \frac{\partial^2 f}{\partial y^2} = 2$

Stationary points	$A = 2$	$B = -1$	$C = 2$	$\Delta = AC - B^2$	Conclusion	Extreme value
(1,0)	$2 > 0$	-1	2	$3 > 0$	Minimum	-1

5. Find the extreme value of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Solution:

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$\frac{\partial f}{\partial x} = 3x^2 - 3$ and $\frac{\partial f}{\partial y} = 3y^2 - 12$

$\frac{\partial f}{\partial x} = 0$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow (x-1)(x+1) = 0$

$\Rightarrow x = 1$ $x = -1$

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3y^2 - 12 = 0$

$\Rightarrow y^2 - 4 = 0$

$\Rightarrow (y-2)(y+2) = 0$

$\Rightarrow y = 2,\ y = -2$

$\therefore$ The stationary points are $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$.

$A = \frac{\partial^2 f}{\partial x^2} = 6x$, $B = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $C = \frac{\partial^2 f}{\partial y^2} = 6y$.

Stationary points	$A = 6x$	$B = 0$	$C = 6y$	$\Delta = AC - B^2$	Conclusion	Extreme value
(1,2)	6 >0	0	12	$72 > 0$	Minimum	2

$(1,-2)$	6	0	-12	$-72<0$	Neither max. nor min. Saddle points	----
$(-1,2)$	-6	0	12	$-72<0$		
$(-1,-2)$	$-6<0$	0	-12	$72>0$	Maximum	38

6. Discuss the maxima and minima of $f(x,y)=x^3y^2(1-x-y)$

Solution:

$f(x,y)=x^3y^2(1-x-y)=x^3y^2-x^4y^2-x^3y^3$

$\frac{\partial f}{\partial x}=3x^2y^2-4x^3y^2-3x^2y^3$ and $\frac{\partial f}{\partial y}=2x^3y-2x^4y-3x^3y^2$

$\frac{\partial f}{\partial x}=0$

$\Rightarrow 3x^2y^2-4x^3y^2-3x^2y^3=0$

$\Rightarrow x^2y^2(3-4x-3y)=0$

$\Rightarrow x=0,\ y=0,\ 3-4x-3y=0$

$\Rightarrow x=0\ y=0,\ 4x+3y=3$ ... (1)

$\frac{\partial f}{\partial y}=0$

$\Rightarrow 2x^3y-2x^4y-3x^3y^2=0$

$\Rightarrow x^3y(2-2x-3y)=0$

$\Rightarrow x=0,\ y=0,\ 2-2x-3y=0$

$\Rightarrow x=0,\ y=0,\ 2x+3y=2$ ... (2)

From (2) $2x=2-3y \Rightarrow 4x=4-6y$

Substituting this in (1), $4-6y+3y=3 \Rightarrow -3y=-1 \Rightarrow y=\frac{1}{3}$

Sub $y=\frac{1}{3}$ in (1), $4x+3\left(\frac{1}{3}\right)=3 \Rightarrow 4x+1=3 \Rightarrow 4x=2 \Rightarrow x=\frac{1}{2}$.

Substituting $x=0,\ y=0\ in$ (1) and (2), we get,

When $x=0$, we have $4(0)+3y=3 \Rightarrow y=1$ and $2(0)+3y=2 \Rightarrow y=\frac{2}{3}$

When $y=0$, we have $4x+3(0)=3 \Rightarrow x=\frac{3}{4}$ and $2x+3(0)=2 \Rightarrow x=1$

$\therefore$ The stationary points are $\left(\frac{1}{2},\frac{1}{3}\right)$, $(0,0)$, $(0,1)$, $\left(0,\frac{2}{3}\right)$, $\left(\frac{3}{4},0\right)$, $(1,0)$.

$A=\frac{\partial^2 f}{\partial x^2}=6xy^2-12x^2y^2-6xy^3$, $B=\frac{\partial^2 f}{\partial x\partial y}=6x^2y-8x^3y-9x^2y^2$ and

$C=\frac{\partial^2 f}{\partial y^2}=2x^3-2x^4-6x^3y$

Stationary points	$A=6xy^2-12x^2y^2-6xy^3$	$B=6x^2y-8x^3y-9x^2y^2$	$C=2x^3-2x^4-6x^3y$	$\Delta=AC-B^2$	Conclusion	Extreme value
$\left(\frac{1}{2},\frac{1}{3}\right)$	$-\frac{1}{9}<0$	$-\frac{1}{12}$	$-\frac{1}{8}$	$\frac{1}{144}>0$	Max	$\frac{1}{432}$
$(0,0)$	0	0	0	0	Doubtful	----

(0,1)	0	0	0	0	case	
$\left(0,\frac{2}{3}\right)$	0	0	0	0		
$\left(\frac{3}{4},0\right)$	0	0	$\frac{27}{128}$	0		
(1,0)	0	0	0	0		

7. Find the local maxima, local minima of the function $f(x,y) = x^3y^2(12 - x - y)$.

Solution:

$f(x,y) = x^3y^2(12 - x - y) = 12x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$ and $\frac{\partial f}{\partial y} = 24x^3y - 2x^4y - 3x^3y^2$

$\frac{\partial f}{\partial x} = 0$

$\Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(36 - 4x - 3y) = 0$

$\Rightarrow x = 0,\ y = 0,\ 36 - 4x - 3y = 0$

$\Rightarrow x = 0\ \ y = 0,\ 4x + 3y = 36$ ... (1)

$\frac{\partial f}{\partial y} = 0$

$\Rightarrow 24x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(24 - 2x - 3y) = 0$

$\Rightarrow x = 0,\ y = 0,\ 24 - 2x - 3y = 0$

$\Rightarrow x = 0,\ y = 0,\ 2x + 3y = 24$ ... (2)

From (2) $2x = 24 - 3y \Rightarrow 4x = 48 - 6y$

Substituting this in (1), $48 - 6y + 3y = 36 \Rightarrow -3y = -12 \Rightarrow y = 4$

Sub $y = 4$ in (1), $4x + 3(4) = 36 \Rightarrow 4x + 12 = 36 \Rightarrow 4x = 24 \Rightarrow x = 6.$

Substituting $x = 0,\ y = 0\ in\ (1)\ and\ (2)$, we get,

When $x = 0$, we have $4(0) + 3y = 36 \Rightarrow y = 12$ and $2(0) + 3y = 24 \Rightarrow y = 8$

When $y = 0$, we have $4x + 3(0) = 36 \Rightarrow x = 9$ and $2x + 3(0) = 24 \Rightarrow x = 12$

$\therefore$ The stationary points are (6,4), (0,0), (0,12), (0,8), (9,0), (12,0).

$A = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 12x^2y^2 - 6xy^3$, $B = \frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 8x^3y - 9x^2y^2$ and

$C = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 2x^4 - 6x^3y$

Stationary points	$A = 72xy^2 - 12x^2y^2 - 6xy^3$	$B = 72x^2y - 8x^3y - 9x^2y^2$	$C = 24x^3 - 2x^4 - 6x^3y$	$\Delta = AC - B^2$	Conclusion	Extreme value
(6,4)	$-2304 < 0$	-1728	-2592	$2985984 > 0$	Max	6912

(0,0)	0	0	0	0		
(0,12)	0	0	0	0		
(0,8)	0	0	0	0	Doubtful case	----
(9, 0)	0	0	4374	0		
(12,0)	0	0	0	0		

Lagrange's method of undetermined multipliers

Let $f(x,y,z)$ be a function of three variables x,y,z and the variable be connected by the relation $g(x,y,z)=0$. Suppose we wish to find the values of x,y,z for which $f(x,y,z)$ is extremum.

For this purpose, we construct an auxiliary equation $u(x,y,z)=f(x,y,z)+\lambda\, g(x,y,z)=0$. Differentiating this w.r.t. x,y,z partially we get,

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0 \quad ...(1)\ ,\ \frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0 \quad ...(2) \text{ and } \frac{\partial u}{\partial z}=\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}=0 \quad ...(3).$$

Eliminating λ from Eqns. (1), (2) and (3), the values of x,y,z are obtained. This method is called Lagrange's method of undetermined multipliers and Eqns. (1), (2) and (3), are called Lagrange's equations. λ is called undetermined multiplier or Lagrange's multiplier.

Note:

This method does not specify whether the extreme value found out is a maximum value or a minimum value. It is decided from the physical consideration of the problem.

Problems:

1. Find the extreme value of $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$.

Solution:

Given $f(x,y,z)=x^2+y^2+z^2$ ... (1) and $g(x,y,z)=x+y+z-3a=0$ ... (2)

Now, $u=f+\lambda g=x^2+y^2+z^2+\lambda(x+y+z-3a)$ ... (3)

Differentiate (3) partially w.r.t x,y,z

$$\frac{\partial u}{\partial x}=0\Rightarrow 2x+\lambda=0 \quad ...(4),\ \frac{\partial u}{\partial y}=0\Rightarrow 2y+\lambda=0 \quad ...(5) \text{ and } \frac{\partial u}{\partial z}=0\Rightarrow 2z+\lambda=0 \quad ...(6)$$

From (4) ,(5) and (6) $x=-\frac{\lambda}{2}$, $y=-\frac{\lambda}{2}$ and $z=-\frac{\lambda}{2}$.

$\Rightarrow x=y=z$

Substitute in (2) $x+x+x=3a \Rightarrow 3x=3a \Rightarrow x=a$

$\Rightarrow y=a \ \ and \ \ z=a$

Extreme value of $x^2+y^2+z^2$ is $3a^2$.

2. Find the maximum value of $x^m y^n z^p$ subject to the condition $x+y+z=a$.

Solution:

Given $f(x,y,z)=x^m y^n z^p$ ... (1) and $g(x,y,z)=x+y+z-a=0$ ... (2)

$u=f+\lambda g=x^m y^n z^p+\lambda(x+y+z-a)$ ... (3)

Differentiate (3) partially w.r.t x,y,z

$\frac{\partial u}{\partial x}=0 \Rightarrow mx^{m-1}y^n z^p+\lambda=0$ ...(4) , $\frac{\partial u}{\partial y}=0 \Rightarrow x^m ny^{n-1}z^p+\lambda=0$ ...(5) and

$\frac{\partial u}{\partial z}=0 \Rightarrow x^m y^n pz^{p-1}+\lambda=0$ ...(6)

From (4), (5) *and* (6) ,

$mx^{m-1}y^n z^p=-\lambda \Rightarrow \frac{mx^m y^n z^p}{x}=-\lambda$ ... (7)

$x^m z^p ny^{n-1}=-\lambda \Rightarrow \frac{nx^m y^n z^p}{y}=-\lambda$ ... (8) *and*

$x^m y^n pz^{p-1}=-\lambda \Rightarrow \frac{px^m y^n z^p}{z}=-\lambda$ ... (9)

From (7), (8) and (9)

$$\frac{mx^m y^n z^p}{x}=\frac{nx^m y^n z^p}{y}=\frac{px^m y^n z^p}{z} \Rightarrow \frac{m}{x}=\frac{n}{y}=\frac{p}{z}$$

From $\frac{m}{x}=\frac{n}{y}$ *we have* $my=nx \Rightarrow y=\frac{n}{m}x$ ... (10)

from $\frac{m}{x}=\frac{p}{z}$ *we have* $mz=px \Rightarrow z=\frac{p}{m}x$ ... (11)

Substitute (10) and (11) in (2)

$$x+\frac{n}{m}x+\frac{p}{m}x=a \Rightarrow x\left(1+\frac{n}{m}+\frac{p}{m}\right)=a \Rightarrow \frac{x(m+n+p)}{m}=a \Rightarrow \boxed{x=\frac{am}{m+n+p}}$$

$$y=\frac{n}{m}x=\frac{n}{m}\left(\frac{am}{m+n+p}\right) \Rightarrow \boxed{y=\frac{an}{m+n+p}}$$

$$z=\frac{p}{m}x=\frac{p}{m}\left(\frac{am}{m+n+p}\right) \Rightarrow \boxed{z=\frac{ap}{m+n+p}}$$

Maximum value of $x^m y^n z^p$ *is* $\left(\frac{am}{m+n+p}\right)^m\left(\frac{an}{m+n+p}\right)^n\left(\frac{ap}{m+n+p}\right)^p$

3. If $x^2+y^2+z^2=r^2$, show that the maximum value of $yz+zx+xy$ is r^2 and the minimum value is $-\frac{r^2}{2}$.

Solution:

Given $f(x,y,z)=yz+zx+xy$ ... (1) and $g(x,y,z)=x^2+y^2+z^2-r^2=0$ ... (2)

$u=f+\lambda g=yz+zx+xy+\lambda(x^2+y^2+z^2-r^2)$ ... (3)

Differentiate (3) partially w.r.t x,y,z

$\frac{\partial u}{\partial x}=0\Rightarrow z+y+\lambda(2x)=0$ ...(4), $\frac{\partial u}{\partial y}=0\Rightarrow z+x+\lambda(2y)=0$ ...(5) and

$\frac{\partial u}{\partial z}=0\Rightarrow y+x+\lambda(2z)=0$ ...(6)

Multiply $Eqn.(4)$ by yz, $Eqn\ (5)$ by xz and $Eqn.(6)$ by xy

$yz^2+y^2z+\lambda(2xyz)=0\Rightarrow yz^2+y^2z=-\lambda(2xyz)$ ...(7),

$xz^2+x^2z+\lambda(2xyz)=0\Rightarrow xz^2+x^2z=-\lambda(2xyz)$ ...(8)

$xy^2+x^2y+\lambda(2xyz)=0\Rightarrow xy^2+x^2y=-\lambda(2xyz)$ ...(9)

From (7), (8) and (9) $yz^2+y^2z=xz^2+x^2z=xy^2+x^2y$

$yz^2+y^2z=xz^2+x^2z$

$\Rightarrow yz^2+y^2z-xz^2-x^2z=0$

$\Rightarrow z^2(y-x)+z(y^2-x^2)=0$

$\Rightarrow z^2(y-x)+z(y-x)(y+x)=0$

$\Rightarrow z(y-x)(z+y+x)=0$

$\Rightarrow y-x=0$ and $z+y+x=0$

$\Rightarrow y=x$ and $x+y+z=0$

$xz^2+x^2z=xy^2+x^2y$

$\Rightarrow xz^2+x^2z-xy^2-x^2y=0$

$\Rightarrow x^2(z-y)+x(z^2-y^2)=0$

$\Rightarrow x^2(z-y)+x(z-y)(z+y)=0$

$\Rightarrow x(z-y)(x+z+y)=0$

$\Rightarrow z-y=0$ and $x+z+y=0$

$\Rightarrow y=z$ and $x+y+z=0$

$\Rightarrow x=y=z$; $x+y+z=0$

$Substitute$ $x=y=z$ in (2) $x^2+x^2+x^2=r^2 \Rightarrow 3x^2=r^2 \Rightarrow x^2=\frac{r^2}{3} \Rightarrow x=\frac{r}{\sqrt{3}}$

$\because x=y=z,$ we $have$ $y=\frac{r}{\sqrt{3}}$ and $z=\frac{r}{\sqrt{3}}$

$\therefore$ $maximum$ $value$ of $yz+zx+xy$ is

$$\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)+\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)+\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)=\frac{r^2}{3}+\frac{r^2}{3}+\frac{r^2}{3}=\frac{3r^2}{3}=r^2$$

Now, consider $x+y+z=0$

$\Rightarrow (x+y+z)^2=0$

$\Rightarrow x^2+y^2+z^2+2xy+2yz+2zx=0$

$\Rightarrow 2(xy+yz+zx)=-(x^2+y^2+z^2)$

$\Rightarrow 2(xy+yz+zx)=-r^2$

$\Rightarrow xy + yz + zx = -\dfrac{r^2}{2}$

$\therefore$ *minimum value of* $yz + zx + xy$ *is* $-\dfrac{r^2}{2}$

4. A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the least material for its construction.

Solution:

Let x, y, z be the length, breadth and height of the box respectively. The material for the construction of the box is least, when the surface area of the box is least.

Hence we have to minimize $xy + 2yz + 2zx$ subject to the condition that the volume of the box is $xyz = 32$.

Here $f(x,y,z) = xy + 2yz + 2zx$ ... (1) and $g(x,y,z) = xyz - 32$ ... (2)

$u = f + \lambda g = xy + 2yz + 2zx + \lambda(xyz - 32)$ ... (3)

Differentiate (3) partially w.r.t x, y, z

$\dfrac{\partial u}{\partial x} = 0 \Rightarrow y + 2z + \lambda(yz) = 0$ ...(4), $\dfrac{\partial u}{\partial y} = 0 \Rightarrow x + 2z + \lambda(zx) = 0$ ...(5) and

$\dfrac{\partial u}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda(xy) = 0$ ...(6)

Multiply $Eqn.(4)$ by x, $Eqn\,(5)$ by y and $Eqn.(6)$ by z

$xy + 2xz + \lambda(xyz) = 0 \Rightarrow xy + 2xz = -\lambda xyz$ ...(7),

$xy + 2yz + \lambda(xyz) = 0 \Rightarrow xy + 2yz = -\lambda(xyz)$ ...(8)

$2xz + 2yz + \lambda(xyz) = 0 \Rightarrow 2xz + 2yz = -\lambda(xyz)$ ...(9)

from (7), (8) *and* (9), $xy + 2xz = xy + 2yz = 2xz + 2yz$

$xy + 2xz = xy + 2yz$	$xy + 2yz = 2xz + 2yz$
$\Rightarrow 2xz = 2yz$	$\Rightarrow xy = 2xz$
$\Rightarrow x = y$	$\Rightarrow y = 2z$

$\Rightarrow x = y = 2z$. Substitute this in (2)

$xyz = 32 \;\Rightarrow\; x \times x \times \dfrac{x}{2} = 32 \;\Rightarrow x^3 = 64 \;\Rightarrow\; x = 4$

$\Rightarrow y = 4$ *and* $z = \dfrac{y}{2} = \dfrac{4}{2} = 2$

The dimensions of the box are $x = 4,\; y = 4,\; z = 2$

5. Find the volume of the greatest rectangular parallelepiped inscribed in the ellipsoid whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution:

Let $2x,\ 2y,\ 2z$ be the dimensions of the required rectangular parallelepiped. By symmetry, the Centre of the parallelepiped coincides with that of the ellipsoid namely, origin and its faces are parallel to the coordinate planes.

Also one of the vertices of the parallelepiped has coordinate (x, y, z) which satisfy the equation of the ellipsoid.

Thus we have to maximize $V = 8xyz$ subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Here $f(x,y,z) = 8xyz$ ... (1) and $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ ...(2).

$$u = f+\lambda g = 8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right) \quad ...(3)$$

Differentiate (3) partially w.r.t x, y, z

$\frac{\partial u}{\partial x}=0 \Rightarrow 8yz+\frac{2\lambda x}{a^2}=0$ ...(4), $\frac{\partial u}{\partial y}=0 \Rightarrow 8xz+\frac{2\lambda y}{b^2}=0$ ...(5) and

$\frac{\partial u}{\partial z}=0 \Rightarrow 8xy+\frac{2\lambda z}{c^2}=0$ ...(6)

Multiply $Eqn.(4)$ by x, $Eqn\ (5)$ by y and $Eqn.(6)$ by z

$$8xyz+\frac{2\lambda x^2}{a^2}=0 \Rightarrow 8xyz=-\frac{2\lambda x^2}{a^2} \Rightarrow \boxed{\frac{8xyz}{-2\lambda}=\frac{x^2}{a^2}} \quad ...(7),$$

$$8xyz+\frac{2\lambda y^2}{b^2}=0 \Rightarrow 8xyz=-\frac{2\lambda y^2}{b^2} \Rightarrow \boxed{\frac{8xyz}{-2\lambda}=\frac{y^2}{b^2}} \quad ...(8)$$

$$8xyz+\frac{2\lambda z^2}{c^2}=0 \Rightarrow 8xyz=-\frac{2\lambda z^2}{c^2} \Rightarrow \boxed{\frac{8xyz}{-2\lambda}=\frac{z^2}{c^2}} \quad ...(8)$$

From (7), (8) and (9) we have

$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=k$ (say) substitute this in (2)

$$k+k+k=1 \Rightarrow 3k=1 \Rightarrow k=\frac{1}{3}$$

$$\therefore \frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$$

$$\Rightarrow \frac{x^2}{a^2}=\frac{1}{3},\ \frac{y^2}{b^2}=\frac{1}{3}\ and\ \frac{z^2}{c^2}=\frac{1}{3}$$

$$\Rightarrow x^2=\frac{a^2}{3},\ y^2=\frac{b^2}{3}\ and\ z^2=\frac{c^2}{3}$$

$$\Rightarrow x=\frac{a}{\sqrt{3}},\ y=\frac{b}{\sqrt{3}}\ and\ z=\frac{c}{\sqrt{3}}$$

$\therefore$ Maximum volume is $8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)=\frac{8abc}{3\sqrt{3}}$