UNIT – II SEQUENCES AND SERIES

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Sequences: Definition and examples – Series: Types and Convergence – Series of positive terms – Tests of convergence: Comparison test, Integral test and D'Alembert's ratio test – Alternating series – Leibnitz's test – Series of positive and negative terms – Absolute and conditional convergence.

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Sequence:

An ordered set of real numbers $u_1, u_2, u_3, \ldots u_n, \ldots$ is called a sequence and it is denoted by $\{u_n\}$. If the number of terms is unlimited then the sequence is an infinite sequence. The general term of a sequence is u_n.

Example:

1) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\ldots \frac{1}{n^2}, \ldots\ldots$ is a sequence.
2) $3, 5, 7, 9, \ldots\ldots (2n+1), \ldots\ldots$ is a sequence.

Convergent sequence:

A sequence $\{u_n\}$ is said to be convergent if $\lim_{n \to \infty} u_n = finite$.

Divergent sequence:

A sequence $\{u_n\}$ is said to be divergent if $\lim_{n \to \infty} u_n = +\infty \ or \ -\infty$.

Bounded sequence:

A sequence $\{u_n\}$ is bounded if there exists a real number such that $|u_n| \leq M$, for every n in N .

Unbounded sequence:

A sequence $\{u_n\}$ is unbounded if there exists no real number M such that $|u_n| \leq M$, for every n in N.

Oscillatory sequence:

If a sequence $\{u_n\}$ neither converges to a finite no. nor diverges to $-\infty$ or $+\infty$, it is called an oscillatory sequence.

Note:

1) A bounded sequence which does not converge is said to be oscillate finitely.
2) An unbounded sequence which does not diverge is said to be oscillate infinitely.

Monotonic sequences

Let $\{a_n\}$ be the given sequence.

S.no.	Types	Condition
1	Monotonically increasing	$a_1 \le a_2 \le a_3 \le ..... \le a_n \le a_{n+1} \le .....$
2	Monotonically decreasing	$a_1 \ge a_2 \ge a_3 \ge ..... \ge a_n \ge a_{n+1} \ge .....$
3	Monotonic	Either monotonically increasing or decreasing
4	Strictly Monotonically increasing	$a_1 < a_2 < a_3 < ..... < a_n < a_{n+1} < .....$
5	Strictly Monotonically decreasing	$a_1 > a_2 > a_3 > ..... > a_n > a_{n+1} > .....$
6	Strictly Monotonic	Either Strictly Monotonically increasing or decreasing

Problems:

1) Determine the general term and prove that the sequences are convergent.

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$..........

ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$..........

iii) $1,-1,1,-1$..........

iv) $\frac{2^1}{1!}, \frac{2^2}{2!}, \frac{2^3}{3!}, \frac{2^4}{4!}$..........

Solution:

i) Given $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.......... Here $u_n = \frac{1}{2^n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$, finite.

The sequence in (i) converges.

ii) Given $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.......... Here $u_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{(1+0)} = 1, \text{ finite.}$$

The given sequence in (ii) converges.

iii) Given $1, -1, 1, -1 \ldots\ldots\ldots.$ Here $u_n = (-1)^{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-1)^{n-1} = 0, \text{ finite.}$$

The sequence in (iii) converges.

iv) Given $\frac{2^1}{1!}, \frac{2^2}{2!}, \frac{2^3}{3!}, \frac{2^4}{4!} \ldots\ldots\ldots.$ Here $u_n = \frac{2^n}{n!}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n}{n!} = 0, \text{ finite.}$$

The sequence in (iv) converges..

2) Give an example of monotonically increasing and decreasing sequences which are convergent and divergent.

Solution:

i) Let $\{u_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\ldots\ldots.\right\}$

Since $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} \ldots\ldots\ldots.$, the sequence is monotonically increasing.

Here $u_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{(1+0)} = 1, \text{ finite.}$$

The given sequence converges.

ii) Let $\{u_n\} = \{1, 2, 3, 4 \ldots\ldots\ldots.\}$

Since $1 < 2 < 3 < 4 < \ldots\ldots\ldots.$, the sequence is monotonically increasing.

Here $u_n = n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n = \infty, \text{ infinite.}$$

The given sequence diverges.

iii) Let $\{u_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\ldots\ldots.\right\}$

Since $1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} \ldots\ldots\ldots.$ the sequence is monotonically decreasing.

Here $u_n = \frac{1}{n}$ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$, finite.

The sequence converges.

iv) Let $\{u_n\} = \{-1, -2, -3, -4 \ldots\ldots\ldots\}$

Since $-1 > -2 > -3 > -4 > ....$, the sequence is monotonically decreasing.
Here $u_n = -n$.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} -n = -\infty$, infinite.

The given sequence diverges.

3) Discuss the convergence of the sequence {u_n} where (i) $u_n = \frac{n+1}{n}$ (ii) $u_n = \frac{n}{n^2+1}$ (iii) $u_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... + \frac{1}{3^n}$.

Solution:

(i) $u_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 + \frac{1}{\infty} = 1 + 0 = 1, \text{ finite}$$

$\Rightarrow \{u_n\}$ is a convergent sequence.

(ii) $u_n = \frac{n}{n^2+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{n\left(1+\frac{1}{n^2}\right)} = \frac{1}{\infty} = 0, \text{ finite}$$

$\Rightarrow \{u_n\}$ is a convergent sequence.

(iii) $u_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... + \frac{1}{3^n} = \frac{\left(1-\left(\frac{1}{3}\right)^{n+1}\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right)$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right) = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{\infty}\right) = \frac{3}{2}(1-0) = \frac{3}{2}, \text{ finite.}$$

$\Rightarrow \{u_n\}$ is a convergent sequence.

Remember:

1) $\lim_{n\to\infty} x^n = 0$ if $x < 1$

2) $\lim_{n\to\infty} x^n = \infty$ if $x > 1$

3) $\lim_{n\to\infty} nx^n = 0$ if $x < 1$

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x

5) $\lim_{n\to\infty}\frac{\log n}{n}=0$

6) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$; $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$ and $\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n}=e$

7) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=e^{x}$

8) $\lim_{n\to\infty} n^{\frac{1}{n}}=1$

9) $\lim_{n\to\infty}(n!)^{\frac{1}{n}}=\infty$

10) $\lim_{n\to\infty}\left(\frac{n!}{n}\right)^{\frac{1}{n}}=\frac{1}{e}$

11) $\lim_{n\to 0}\left(\frac{a^{n}-1}{n}\right)=\log a$

12) $\lim_{n\to\infty}\left(\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}\right)=\log a$

13) $\lim_{n\to\infty} n^{h}=\infty$

14) $\lim_{n\to\infty}\frac{1}{n^{h}}=0$

15) $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\tan x}{x}=1$

16) $\log\infty=\infty$ and $\log 1=0$

17) $(1+x)^{n}=1+nx+\frac{n(n-1)}{2!}x^{2}+\frac{n(n-1)(n-2)}{3!}x^{3}+.........+x^{n}$

Series:

A series is the sum of terms of a sequence. Let $\{u_n\}$ be a sequence of real nos. Then the expression $u_1+u_2+...+u_n+......$ is called the series associated with the sequence.

If the no. of terms of a series is limited, the series is called finite. When the no. of terms of a series are unlimited, it is called an infinite series.

The infinite series $u_1+u_2+...+u_n+......$ is usually denoted by $\sum_{n=1}^{\infty}u_n$ or $\sum u_n$.

Series of positive terms:

If all the terms of the series $\sum u_n = u_1 + u_2 + \ldots + u_n + \ldots\ldots$ are positive i.e., if $u_n > 0$, $\forall n$, then the series $\sum u_n$ is called a series of positive terms.

Properties of infinite series:

1. the nature of the infinite series does not change (unaffected) (i) by multiplication of all terms by a constant k. (ii) by addition or deletion of a finite no. of terms.
2. If a series in which all terms are +ve is convergent the series remains convergent even when some or all of its terms are – ve.
3. If $\sum u_n$ and $\sum v_n$ are converges to S_1 and S_2 resp., then $\sum(u_n + v_n)$ and $\sum(u_n + v_n)$ also converges to $S_1 + S_2$ and $S_1 - S_2$ resp.

Partial sums:

Let $\sum u_n = u_1 + u_2 + \ldots + u_n + \ldots\ldots$ be an infinite series, where the terms may be +ve or – ve , then $S_n = u_1 + u_2 + \ldots + u_n$ is called the n[th] partial sum of $\sum u_n$. Sequence $\{S_n\}$ is called sequence of partial sums.

Note:

To every infinite series $\sum u_n$, there corresponds a sequence $\{S_n\}$ of its partial sums.

Convergence, divergence and oscillation of series:

Let $\sum u_n = u_1 + u_2 + \ldots + u_n + \ldots\ldots$ be an infinite series and $S_n = u_1 + u_2 + \ldots + u_n$ be its n^{th} partial sum.

$\sum u_n$ is convergent	$\lim_{n\to\infty} S_n = finite$
$\sum u_n$ is divergent	$\lim_{n\to\infty} S_n = \pm\infty$
$\sum u_n$ is oscillatory (finite or infinite)	$\lim_{n\to\infty} S_n \neq$ a unique finite or infinite

Problems:

1. Test the convergence of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \infty$

   Solution:

   Let $\sum u_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^{n-1}} + \ldots + \infty$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^{n-1}} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\left(\frac{1}{2}\right)^n\right) = 2\left(1-\left(\frac{1}{2}\right)^\infty\right) = 2(1-0) = 2$$

$\Rightarrow \sum u_n$ is convergent.

2. Examine the nature of series $1+2+3+\ldots+n+\ldots+\infty$.

Solution:

Let $\sum u_n = 1+2+3+\ldots+n+\ldots+\infty$

$$S_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty \Rightarrow \sum u_n \text{ is divergent.}$$

3. Discuss the nature of the series $2-2+2-2+\ldots.+\infty$

Solution:

$$\sum u_n = 2-2+2-2+\ldots.+\infty$$

$$S_n = 2-2+2-2+\ldots..+2 = \begin{cases} 0 & if \;\; n \;\; is \;\; even \\ 2 & if \;\; n \;\; is \;\; odd \end{cases}$$

$\lim_{n\to\infty} S_n = 0$ or $\lim_{n\to\infty} S_n = 2$

$\Rightarrow \sum u_n$ Oscillates finitely.

4. Discuss the convergence or otherwise of the series $1-2+4-8+16-\ldots.+\infty$.

Solution:

$$\sum u_n = 1-2+4-8+16+\ldots+\infty = 1-2+2^2-2^3+2^4+\ldots+(-2)^{n-1}+\ldots.+\infty.$$

$$S_n = 1-2+2^2-2^3+2^4+\ldots+(-2)^{n-1} = \frac{1\left(1-(-2)^n\right)}{1-(-2)} = \frac{1-(-2)^n}{3}$$

$$\Rightarrow S_n = \begin{cases} \frac{1+2^n}{3} & if \;\; n \;\; is \;\; odd \\ \frac{1-2^n}{3} & if \;\; n \;\; is \;\; even \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_n \frac{1+2^n}{3} = \frac{1+\infty}{3} = +\infty \text{ if n is odd and}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_n \frac{1-2^n}{3} = \frac{1-\infty}{3} = -\infty \text{ if n is even}$$

$\therefore \sum u_n$ oscillates infinitely.

5. Discuss the convergence or otherwise of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots\infty$

Solution:

Consider $u_n = \frac{1}{n(n+1)} = \frac{A}{n}+\frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)}$

$\Rightarrow A(n+1)+Bn = 1$

Put $n=-1$

$A(-1+1)+B(-1)=1 \Rightarrow \boxed{B=-1}$

Put $n=0$

$A(0+1)+B(0)=1 \Rightarrow \boxed{A=1}$

$$\therefore u_n = \frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$$

$$\Rightarrow u_1 = 1-\frac{1}{2},\ u_2 = \frac{1}{2}-\frac{1}{3},\ u_3 = \frac{1}{3}-\frac{1}{4}\ \ldots\ldots\ u_n = \frac{1}{n}-\frac{1}{n+1}.$$

Now, $S_n = u_1+u_2+u_3+\ldots+u_n = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}\ldots\ldots+\frac{1}{n}-\frac{1}{n+1} \Rightarrow S_n = 1-\frac{1}{n+1}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1-\frac{1}{\infty} = 1-0 = 1$$

$\sum u_n$ converges to 1.

6. Show that the series $1^2+2^2+3^2+\ldots+n^2+\ldots\infty$ diverges to $+\infty$.

Solution:

Let $\sum u_n = 1^2+2^2+3^2+\ldots+n^2+\ldots\infty$

$$S_n = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6} = +\infty$$

$\therefore$ The series $\sum u_n$ diverges to $+\infty$.

Necessary condition for convergence of a series:

If a series $\sum u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$ (i.e.) $\sum u_n$ is convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$.

Note:

(1) $\lim_{n\to\infty} u_n = 0 \Rightarrow \sum u_n$ may or may not be convergent.

(2) $\lim\limits_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ is divergent.

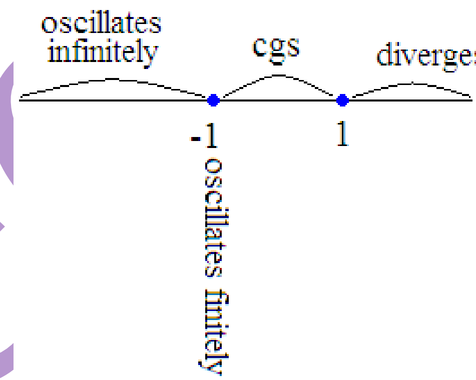
Series of positive terms:

If all the terms of the series $\sum u_n = u_1 + u_2 + \ldots + u_n + \ldots\infty$ are positive (i.e.) if $u_n > 0$, $\forall n$, then the series $\sum u_n$ is called the series of positive terms.

Note:

A positive term of the series either converges or diverges to $+\infty$. It cannot oscillate.

Problems:



1. Prove that the geometric series $1 + x + x^2 + \ldots + \infty$
(i) converges if $-1 < x < 1$ i.e. $|x| < 1$.
(ii) diverges if $x \geq 1$.
(iii) oscillates finitely if $x = -1$.
(iv) oscillates infinitely if $x < -1$.

Solution:

(i) Given $-1 < x < 1$ i.e. $|x| < 1$.

Consider $S_n = 1 + x + x^2 + \ldots + x^n = \dfrac{1 - x^{n+1}}{1 - x} = \dfrac{1}{1 - x} - \dfrac{x^{n+1}}{1 - x}$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x^{n+1}}{1-x}\right) = \frac{1}{1-x} - 0 = \frac{1}{1-x}. \quad \left(\because |x|<1,\ x^n \to 0 \ as\ n \to \infty\right).$$

$\Rightarrow \sum u_n$ is convergent.

(ii) Given $x \geq 1$.

For $x > 1$, Consider $S_n = 1 + x + x^2 + \ldots + x^n = \dfrac{x^{n+1} - 1}{x - 1} = \dfrac{x^{n+1}}{x - 1} - \dfrac{1}{x - 1}$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{x^{n+1}}{x-1} - \frac{1}{x-1}\right) = \infty. \quad \left(\because x>1,\ x^n \to \infty \ as\ n \to \infty\right).$$

For $x = 1$, $S_n = 1 + 1 + 1 + \ldots + 1\,(n\ times) = n$.

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} n = \infty$.

$\Rightarrow \sum u_n$ is divergent.

(iii) Given $x = -1$.

$$S_n = 1 + (-1) + (-1)^2 + \ldots + (-1)^n = 1 - 1 + 1 + \ldots + (-1)^n = \begin{cases} 1 & if\ \ n\ is\ odd \\ 0 & if\ \ n\ is\ even \end{cases}$$

$\therefore \lim\limits_{n\to\infty} S_n = 1$ or $\lim\limits_{n\to\infty} S_n = 0$

$\Rightarrow \sum u_n$ oscillates finitely.

(iv) Given $x < -1 \Rightarrow -x > 1$. Let $r = -x$ then $r > 1.\left(\because r > 1, r^n \to \infty \; as \; n \to \infty\right)$.

$$S_n = 1 + x + x^2 + \ldots + x^n = \frac{1\left(1 - x^n\right)}{1 - x} = \frac{1\left(1 - (-r)^n\right)}{1 - (-r)} = \frac{1 - (-r)^n}{1 + r}$$

$$\Rightarrow S_n = \begin{cases} \dfrac{1 + r^n}{1 + r} & if \;\; n \;\; is \;\; odd \\ \dfrac{1 - r^n}{1 + r} & if \;\; n \;\; is \;\; even \end{cases}$$

$\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} S_n \dfrac{1 + r^n}{1 + r} = \dfrac{1 + \infty}{1 + r} = +\infty$ if n is odd and

$\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} S_n \dfrac{1 - r^n}{1 + r} = \dfrac{1 - \infty}{1 + r} = -\infty$ if n is even

$\therefore \sum u_n$ oscillates infinitely.

2. Prove that Hyper harmonic series or p – series $\sum \dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \ldots + \dfrac{1}{n^p} + \ldots \infty$ converges if $p > 1$, diverges if $p \le 1$.

Solution:

Consider $S_n = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \ldots + \dfrac{1}{n^p}$.

Case (i) $p > 1$.

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{\left(2^{p-1}\right)^2}$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \ldots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \ldots + \frac{1}{8^p} = \frac{8}{8^p} = \frac{1}{8^{p-1}} = \frac{1}{\left(2^{p-1}\right)^3} \text{ and so on.}$$

Now $\sum \dfrac{1}{n^p} < \dfrac{1}{2^{p-1}} + \dfrac{1}{(2^{p-1})^2} + \dfrac{1}{(2^{p-1})^3} \ldots\ldots \infty$, which is G.P. with common ratio $\dfrac{1}{2^{p-1}}$.

Common ratio = $\dfrac{1}{2^{p-1}} < 1, \because p > 1$.

$\frac{1}{2^{p-1}}+\frac{1}{(2^{p-1})^2}+\frac{1}{(2^{p-1})^3}......\infty$ converges (by geometric series condition (i)).

$\Rightarrow \sum \frac{1}{n^p}$ converges.

Case (ii) $p \le 1$

(a) Consider $p=1$.

$$\sum \frac{1}{n^p}=\sum \frac{1}{n}=1+\frac{1}{2}+...+\frac{1}{n}+...\infty$$

$$1+\frac{1}{2}>\frac{1}{2}$$

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

$\frac{1}{8^p}+\frac{1}{9}+\frac{1}{10}+...+\frac{1}{16}<\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+...+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}$ and so on.

$$\sum \frac{1}{n^p}=\sum \frac{1}{n}>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....\infty=\frac{1}{2}(1+1+....\infty)=\frac{1}{2}\sum v_n .$$

$\sum v_n$ is geometric series with common ratio 1 which is a divergent series (by condition (ii) of geometric series).

$\therefore \sum \frac{1}{n^p}=\sum \frac{1}{n}$ is also a divergent series.

(b) Consider $p<1$.

When $p<1$, $n^p<n \Rightarrow \frac{1}{n^p}>\frac{1}{n}$ for all n.

$\sum \frac{1}{n}$ is divergent by case (ii) (a). $\Rightarrow \sum \frac{1}{n^p}$ is also divergent.

3. Test the convergence or divergence of the series $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+...\infty$.

Solution:

Given $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+...\infty=\left(\frac{1}{2}+\frac{1}{2^3}+...\infty\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+...\infty\right)=\sum u_n+\sum v_n$

$\sum u_n$ is a geometric series with common ration = $\frac{1}{2^2}=\frac{1}{4}<1$

$\Rightarrow \sum u_n$ is a convergent series.

$\sum v_n$ is a geometric series with common ration = $\frac{1}{3^2}=\frac{1}{9}<1$

$\Rightarrow \sum v_n$ is a convergent series.

$\Rightarrow \sum u_n + \sum v_n$ is also convergent.

4. Examine the convergence of the series $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\infty$.

Solution:

Let $\sum u_n = 1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\infty$

$$=1+\frac{1}{\left(2^2\right)^{2/3}}+\frac{1}{\left(3^2\right)^{2/3}}+\frac{1}{\left(4^2\right)^{2/3}}+\ldots\infty$$

$$=1+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\ldots\infty=\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}.$$

Here $p=\frac{4}{3}>1$. $\therefore$ by p – series $\sum u_n = 1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\infty$ is convergent.

Comparison test: (form - I i.e. limit form)

Let $\sum u_n$ and $\sum v_n$ be two positive term series.

(i) If $\lim_{n\to\infty}\frac{u_n}{v_n}=\ell$ (finite and non zero), then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

(ii) If $\lim_{n\to\infty}\frac{u_n}{v_n}=0$ and $\sum v_n$ converges, then $\sum u_n$ converges.

(iii) If $\lim_{n\to\infty}\frac{u_n}{v_n}=\infty$ and $\sum v_n$ diverges, then $\sum u_n$ diverges.

Note:

Select the series $\sum v_n$ as follows: $v_n=\frac{n^a}{n^b}$ where a = highest power term of n in u_n numerator and b = highest power term of n in u_n denominator.

For example, let $u_n=\frac{2n+3}{n(n-5)(n+3)}$. Here a = 1 (highest power term of n in u_n numerator) and b = 3 (highest power term of n in u_n denominator). Then $v_n=\frac{n^1}{n^3}=\frac{1}{n^2}$.

$\therefore \sum v_n=\sum\frac{1}{n^2}$.

Problems:

1. Test convergence of the series $\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\ldots+\infty$

Solution:

Let $\sum u_n=\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\ldots+\infty$. Then $u_n=\frac{2n-1}{n(n+1)(n+2)}$, $n=1,2,3,\ldots\infty$.

$v_n=\frac{n^a}{n^b}=\frac{n^1}{n^3}=\frac{1}{n^2}$. (Here a = 1 and b = 3).

$$\frac{u_n}{v_n}=\frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}}=\frac{2n-1}{n(n+1)(n+2)}\times\frac{n^2}{1}$$

$$\Rightarrow\frac{u_n}{v_n}=\frac{n\left(2-\frac{1}{n}\right)}{n\cdot n\left(1+\frac{1}{n}\right)\cdot n\left(1+\frac{1}{n}\right)}\times n^2=\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}=\frac{\left(2-\frac{1}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)\left(1+\frac{1}{\infty}\right)}=\frac{2-0}{(1+0)(1+0)}=2,\text{ a finite value.}$$

$\Rightarrow\sum u_n$ and $\sum v_n$ both converge or diverge together. By p – series $\sum v_n=\frac{1}{n^2}$ converges (since $p=2>1$).

$\Rightarrow\sum u_n$ also converges.

2. Test the convergence of the following series

(i) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots\infty$. (ii) $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2(n+1)}}+\ldots+\infty$.

Solution:

(i) Let $\sum u_n=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots\infty$. Then $u_n=\frac{1}{\sqrt{n}+\sqrt{n+1}}$.

$v_n=\frac{n^a}{n^b}=\frac{1}{n^{1/2}}=\frac{1}{\sqrt{n}}$ (here a = 0 and b = ½).

$$\frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \sqrt{n} = \frac{1}{\sqrt{n}\left(1+\sqrt{\frac{n+1}{n}}\right)} \times \sqrt{n} = \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)} = \frac{1}{\left(1+\sqrt{1+\frac{1}{\infty}}\right)} = \frac{1}{\left(1+\sqrt{1+0}\right)} = \frac{1}{2}$$, a finite value.

$\Rightarrow \sum u_n$ and $\sum v_n$ both converge or diverge together. By p – series $\sum v_n = \frac{1}{\sqrt{n}}$ diverges (since $p = \frac{1}{2} < 1$).

$\Rightarrow \sum u_n$ also diverges.

(ii) Let $\sum u_n = \sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots + \infty$. Then $u_n = \sqrt{\frac{n}{2(n+1)}}$.

$v_n = \frac{n^a}{n^b} = \frac{n^{1/2}}{n^{1/2}} = 1$ (here a = ½ and b = ½).

$$\frac{u_n}{v_n} = \frac{\sqrt{\frac{n}{2(n+1)}}}{1} = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{n}{2n\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{\infty}\right)}} = \sqrt{\frac{1}{2}}$$, a finite value.

$\Rightarrow \sum u_n$ and $\sum v_n$ both converge or diverge together. By geometric series $\sum v_n = \sum 1$ diverges (condition (ii) of geometric series).

$\Rightarrow \sum u_n$ is also divergent.

3. Test the convergence of the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots\infty$.

Solution:

Leaving aside the first term (since addition or deletion of a finite no. of terms does not alter the nature of the series), we have $\sum u_n = \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots\infty$.

Then $u_n = \frac{n^n}{(n+1)^{n+1}}$. Now, $v_n = \frac{n^a}{n^b} = \frac{n^n}{n^{n+1}} = \frac{1}{n}$.

$$\frac{u_n}{v_n} = \frac{n^n}{(n+1)^{n+1}} \times n = \frac{\cancel{n^{n+1}}}{\cancel{n^{n+1}}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e}, \text{ a finite value.} \quad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e\right)$$

$\Rightarrow \sum u_n$ and $\sum v_n$ both converge or diverge together. By p – series $\sum v_n = \sum \frac{1}{n}$ is divergent (since p = 1).

$\Rightarrow \sum u_n$ is also divergent.

4. Using Comparison test, prove that the series $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \ldots\ldots\ldots + \infty$ is divergent.

Solution:

Given $\sum u_n = \frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \ldots\ldots\ldots + \infty \Rightarrow u_n = \frac{n}{(2n-1)(2n+1)}$

$$v_n = \frac{n^a}{n^b} = \frac{n^1}{n^2} = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{n}{(2n-1)(2n+1)} \times n = \frac{\cancel{n^2}}{\cancel{n^2}\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{4}, \text{ a finite value.}$$

$\Rightarrow \sum u_n$ and $\sum v_n$ both converge or diverge together. By p – series $\sum v_n = \sum \frac{1}{n}$ is divergent (since p = 1).

$\Rightarrow \sum u_n$ is also divergent.

5. Test the convergence of the series $\frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}+\ldots\ldots+\infty$

Solution:

$$\sum u_n=\frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}+\ldots\ldots+\infty \Rightarrow u_n=\frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

$$v_n=\frac{n^a}{n^b}=\frac{n^2}{n^3}=\frac{1}{n}$$

$$\frac{u_n}{v_n}=\frac{n^2}{(3n+1)(3n+4)(3n+7)}\times n=\frac{\cancel{n^3}}{\cancel{n^3}\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}=\frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}=\frac{1}{27}$$, a finite value.

$\Rightarrow \sum u_n$ and $\sum v_n$ both converge or diverge together. By p – series $\sum v_n=\sum\frac{1}{n}$ is divergent (since p = 1).

$\Rightarrow \sum u_n$ is also divergent.

Comparison test (Form II)

◊ If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \le v_n$ for all n = 1,2,3,...∞ and if $\sum v_n$ is convergent then $\sum u_n$ is also convergent.

◊ If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \ge v_n$ for all n = 1,2,3,...∞ and if $\sum v_n$ is divergent then $\sum u_n$ is also divergent.

Problems:

1. Test the convergence of the series $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\cdots\infty$

Solution:

Here $u_n=\frac{\sqrt{n}}{2n+3}$. $v_n=\frac{n^a}{n^b}=\frac{n^{1/2}}{n^1}=\frac{1}{n^{1/2}}=\frac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n}=\frac{\sqrt{n}}{2n+3}\sqrt{n}=\frac{n}{2n+3}\ge 1$$

$\Rightarrow u_n \ge v_n,\ \forall n$ and $\sum v_n=\frac{1}{\sqrt{n}}$ is a divergent series (p – series, p = $\frac{1}{2}$<1).

$\Rightarrow \sum u_n$ is also divergent.

2. Test the convergence of the series $\sum\left(\frac{1}{n}-\log\frac{n+1}{n}\right)$.

Solution:

Given $\sum u_n = \sum\left(\frac{1}{n}-\log\frac{n+1}{n}\right) = \sum\left(\frac{1}{n}-\log\left(1+\frac{1}{n}\right)\right)$.

$$\frac{1}{n}-\log\left(1+\frac{1}{n}\right) = \frac{1}{n}-\left(\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\frac{1}{3}\cdot\frac{1}{n^3}-\frac{1}{4}\cdot\frac{1}{n^4}+\cdots\infty\right)$$

$$= \cancel{\frac{1}{n}}-\cancel{\frac{1}{n}}+\frac{1}{2}\cdot\frac{1}{n^2}-\frac{1}{3}\cdot\frac{1}{n^3}+\frac{1}{4}\cdot\frac{1}{n^4}-\cdots\infty = \frac{1}{n^2}\left(\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{n}+\frac{1}{4}\cdot\frac{1}{n^2}-\cdots\infty\right)$$

Choose $v_n = \frac{1}{n^2}$.

$$\frac{u_n}{v_n} = \frac{1}{\cancel{n^2}}\left(\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{n}+\frac{1}{4}\cdot\frac{1}{n^2}-\cdots\infty\right)\cdot\cancel{n^2} = \frac{1}{2}-\frac{1}{3}\cdot\frac{1}{n}+\frac{1}{4}\cdot\frac{1}{n^2}-\cdots\infty \le 1$$

$\Rightarrow u_n \le v_n$, $\forall n$ and $\sum v_n = \frac{1}{n^2}$ is a convergent series (p – series, p =2 > 1).

$\Rightarrow \sum u_n$ is also convergent.

D'Alembert's Ratio Test:

If $\sum u_n$ is a positive term series, and $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \ell$, then

(i) $\sum u_n$ is convergent if $\ell > 1$ (ii) $\sum u_n$ is divergent if $\ell < 1$.

Note:

If $\ell = 1$, the test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it mat diverge.

Problems:

1. Discuss the convergence of the following series:

(i) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots\cdots(p>0)$

(ii) $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2^{n-1}+1}+\cdots$

(iii) $\frac{1^2\cdot 2^2}{1!}+\frac{2^2\cdot 3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\frac{4^2\cdot 5^2}{4!}+\cdots\cdots$

Solution:

(i) Here $u_n = \frac{n^p}{n!}$. $\therefore u_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!}\frac{(n+1)!}{(n+1)^p} = \frac{n^p}{n!}\frac{(n+1)n!}{(n+1)^p} = \frac{n^p}{(n+1)^{p-1}} = \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}}.$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n}{\left(1+\frac{1}{n}\right)^{p-1}} = \infty > 1.$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ is convergent.

(ii) Here $u_n = \frac{1}{2^{n-1}+1}$. $\therefore u_{n+1} = \frac{1}{2^n+1}$.

$$\frac{u_n}{u_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+\frac{1}{2^n}\right)}{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)} = 2\cdot\frac{\left(1+\frac{1}{2^n}\right)}{\left(1+\frac{1}{2^{n-1}}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} 2\cdot\frac{\left(1+\frac{1}{2^n}\right)}{\left(1+\frac{1}{2^{n-1}}\right)} = 2 > 1.$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ is convergent.

(iii) Here $u_n = \frac{n^2(n+1)^2}{n!}$. $\therefore u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2\,\cancel{(n+1)^2}}{n!}\frac{(n+1)!}{\cancel{(n+1)^2}\,(n+2)^2} = \frac{n^2\cdot(n+1)\,\cancel{n!}}{\cancel{n!}\,(n+2)^2} = \frac{\cancel{n^3}\left(1+\frac{1}{n}\right)}{\cancel{n^2}\left(1+\frac{2}{n}\right)^2} = n\cdot\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)^2}.$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\cdot\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)^2} = \infty > 1.$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ is convergent.

2. Discuss the convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$.

Solution:

Here $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$. $\therefore u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1}$.

$$\frac{u_n}{u_{n+1}} = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \cdot \frac{\sqrt{(n+1)^2+1}}{\sqrt{n+1}} \frac{1}{x^{n+1}} = \frac{1}{x}\sqrt{\frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1}} = \frac{1}{x}\sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}}.$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} = \frac{1}{x}.$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ is convergent if $\frac{1}{x} > 1$ i.e. $x < 1$ and diverges if $\frac{1}{x} < 1$ i.e. $x > 1$. When $x = 1$, the Ratio test fails.

When $x = 1$, $u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}}$.

Take $v_n = \frac{1}{\sqrt{n}}$. $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$, which is finite and $\neq 0$. $\therefore$ By Comparison Test, $\sum u_n$ and $\sum v_n$ both converge or diverge together.

Since, $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2} (<1)$, $\sum v_n$ is a divergent series (p – series). $\Rightarrow \sum u_n$ diverges.

Hence the given series $\sum u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

3. Examine the convergence or divergence of the following series

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots.. + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots. \quad (x > 0)$$

Solution:

Leaving the first term, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1} x^n$ $\therefore u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1} x^{n+1}$.

$$\frac{u_n}{u_{n+1}}=\frac{2^{n+1}-2}{2^{n+1}+1}x^n\cdot\frac{2^{n+2}+1}{2^{n+2}-2}\frac{1}{x^{n+1}}=\frac{1}{x}\frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{2^{n+2}\left(1+\frac{1}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)}$$

$$\frac{u_n}{u_{n+1}}=\frac{1}{x}\frac{\left(1-\frac{2}{2^{n+1}}\right)}{\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{\left(1+\frac{1}{2^{n+2}}\right)}{\left(1-\frac{2}{2^{n+2}}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{x}\frac{\left(1-\frac{2}{2^{n+1}}\right)}{\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{\left(1+\frac{1}{2^{n+2}}\right)}{\left(1-\frac{2}{2^{n+2}}\right)}=\frac{1}{x}$$

By Ratio test, $\sum u_n$ converges if $\frac{1}{x}>1$ *i.e.*, $x<1$ and diverges if $\frac{1}{x}<1$ *i.e.*, $x>1$.

If $x=1$, then $u_n=\frac{2^{n+1}-2}{2^{n+1}+1}=\frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}=\frac{\left(1-\frac{2}{2^{n+1}}\right)}{\left(1+\frac{1}{2^{n+1}}\right)}$.

$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\left(1-\frac{2}{2^{n+1}}\right)}{\left(1+\frac{1}{2^{n+1}}\right)}=1\neq 0\Rightarrow\sum u_n$ does not converge. Being a series of +ve terms, it must diverge.

Hence, $\sum u_n$ converges if $x<1$ and diverges if $x\geq 1$.

4. Test the convergence of the series $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+.........$ by D'Alembert's ratio test.

Solution:

Leaving the first term, $u_n=\frac{3\cdot 6\cdots(3n)}{7\cdot 10\cdots(3n+4)}x^n \quad \therefore u_{n+1}=\frac{3\cdot 6\cdots(3n+3)}{7\cdot 10\cdots(3n+7)}x^{n+1}$.

$$\frac{u_n}{u_{n+1}}=\frac{3\cdot 6\cdots(3n)}{7\cdot 10\cdots(3n+4)}x^n\cdot\frac{7\cdot 10\cdots(3n+7)}{3\cdot 6\cdots(3n+3)}\frac{1}{x^{n+1}}=\frac{1}{x}\frac{(3n+7)}{(3n+3)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{x}\frac{3n\left(1+\frac{7}{3n}\right)}{3n\left(1+\frac{1}{n}\right)}=\frac{1}{x}.$$ By Ratio test, $\sum u_n$ converges if $\frac{1}{x}>1$ *i.e.*, $x<1$ and diverges if $\frac{1}{x}<1$ *i.e.*, $x>1$.

If $x=1$, Ratio test fails.

(by Raabe's test $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(3n+7)}{(3n+3)}-1\right)=\lim_{n\to\infty}\frac{4n}{3n+3}=\frac{4}{3}>1$ $\sum u_n$ converges if $x=1$ (this test is not in syllabus))

Hence, $\sum u_n$ converges if $x<1$ and diverges if $x>1$.

Integral test:

If for $x\geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n)=u_n$ for all positive integral values of n, then the series $\sum u_n$ and the integral $\int_1^\infty f(x)dx$ converge or diverge together.

Note:

1. If $\int_1^\infty f(x)dx$ = finite then $\int_1^\infty f(x)dx$ converges.
2. If $\int_1^\infty f(x)dx$ = $+\infty$ then $\int_1^\infty f(x)dx$ diverges.

Problems:

1. Test for convergence the series: $\sum\frac{1}{n^2+1}$

Solution:

Here $u_n=\frac{1}{n^2+1}=f(n)\Rightarrow f(x)=\frac{1}{x^2+1}$.

For $x\geq 1$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

Now, $\int_1^\infty f(x)dx=\int_1^\infty\frac{1}{x^2+1}dx=\left(\tan^{-1}x\right)_1^\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}=fnite$.

$\Rightarrow \int_1^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

2. Show that the series $\sum_1^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Solution:

Here $u_n = \frac{1}{n^p} = f(n) \Rightarrow f(x) = \frac{1}{x^p}$.

For $x \ge 1$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

Case 1: $\boxed{p > 1}$ $\Rightarrow p - 1 > +ve$.

$$\therefore \int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx = \left(\frac{x^{-p+1}}{-p+1}\right)_1^\infty$$

$$= \left(\frac{x^{-(p-1)}}{-p+1}\right)_1^\infty = \frac{1}{1-p}\left(\frac{1}{x^{(p-1)}}\right)_1^\infty = \frac{1}{1-p}(0-1) = \frac{1}{p-1} = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

Case 2: $\boxed{0 < p < 1}$ $\Rightarrow 1 - p > +ve$.

$$\therefore \int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx = \left(\frac{x^{-p+1}}{-p+1}\right)_1^\infty = \left(\frac{x^{(1-p)}}{1-p}\right)_1^\infty = \frac{1}{1-p}(\infty - 1) = \infty$$

$\Rightarrow \int_1^\infty f(x)dx$ diverges and hence by integral test, $\sum u_n$ also diverges.

Case 3: $\boxed{p = 1}$

$$\therefore \int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x}dx = (\log x)_1^\infty = (\infty - 0) = \infty$$

$\Rightarrow \int_1^\infty f(x)dx$ diverges and hence by integral test, $\sum u_n$ also diverges.

Hence $\sum u_n$ converges if $p > 1$ and diverges if $0 < p \le 1$.

3. Test for convergence the series: $\sum ne^{-n^2}$.

Solution:

Here $u_n = ne^{-n^2} = f(n) \Rightarrow f(x) = xe^{-x^2}$.

For $x \geq 1$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

Now, $\int_1^\infty f(x)dx = \int_1^\infty xe^{-x^2}dx$.

Put $x^2 = t \Rightarrow 2xdx = dt$. When $x=1$, $t = x^2 = 1$. When $x = \infty$, $t = x^2 = \infty$.

$$\int_1^\infty f(x)dx = \int_1^\infty xe^{-x^2}dx = \frac{1}{2}\int_1^\infty e^{-t}dt = \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_1^\infty = \frac{-1}{2}\left(e^{-\infty} - e^{-1}\right) = \frac{-1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e} = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

4. Using the integral test, discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$.

Solution:

Here $u_n = \frac{1}{n\sqrt{n^2-1}} = f(n) \Rightarrow f(x) = \frac{1}{x\sqrt{x^2-1}}$.

For $x \geq 2$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

$$\text{Now, } \int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x\sqrt{x^2-1}}dx = \left(\sec^{-1} x\right)_2^\infty = \left(\cos^{-1}\left(\frac{1}{x}\right)\right)_2^\infty$$

$$= \cos^{-1}\left(\frac{1}{\infty}\right) - \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} = \text{finite}$$

$\Rightarrow \int_2^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

5. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

Solution:

Here $u_n = \frac{1}{(n+1)^2} = f(n) \Rightarrow f(x) = \frac{1}{(x+1)^2}$.

For $x \geq 1$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

Now, $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{(x+1)^2}dx = \left(\frac{(x+1)^{-1}}{-1}\right)_1^\infty = -\left(\frac{1}{x+1}\right)_1^\infty = -\left(\frac{1}{\infty}-\frac{1}{2}\right) = \frac{1}{2}$ = finite

$\Rightarrow \int_1^\infty f(x)dx$ converges and hence by integral test, $\sum u_n$ also converges.

6. Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{2n^3}{n^4+3}$

Solution:

Here $u_n = \frac{2n^3}{n^4+3} = f(n) \Rightarrow f(x) = \frac{2x^3}{x^4+3}$.

For $x \geq 1$, $f(x)$ is a positive and monotonic decreasing. $\therefore$ Integral test is applicable.

Now, $\int_1^\infty f(x)dx = \int_1^\infty \frac{2x^3}{x^4+3}dx$

Put $x^4+3=t \Rightarrow 4x^3dx = dt \Rightarrow x^3dx = \frac{dt}{4}$. When $x=1$, $t=x^4+3=1+3=4$. When $x=\infty$, $t=x^4+3=\infty$.

$\int_1^\infty f(x)dx = \int_1^\infty \frac{2x^3}{x^4+3}dx = 2\int_4^\infty \frac{1}{4}\frac{dt}{t} = \frac{1}{2}(\log t)_4^\infty = \frac{1}{2}(\log\infty - \log 4) = \infty$ = infinite

$\Rightarrow \int_1^\infty f(x)dx$ diverges and hence by integral test, $\sum u_n$ also diverges.

Alternating series:

A series in which the terms are alternate positive and negative is called an alternating series. Thus, the series $\sum(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n + \cdots\cdots$ where $u_n > 0$, for every n is an alternating series.

Leibnitz's Test on alternating series:

The alternating series $\sum(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}u_n + \cdots\cdots$ ($u_n > 0$, $\forall n$) converges if (i) $u_n > u_{n+1}$, $\forall n$ (ii) $\lim_{n\to\infty} u_n = 0$.

Note:

The alternating series will not convergent if any one of the two conditions is not satisfied. If $\lim_{n\to\infty} u_n \neq 0$, then the series is oscillatory.

Problems:

1. Examine the convergence of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

   Solution:
   It is an alternating series.

   (i) Here $u_n=\frac{1}{n},\ u_{n+1}=\frac{1}{n+1}$. $\because \frac{1}{n}>\frac{1}{n+1}\quad \forall n,\ \therefore u_n>u_{n+1},\ \forall n.$

   (ii) $\lim_{n\to\infty} u_n=\lim_{n\to\infty}\frac{1}{n}=0.$

   Both conditions of Leibnitz's Test are satisfied. Hence the given series is convergent.

2. Examine the convergence of the following series :

   (a) $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots$

   (b) $\frac{1}{2^3}-\frac{1}{3^3}(1+2)+\frac{1}{4^3}(1+2+3)-\frac{1}{5^3}(1+2+3+4)+\cdots\cdots$

   Solution:

   (a) It is an alternating series.

   (i) $u_n=\frac{n+1}{n},\ \ u_{n+1}=\frac{n+2}{n+1}.$

   $$u_n-u_{n+1}=\frac{n+1}{n}-\frac{n+2}{n+1}=\frac{(n+1)^2-n(n+2)}{n(n+1)}=\frac{1}{n(n+1)}>0,\ \forall n.$$

   $\Rightarrow u_n>u_{n+1},\ \forall n.$

   (ii) $$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1\neq 0.$$

   Since the second condition of Leibnitz's test is not satisfied, the series is not convergent.

   (b) It is an alternating series.

   (i) Here $u_n=\frac{1}{(n+1)^3}(1+2+3+\cdots+n)=\frac{1}{(n+1)^3}\frac{n(n+1)}{2}=\frac{1}{2}\frac{n}{(n+1)^2}$, and

   $u_{n+1}=\frac{1}{2}\frac{n+1}{(n+2)^2}.$

   $$u_n-u_{n+1}=\frac{1}{2}\frac{n}{(n+1)^2}-\frac{1}{2}\frac{n+1}{(n+2)^2}=\frac{1}{2}\frac{n^2+n-1}{(n+1)^2(n+2)^2}>0,\ \forall n.$$

   $\Rightarrow u_n>u_{n+1},\ \forall n.$

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{2}\frac{n}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{1}{2}\frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} = 0$. Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

3. Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$.

Solution:

The given series is $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1} = \frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \cdots \infty$.

It is an alternating series.

(i) Here $u_n = \frac{n}{2n-1}$ and $u_{n+1} = \frac{n+1}{2(n+1)-1} = \frac{n+1}{2n+1}$.

$u_n - u_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1} = \frac{1}{4n^2-1} > 0,\ \forall n$.

$\Rightarrow u_n > u_{n+1},\ \forall n$.

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n}{2n-1} = \lim\limits_{n\to\infty} \frac{\not{n}}{\not{n}\left(2-\frac{1}{n}\right)} = \frac{1}{2} \neq 0$. Here the second condition of Leibnitz's Test is not satisfied. Hence the given series is not convergent.

Absolute convergence of a series:

If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an absolutely convergent series, i.e., The convergent series $\sum u_n$ is said to be absolutely convergent if $\sum |u_n|$ is a convergent series.

Conditionally convergent:

A series is said to be conditionally convergent if it is convergent but does not converge absolutely.

Note:

Absolutely convergent	Conditionally convergent
i) $\sum u_n$ is convergent.	i) $\sum u_n$ is convergent.

ii) $\sum \lvert u_n \rvert$ is convergent.	ii) $\sum \lvert u_n \rvert$ is divergent.

Problems:

1. Test whether the following series are absolutely convergent or conditionally convergent?

(a) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\infty$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$.

Solution:

(a) Given $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\infty$

(i) It is an alternating series. Here $u_n=\frac{1}{n^2}$ and $u_{n+1}=\frac{1}{(n+1)^2}$.

$$u_n-u_{n+1}=\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{n^2+2n+1-n^2}{n^2(n+1)^2}=\frac{2n+1}{n^2(n+1)^2}>0\ \forall n.$$

$\Rightarrow u_n>u_{n+1},\ \forall n$.

(ii) $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n^2}=0$.

Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

(iii) $\sum \lvert u_n \rvert=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\infty=\sum\frac{1}{n^2}$ which we know is a convergent series.

Thus the given series converges absolutely.

(b) Given $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty$.

It is an alternating series.

(i) $u_n=\frac{1}{2n-1}$ and $u_{n+1}=\frac{1}{2(n+1)-1}=\frac{1}{2n+1}$

$$u_n-u_{n+1}\frac{1}{2n-1}-\frac{1}{2n+1}=\frac{\cancel{2n}+1-\cancel{2n}+1}{4n^2-1}=\frac{2}{4n^2-1}>0\ \forall n.$$

$\Rightarrow u_n>u_{n+1},\ \forall n$.

(ii) $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{2n-1}=0$.

Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

(iii) $\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \infty$. Here $u_n = \frac{1}{2n-1}$. Take $v_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n}{n\left(2-\frac{1}{n}\right)} = \frac{1}{2} =$ finite. Hence by comparison test, $\sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n}$, is a divergent series. (by p –series since p = 1). $\therefore \sum u_n$ also diverges. Hence the given series converges, and the series of absolute terms diverges, therefore the given series converges conditionally.

Result:

Every absolutely convergent series is convergent.

Problems:

1. Prove that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots \infty$ converge absolutely.

Solution:

The given series is $\sum_{n=1}^{\infty} u_n = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots \infty = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n^3}$.

Here $|u_n| = \frac{|\sin nx|}{n^3}$. Choose $v_n = \frac{1}{n^3}$. Since $\frac{|\sin nx|}{n^3} \le \frac{1}{n^3} \Rightarrow |u_n| \le v_n, \quad \forall n$ by comparison test the series $\sum |u_n|$ converges.

$\Rightarrow$ the given series converges absolutely.

2. Examine the convergence of $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \cdots \infty$ (or) Find the interval of convergence of the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \cdots \infty$.

Solution:

The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{\sqrt{n}}$. Here $|u_n| = \frac{|x^n|}{\sqrt{n}} = \frac{|x|^n}{\sqrt{n}}$ and $|u_{n+1}| = \frac{|x|^{n+1}}{\sqrt{n+1}}$.

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x^n|}{\sqrt{n}} \times \frac{\sqrt{n+1}}{|x|^{n+1}} = \sqrt{\frac{n+1}{n}} \cdot \frac{1}{|x|} = \sqrt{1+\frac{1}{n}} \cdot \frac{1}{|x|}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{1}{|x|} = \frac{1}{|x|}.$$

By ratio test, $\sum |u_n|$ (i) Converges if $\frac{1}{|x|} > 1$ i.e. $|x| < 1$ i.e. if $-1 < x < 1$

(ii) Diverges if $\frac{1}{|x|} < 1$ i.e. $|x| > 1$ i.e. x > 1 or x < - 1.

(iii) Test fails when $|x| = 1$ i.e. x = 1 or x = - 1.

Case (i) When x = 1, the series becomes $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \infty$ which is an alternating series and is convergent.

Case (ii) When x = - 1, the series becomes

$-1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \infty = -\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \infty\right) = -\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent by p – series. Hence the given series for $-1 < x \le 1$.

3. Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$.

Solution:

Here $|u_n| = \left|\frac{\cos n\pi}{n^2+1}\right| = \frac{1}{n^2+1}$. Choose $v_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{1}{n^2+1} n^2 = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = 1, \text{ finite and non zero.}$$

$\therefore \sum |u_n|$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is a convergent series (p series with p = 2 > 1), we have $\sum |u_n|$ is convergent , by comparison test.

$\Rightarrow$ the given series converges absolutely.