SRI RAMAKRISHNA INSTITUTE OF TECHNOLOGY, COIMBATORE- 10

DEPARTMENT OF SCIENCE AND HUMANITIES

COMMON TO BE (CSE)

SUBJECT: NUMERICAL METHODS (SEMESTER - IV)

UNIT III

NUMERICAL DIFFERENTIATION & INTEGRATION

1. Given $f(x) = -1,\ f(1) = 1\ and\ f(2) = 4$, find the roots of the polynomial equation $f(x) = 0$.

Answer :

Let us known $f(x_0) = y_0$.

Here $y_0 = -1,\ \Delta y_0 = 2,\ \Delta y_1 = 3,\ \Delta^2 y_0 = 1,\ p = x$

$$f(x) = -1 + 2x + \frac{x(x-1)}{2}(1) = \frac{1}{2}[x^2 + 3x - 2]$$

$$f(x) = 0 \Rightarrow x = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}$$

2. State Newton's forward Difference formula to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_0$.

Answer :

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left\{\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \cdots\right\} \quad and$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right\}$$

3. Find the parabola of the form $y = a x^2 + bx + c$ passing through the points $(0,0), (1,1)\ \&\ (2,20)$.

Answer :

Let us known $f(x_0) = y_0$.

Here $y_0 = 0,\ \Delta y_0 = 1,\ \Delta y_1 = 19,\ \Delta^2 y_0 = 18,\ p = x$

$$f(x) = 0 + x(1) + \frac{x(x-1)}{2}(18) \Rightarrow y = 9x^2 - 8x$$

4. Write the formula to compute $\frac{dy}{dx}$ at $x = x_0 + ph$ for the given data $(x_i, y_i), i = 0,1,2, \ldots n$.

Answer :

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left\{\Delta y_0 - \frac{2p-1}{2}\ \Delta^2 y_0 + \frac{3p^2-6p+2}{6}\ \Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12}\ \Delta^4 y_0 - \cdots\right\}$$

$where\ p = \frac{x-x_0}{h.}$

5. Write the formula to compute $\frac{d^2y}{dx^2}$ at $x = x_0 + ph$ for the given data $(x_i, y_i), i = 0,1,2, \dots n.$

Answer :

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left\{\Delta^2 y_0 + (p-1)\ \Delta^2 y_0 + \frac{6p^2 - 18\,p + 11}{12}\ \Delta^4 y_0 + \cdots\right\}$$

$where\ p = \frac{x-x_0}{h.}$

6. State Newton's Backward interpolation formula to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$.

Answer :

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left\{\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right\} \quad and$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\ \nabla^4 y_0 - \cdots\right\}$$

7. Write the formula to compute $\frac{dy}{dx}$ at $x = x_n + ph$ for the given data $(x_i, y_i), i = 0,1,2, \dots n.$

Answer :

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left\{\nabla y_n - \frac{2p+1}{2}\ \nabla^2 y_n + \frac{3+6p+2}{6}\ \nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\ \nabla^4 y_0 - \cdots\right\}$$

$where\ p = \frac{x-x_n}{h.}$

8. Write the formula to compute $\frac{d^2y}{dx^2}$ at $x = x_n + ph$ for the given data $(x_i, y_i), i = 0,1,2, \dots n.$

Answer :

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left\{\nabla^2 y_n + (p+1)\ \nabla^3 y_n + \frac{6p^2 + 18\,p + 11}{12}\ \nabla^4 n + \cdots\right\}$$

$where\ p = \frac{x-x_n}{h.}$

9. Find $\frac{dy}{dx}$ at $x = 2$ from the following data.

$x:$	2	3	4
$y:$	26	58	112

Answer : $\Delta y_0 = 32, \Delta y_1 = 54\,, \Delta^2 y_0 = 22$

$$\frac{dy}{dx} = 32 - \frac{1}{2}\,(22) = 21.$$

10. Find $\frac{dy}{dx}$ at $x = 6$ from the following data.

$x:$	2	4	6
$y:$	3	11	27

Answer : $\nabla y_n = 16,\ \nabla y_{n-1} = 8,\ \nabla^2 y_n = 16 - 8 = 8$

$$\left(\frac{d^2y}{dx^2}\right)_{x=6} = \frac{1}{2}\left(16 + \frac{8}{2}\right) = 10.$$

11. A curve passing through the points $(1,0), (2,1)$ and $(4,5)$. Find the slope of the curve at $x = 3$.

Answer :

$$f(1,2) = 1, \qquad f(2,4) = 2, \qquad f(1,2,4) = \frac{1}{3}.$$

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$

$$f(x) = 0 + (x-1)(1) + (x-1)(\,x-2)\frac{1}{3} \;=\; x - 1 + \frac{1}{3}\,(x^2 - 3x + 2)$$

$$f'^{(x)} = 1 + \frac{2x}{3} - 1 = \frac{2x}{3}$$

Slope at $x = 3$ is $\frac{2(3)}{3} = 2.$

12. State Trapezoidal rule with the error order.

Answer :

For the given data (x_i, y_i) where $x_i = x_0 + ih,\ \ i = 0,1,2 \ldots.n$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\,[(y_0 + y_n) + 2(y_1 + y_3 + \cdots + y_{n-1})] \qquad and$$

$$Error\ is\ of\ order\ h^2\,.$$

13. State Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule.

Answer :

If (x_i, y_i) $i = 0,1,2, \ldots n$ where $x_i = x_0 + ih,\ then$

Simpson's $\frac{1}{3}$ rule :

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots +) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

Simpson's $\frac{3}{8}$ rule :

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots +) + 2(y_3 + y_6 + y_9 \ldots)\right]$$

14. State the basic principle for deriving Simpson's $\frac{1}{3}$ rule.

Answer :

The curve passing through the consecutive points is replaced by a parabola.

15. State the order of error in Simpson's $\frac{1}{3}$ rule.

Answer :

Error in Simpson's $\frac{1}{3}$ rule is of order h^4.

16. Using Simpson's rule, find $\int_0^4 e^x\,dx$ given $e^0 = 1, e^1 = 2.72,\ e^2 = 7.39,\ e^3 = 20.09\ \&\ e^4 = 54.6.$

Answer :

$$\int_0^4 e^x\,dx = \frac{1}{3}\left[(1 + 54.6) + 4(2.72 + 20.09) + 2(7.39)\right] = 53.873.$$

17. A curve passes through $(2,8),\ (3,27),\ (4,64)\ \&\ (5,125).$ find the area of the curve between x- axis and the line $x = 2\ and\ x = 5$, by Trapezoidal rule.

Answer :

$$\int_2^5 y\,dx = \frac{1}{2}\left[(8 + 125) + 2(27 + 64)\right] = 157.5\ sq.units.$$

18. Find $\int_{-2}^{+2} x^4\,dx$ by Simpson's rule, taking h=1.

Answer :

$x:$	-2	-1	0	1	2
$y:$	16	1	0	1	16

$$\therefore \int_{-2}^{+2} x^4\,dx = \frac{1}{3}\left[(16 + 16) + 4(2)\right] = 13.3\ sq.units.$$

19. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Trapezoidal rule with $h = 0.5$.

Answer :

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.5}{2}\ [1.5 + 2(0.8)] = 0.775.$$

20. Use Simpson's $\frac{1}{3}$ rule with $h = 0.5$ to evaluate $\int_0^1 \frac{dx}{1+x}$.

Answer :

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{6}\left[1 + \frac{4}{1.5} + \frac{1}{2}\right] = 0.6944.$$

21. Evaluate $\int_{-1}^{+1} |x| dx$ with two subintrevals by Simpson's $\frac{1}{3}$ rule and by Trapezoidal rule.

Answer :

By Simpson's $\frac{1}{3}$ rule $\ I = \frac{1}{3}\ [1 + 0 + 1] = \frac{2}{3}$

By Trapezoidal rule $\ I = \frac{1}{2}\ [1 + 1] = 1$

22. If $I = \int_0^1 e^{-x^2}\ dx, then\ \ I_1 = 0.731\ \ \&\ \ I_2 = 0.7430\ \ \ with\ \ h = 0.5\ \ \&\ \ h = 0.25.$ Find I by using Romberg's method.

Answer :

$$I = \int_0^1 e^{-x^2}\ dx = 0.7430 + \left(\frac{0.7430 - 0.7314}{3}\right) = 0.7469.$$

23. Find $I = \int_{-1}^{+1} \frac{dx}{1+x^2}$ by Two – Point Gaussian formula.

Answer :

$$I = \int_{-1}^{+1} \frac{dx}{1+x^2} = \frac{1}{1+1/3} + \frac{1}{1+1/3} = \frac{3}{2} = 1.5$$

24. Find $I = \int_0^{+1} \frac{dx}{1+x}$ by Two – Point Gaussian formula.

Answer :

$$I = \int_{-1}^{+1} \frac{dx}{t+3}\ \ using\ \ x = \frac{1+t}{2}.\ Then\ \ I = \frac{1}{3+\frac{1}{\sqrt{3}}} + \frac{1}{3+\frac{1}{\sqrt{3}}} = 0.6923.$$

25. Evaluate $I = \int_{-1}^{+1} \cos x\ \ dx$ by Two – Point Gaussian formula.

Answer :

$$I = \int_{-1}^{+1} \cos x\ dx\ = 2\cos\left(\frac{1}{\sqrt{3}}\right) = 1.67585$$

26. State three point Gaussin quadrature formula.

Answer :

The three point Gaussian quadrature formula is

$$\int_{-1}^{+1} f(x)\ dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

This formula is exact for polynamials upto degree 5.

27. State two point Gaussin quadrature formula.

Answer :

The three point Gaussian quadrature formula is

$$\int_{-1}^{+1} f(x)\ dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

This formula is exact for polynamials upto degree 3.

28. If the range is not $(-1, +1)$ then what is the idea to solve the Gaussian quadrature problems.

Answer : $x = \frac{b-a}{2} z + \frac{b+a}{2}$

PART - B

1. Find $f'(3)$ & $f''(3)$ for the following data :

x :	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	-14	- 10.032	- 5.296	- 0.256	6.672	14

2. Compute $f'(0)$ & $f''(4)$ from the data.

x :	0	1	2	3	4
$f(x)$:	1	2.718	7.381	20.086	54.598

3. Find the maximum and minimum value of y tabulated below.

x :	-2	-1	0	1	2	3	4
$f(x)$:	2	-0.25	0	-0.25	2	15.75	56

4. Consider the following table of data :

x :	0.2	0.4	0.6	0.8	1.0
$f(x)$:	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find $f'(0.25)$ using Newton's Forward difference approximation, $f'(0.6)$ using Stirling's approximation and $f'(0.95)$ using Newton's Backward difference approximation.

5. Obtain the value of $f'(0.04)$ using Bessel's formula given in the table below :

6. Using Trapezoidal rule, evaluate $\int_{-1}^{+1} \frac{dx}{1+x^2}$ taking 8 intervals.

7. Evaluate $\int_{0}^{+1} \frac{dx}{1+x^2}$ with $h = \frac{1}{6}$ by Trapezoidal rule.

8. Evaluate $\int_{0}^{5} \frac{dx}{4x+5}$ by Simpson's one-third and hence find the value of $\log_e 5$ $(n = 5)$.

9. Evaluate $\int_{0}^{6} \frac{dx}{1+x^2}$ by (1). Trapezoidal rule (2). Simpson's rule. Also check up the results by actual integration.

10. By dividing the range into ten equal parts, evaluate $\int_{0}^{\pi} \sin x \, dx$ by Trapezoidal rule and Simpson's rule. Verify the answer by actual integration.

11. Apply Gauss two – point formula to evaluate $\int_{-1}^{+1} \frac{1}{1+x^2} \, dx$.

12. Use Three-Point Gaussian quadrature formula, evaluate

    (1). $\int_{-1}^{+1} \frac{1}{1+x^2} \, dx$ (2). $\int_{0}^{+1} \frac{1}{1+t^2} \, dt$.

13. Evaluate $\int_{0}^{2} \frac{x^2+2x+1}{1+(x+1)^2} \, dx$ by Gaussian three point formula.

14. Evaluate $\int_{0.2}^{1.5} e^{-x^2} \, dx$ using the three point Gaussian quadrature formula.

15. Evaluate $\int_{-1}^{+1} \frac{x^2}{1+x^4} \, dx$ by using the three point Gaussian quadrature formula.

16. Evaluate $\int_{0}^{2} \int_{0}^{2} f(x, y) dx \, dy$ by Trapezoidal rule for the following data :

$y \backslash x$	0	0.2	1.0	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

17. Use Simpson's $\frac{1}{3}$ rule evaluate $\int_{0}^{1} \int_{0}^{1} \frac{1}{1+x+y} \, dx \, dy$ taking $h = k = 0.5$.

18. Evaluate $\int_{0}^{1} \int_{0}^{2} \frac{2\,x\,y}{(1+x^2)\,(1+y^2)} dx \, dy$ by Trapezoidal rule with $h = k = 0.25$.

19. Evaluate $\int_{1}^{2} \int_{1}^{2} \frac{1}{x^2+y^2} \, dx \, dy$ numerically with $h = 0.2$ along x - direction and $k = 0.25$ along y - direction.

20. The function $f(x, y)$ is defined by the following table. Compute $\int_{1}^{3} \int_{0}^{2} f(x, y) \, dx \, dy$, using Simpson's rule in both direction.