UNIT – V

MULTIPLE INTEGRALS

Double Integrals:

Area of the region by Cartesian co-ordinates

$$Area = \iint_R dxdy$$

Area of the region by Polar co-ordinates

$$Area = \iint_R rdrd\theta$$

$Evaluate \int_0^5 \int_0^2 (x^2 + y^2)dxdy$.

Solution:

$$\int_0^5 \int_0^2 (x^2 + y^2)dxdy = \int_0^5 \left(\frac{x^3}{3} + xy^2\right)_0^2 dy = \int_0^5 \left(\frac{8}{3} + 2y^2\right) dy = \left[\frac{8}{3}y + \frac{2y^3}{3}\right]_0^5$$

$$= \frac{40}{3} + \frac{250}{3} = \frac{290}{3}$$

$Evaluate \int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} rd\theta dr$

Solution:

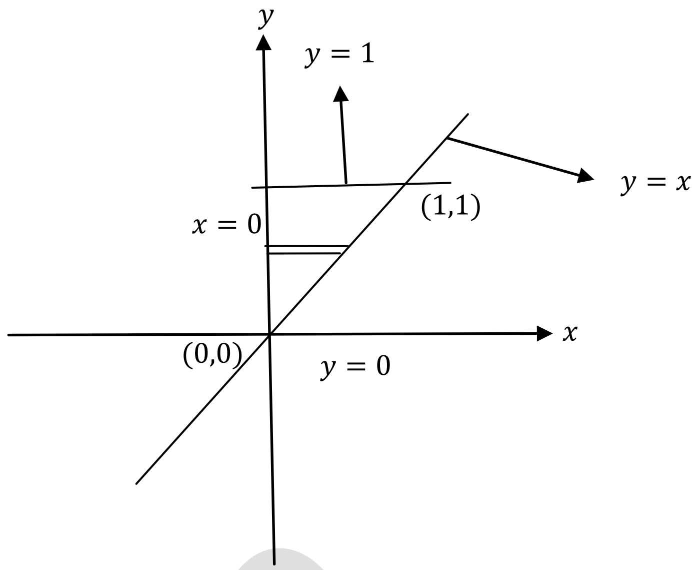
$$\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} rdrd\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{\sin^2\theta}{2}\right] d\theta = \frac{1}{2}\frac{1}{2}\frac{\pi}{2} = \frac{\pi}{8}$$

Find the area bounded by the lines $x = 0, y = 1$ and $y = x$, using double integration.

Solution:

$$Area = \int_0^1 \int_x^1 dydx = \int_0^1 [y]_x^1 dx = \int_0^1 [1 - x]dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$



Evaluate $\int_0^{\pi} \int_0^{a} r dr d\theta$.

Solution:

$$\int_0^{\pi} \int_0^{a} r dr d\theta = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^a d\theta = \frac{a^2}{2}\int_0^{\pi} d\theta = \frac{a^2}{2}[\theta]_0^{\pi} = \frac{\pi a^2}{2}$$

Evaluate $\int_0^{2} \int_0^{\pi} r \sin^2\theta \ dr d\theta$.

Solution:

$$\int_0^{2} \int_0^{\pi} r \sin^2\theta \, d\theta dr = \int_0^{2} \int_0^{\pi} r \frac{(1-\cos 2\theta)}{2} \, d\theta dr = \frac{1}{2}\int_0^{2} r\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} dr$$

$$= \frac{1}{2}\int_0^{2} r[\pi]dr = \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^2 = \pi$$

Plot the region of integration to evaluate the integral $\iint_D f(x,y)dx\,dy$ *whrere D is the region bnded by th line* $y = x - 1$ *and the parabola* $y^2 = 2x + 6$.

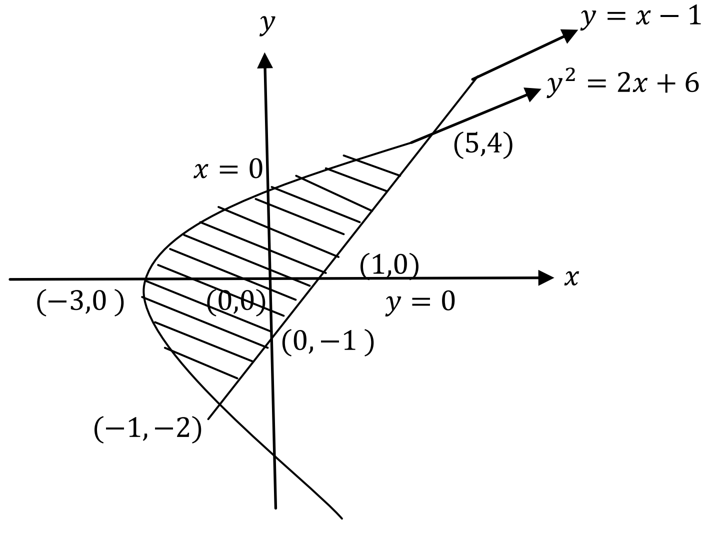
Solution:

$y = x - 1 \ldots (1)$

$y^2 = 2x + 6 \ldots (2) \Rightarrow y^2 = 2(x + 3)$

From (1) and (2), we get

$(x-1)^2 = 2x + 6 \Rightarrow x^2 - 2x + 1 = 2x + 6$



$\Rightarrow x^2 - 4x - 5 = 0 \Rightarrow x = -1,5$

$y = -2,4\ [from\ (1)]$

$Evaluate \int_1^b \int_1^a \frac{dxdy}{xy}$

$$\int_1^b \int_1^a \frac{dxdy}{xy} = \int_1^b [\log x]_1^a \frac{dy}{y} = (\log a - \log 1) \int_1^b \frac{dy}{y} = (\log a)[\log y]_1^b$$

$= (\log a)(\log b - \log 1) = \log a \log b$

$Evaluate \iint (x-y)dxdy\ over\ the\ region\ between\ the\ line\ y = x\ and\ the\ parabola$

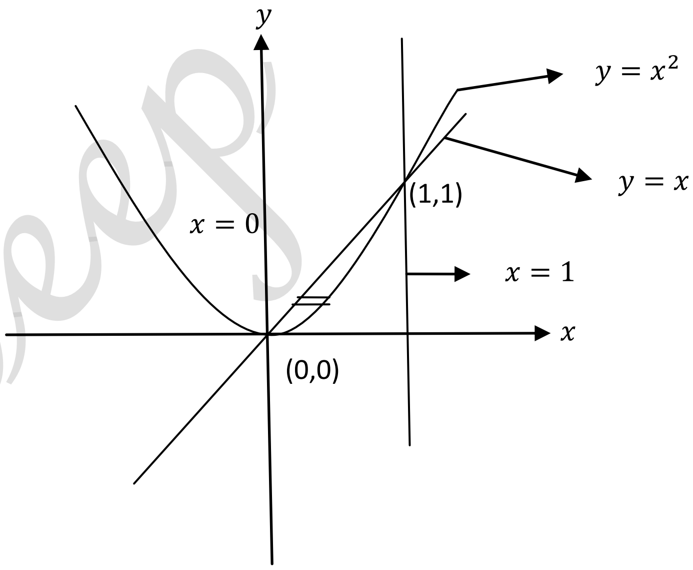
$y = x^2$.

Solution:

The limits are

$x = 0\ to\ x = 1$

$y = x^2\ to\ y = x$



$$\iint (x-y)dxdy = \int_0^1 \int_{x^2}^x (x-y)dydx$$

$$= \int_0^1 \left(xy - \frac{y^2}{2}\right)_{x^2}^x dx$$

$$= \int_0^1 \left(x^2 - \frac{x^2}{2} - \left(x^3 - \frac{x^4}{2}\right)\right) dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2}\right) dx = \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right]_0^1 = \left[\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right] = \frac{1}{60}$$

Using double integral find the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution:

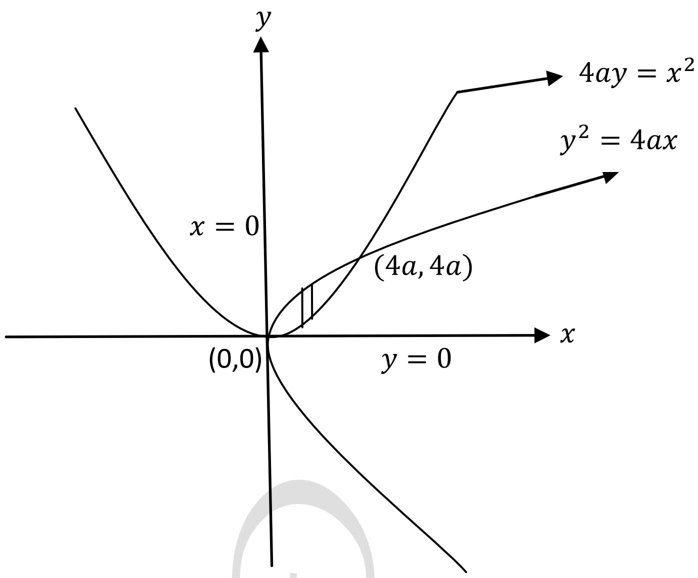
$y^2 = 4ax \ldots (1)$

$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a} \ldots (2)$

From (1) and (2), we get

$$\frac{x^4}{16a^2} = 4ax \Rightarrow x^3 = (4a)^3 \Rightarrow x = 4a$$

Put $x = 4a$ in (2), we get $y = 4a$.



The limits are

$x = 0 \; to \; x = 4a$

$y = \frac{x^2}{4a} \; to \; y = 2\sqrt{ax}$

$$Area = \iint_R dxdy = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$$

$$= \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy = \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dy = \left[\frac{2(ax)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^{4a}$$

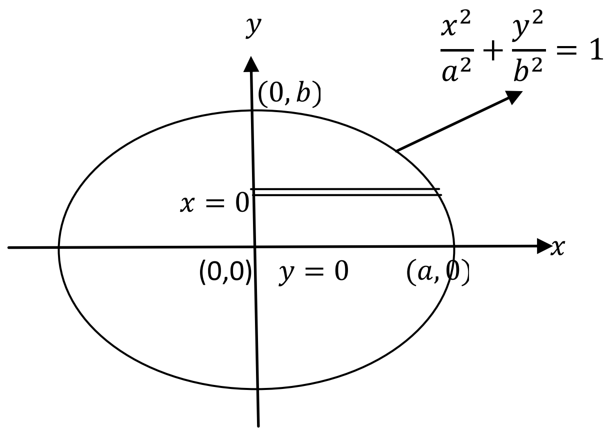
$$= \frac{2(4a^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(4a)^3}{12a} = \frac{32a^3}{3} - \frac{64a^3}{12} = \frac{32a^3}{3} - \frac{16a^3}{3}$$

$$Area = \frac{16a^3}{3}$$

Using double integral find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

The ellipse equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The limits are

$y = -\frac{b}{a}\sqrt{a^2 - x^2} \; to \; y = \frac{b}{a}\sqrt{a^2 - x^2}$

$x = -a \; to \; x = a$

$$Area = \iint_R dxdy = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_{0}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_{0}^{a} [y]_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx \ = 4\int_{0}^{a} \left[\frac{b}{a}\right] dx$$

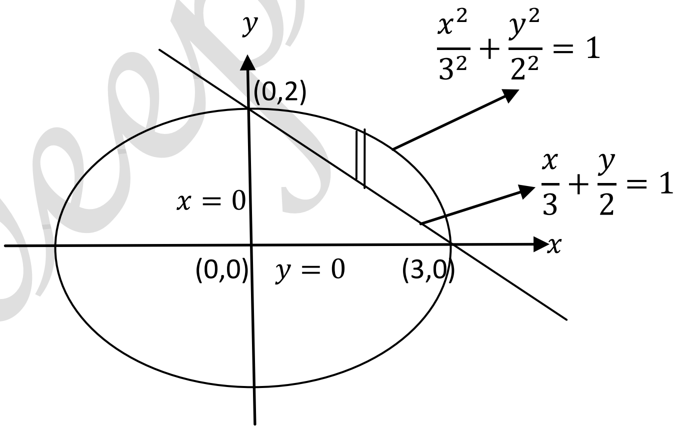
$$= \frac{4b}{a}\int_{0}^{a} \sqrt{a^2-x^2}dx \ = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right] = \frac{4b}{a}\left[\frac{a^2}{2}\frac{\pi}{2}\right]$$

$Area = \pi ab$

Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$.

Solution:



The ellipse equation is $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$

The straight line is $\frac{x}{3} + \frac{y}{2} = 1$

The limits are

$y = 0 \ to \ y = 2$

$x = \frac{3}{2}(2-y) \ to \ x = \frac{3}{2}\sqrt{2^2-y^2}$

$$Area = \iint_R dxdy = \int_{0}^{2}\int_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{2^2-y^2}} dxdy$$

$$= \int_{0}^{2} [x]_{\frac{3}{2}(2-y)}^{\frac{3}{2}\sqrt{2^2-y^2}} dy \ = \int_{0}^{2} \left[\frac{3}{2}\sqrt{2^2-y^2} - \frac{3}{2}(2-y)\right] dy$$

$$= \frac{3}{2}\int_{0}^{2} \left[\sqrt{2^2-y^2} - (2-y)\right] dy \ = \frac{3}{2}\left[\frac{y}{2}\sqrt{2^2-y^2} + \frac{2^2}{2}\sin^{-1}\frac{y}{2} - \left(2y - \frac{y^2}{2}\right)\right]_0^2$$

$$= \frac{3}{2}[2\sin^{-1}1 - (4-2)] = \frac{3}{2}\left[\frac{2\pi}{2} - 2\right]$$

$= \frac{3}{2}[\pi - 2]$

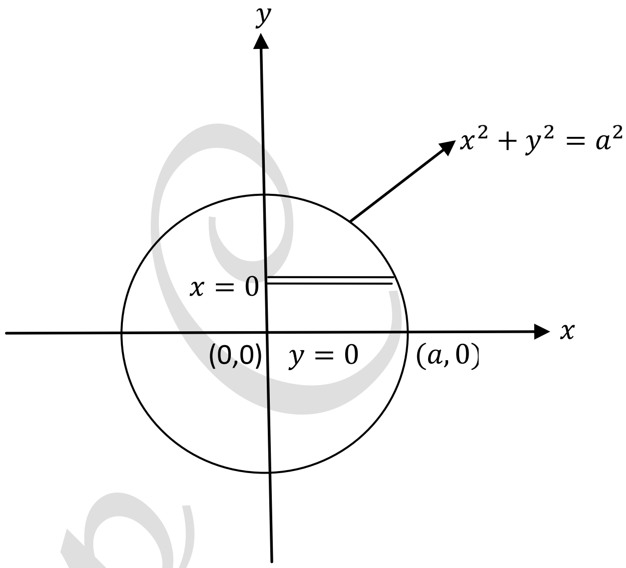
Evaluate $\iint xy\, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Solution:

The limits are

$x = 0 \; to \; x = a$

$y = 0 \; to \; y = \sqrt{a^2 - x^2}$



$$\iint xy\, dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xydydx$$

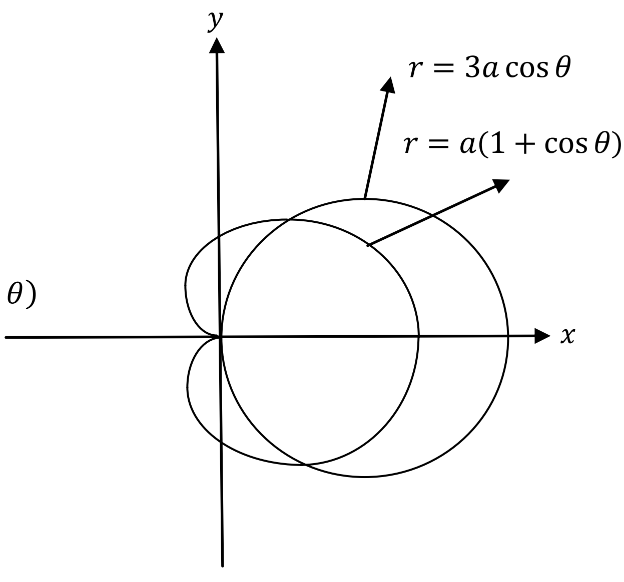
$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a x(a^2 - x^2)dx$$

$$= \frac{1}{2}\int_0^a (a^2x - x^3)dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardiod $r = a(1 + \cos\theta)$.

Solution:



$r = 3a\cos\theta \ldots (1)$

$r = a(1 + \cos\theta) \ldots (2)$

From (1) and (2), we get

$3a\cos\theta = a(1 + \cos\theta) \Rightarrow 3\cos\theta = (1 + \cos\theta)$

$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$

The limits are

$r = a(1 + \cos\theta) \; to \; r = 3a\cos\theta$

$\theta = -\frac{\pi}{3} \; to \; \theta = \frac{\pi}{3}$

$$Area = \iint r dr d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{a(1+\cos\theta)}^{3a\cos\theta} r dr d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_{a(1+\cos\theta)}^{3a\cos\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [9a^2\cos^2\theta - a^2(1+\cos\theta)^2] d\theta$$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [8\cos^2\theta - 2\cos\theta - 1] d\theta = \frac{2a^2}{2}\int_{0}^{\frac{\pi}{3}} [8\cos^2\theta - 2\cos\theta - 1] d\theta$$

$$= a^2\int_{0}^{\frac{\pi}{3}} \left[8\frac{(\cos 2\theta + 1)}{2} - 2\cos\theta - 1\right] d\theta = a^2\int_{0}^{\frac{\pi}{3}} [4\cos 2\theta - 2\cos\theta + 3] d\theta$$

$$= a^2\left[4\frac{(\sin 2\theta)}{2} - 2\sin\theta + 3\theta\right]_0^{\frac{\pi}{3}} = a^2\left[2\ \sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} + 3\left(\frac{\pi}{3}\right)\right]$$

$$= a^2\left[2\ \sin\frac{\pi}{3} - 2\sin\frac{\pi}{3} + \pi\right] = \pi a^2.$$

Find the area of the cardioid $r = a(1+\cos\theta)$

Solution:

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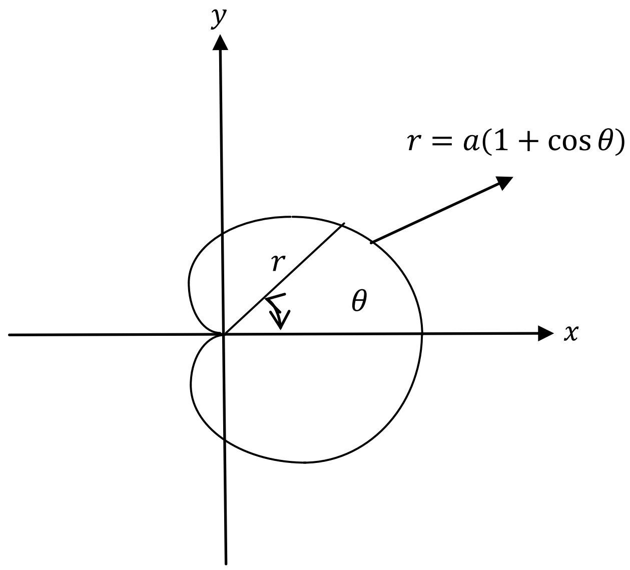


$r = a(1+\cos\theta) \dots (1)$

Put $r = 0$ in (1), we get

$a(1+\cos\theta) = 0 \Rightarrow \cos\theta = -1$

$\Rightarrow \theta = \pm\pi$

The limits are

$r = 0\ \ to\ r = a(1+\cos\theta)$

$\theta = -\pi\ to\ \theta = \pi$

$$Area = \iint r dr d\theta = \int_{-\pi}^{\pi} \int_{0}^{a(1+\cos\theta)} r dr d\theta$$

$$= \int_{-\pi}^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\pi}^{\pi} [a^2(1+\cos\theta)^2]d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} [1 + 2\cos\theta + \cos^2\theta]d\theta = \frac{2a^2}{2}\int_{0}^{\pi} [1 + 2\cos\theta + \cos^2\theta]d\theta$$

$$= a^2\int_{0}^{\pi} \left[1 + 2\cos\theta + \frac{(1+\cos 2\theta)}{2}\right]d\theta = a^2\int_{0}^{\pi} \left[\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{3}{2}\pi a^2$$

Change the order of integration

Change the order of integration in $\int_0^a \int_0^y f(x,y)dxdy.$

Solution:

The limits are

$x = 0 \; to \; x = y$

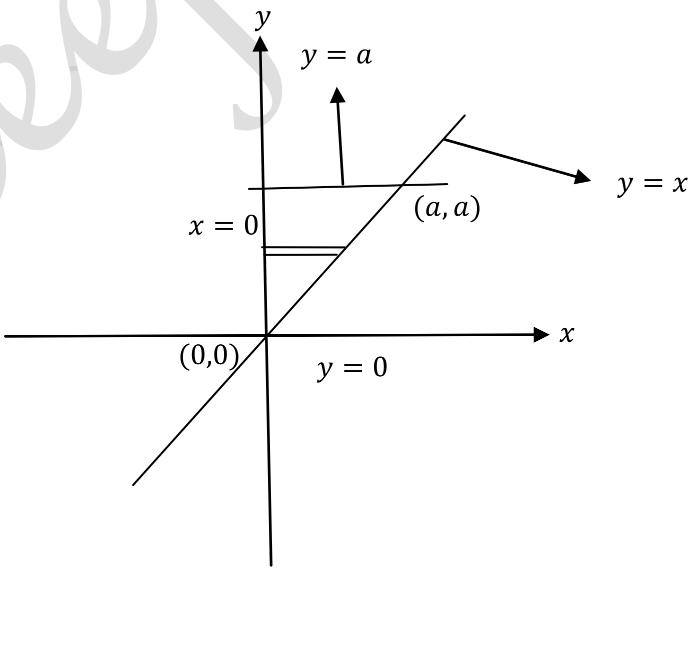
$y = 0 \; to \; y = a$

Changing the order of integration

The limits are

$y = x \; to \; y = a$

$x = 0 \; to \; x = a$



$$\int_0^a \int_0^y f(x,y)dxdy = \int_0^a \int_x^a f(x,y)dxdy.$$

Change the order of integration $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xydxdy$ and hence evaluate it.

Solution:

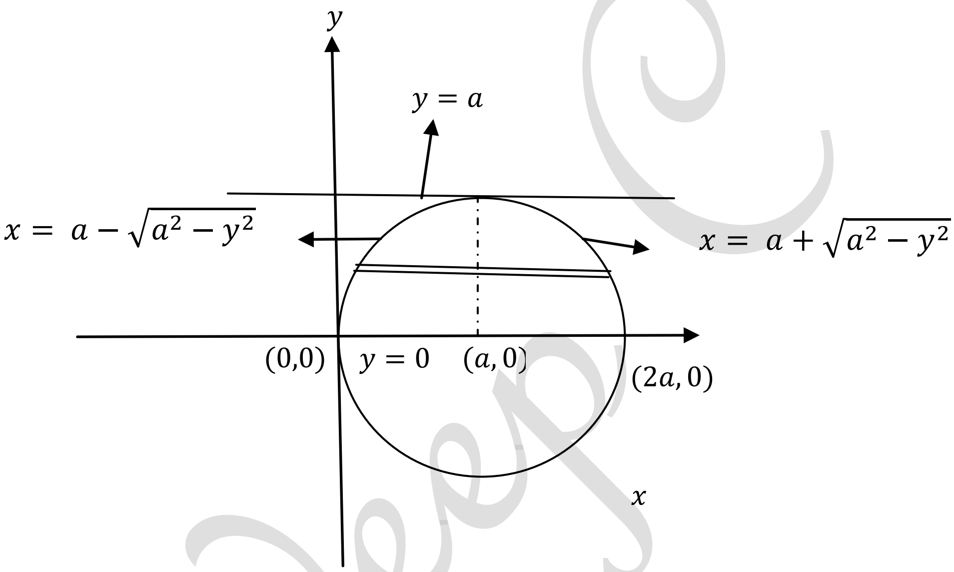
The limits are

$y = 0 \ to \ y = a$

$x = a - \sqrt{a^2 - y^2} \ \ to \ x = \ a + \sqrt{a^2 - y^2}$

$x - a = \sqrt{a^2 - y^2} \Rightarrow (x-a)^2 = a^2 - y^2 \Rightarrow (x-a)^2 + y^2 = a^2$ is a circle with centre at $(a, 0)$ and radius a.

$$(x-a)^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - (x-a)^2} \Rightarrow y = \sqrt{2ax - x^2}$$



Change of order of integration:

The limits are

$$x = 0 \ to \ x = 2a$$

$$y = 0 \ to \ y = \sqrt{2ax - x^2}$$

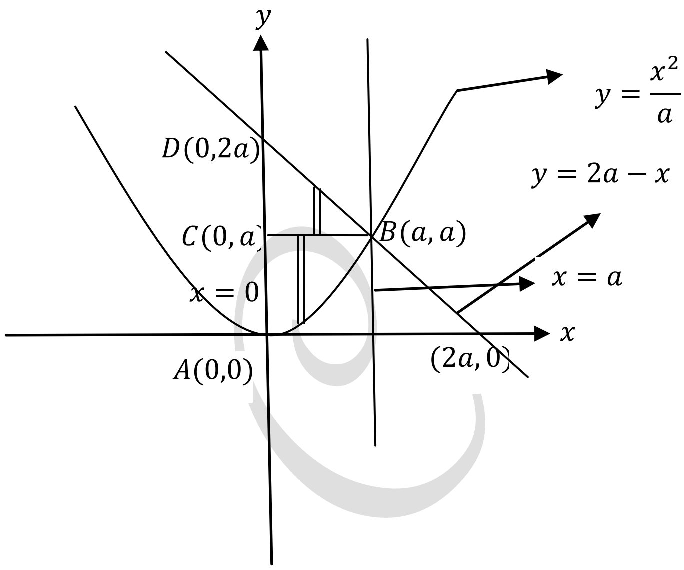
$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy dx dy = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy \ dy dx$$

$$= \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx = \frac{1}{2}\int_0^{2a} x[2ax - x^2] \ dx = \frac{1}{2}\int_0^{2a} [2ax^2 - x^3] \ dx$$

$$= \frac{1}{2}\left[\frac{2ax^3}{3} - \frac{x^4}{4}\right]_0^{2a} = \frac{8a^4}{3} - \frac{8a^4}{4} = \frac{8a^4}{12} = \frac{2a^4}{3}$$

Change the order of integration in the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dydx$ and evaluate it.

Solution:



The limits are

$x = 0\ to\ x = a$

$y = \frac{x^2}{a}\ to\ y = 2a - x$

To find the point of intersection

$\frac{x^2}{a} = 2a - x \Rightarrow x^2 + ax - 2a^2 = 0$

$x = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = \frac{-a \pm 3a}{2} = -2a, a$

$\therefore x = a \Rightarrow y = a$

Change the order of integration

The area of integration is divided into two regions ABC and BCD.

The limits of the region ABC are

$x = 0\ to\ x = \sqrt{ay}$

$y = 0\ to\ y = a$

The limits of the region BCD are

$x = 0\ to\ x = 2a - y$

$y = a\ to\ y = 2a$

$$\int_0^1 \int_{\frac{x^2}{a}}^{2a-x} xy\, dydx = \int_0^a \int_0^{\sqrt{ay}} xy\, dxdy + \int_a^{2a} \int_0^{2a-y} xy\, dxdy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\, dy + \int_a^{2a} \left[\frac{x^2}{2}\right]_0^{2a-y} y\, dy$$

$$=\frac{1}{2}\int_{0}^{a} y(ay)\,dy+\frac{1}{2}\int_{a}^{2a} [2a-y]^2 y\,dy$$

$$=\frac{a}{2}\int_{0}^{1} y^2\,dy+\frac{1}{2}\int_{1}^{2} [4a^2y-4ay^2+y^3]\,dy$$

$$=\frac{a}{2}\left[\frac{y^3}{3}\right]_0^a+\frac{1}{2}\left[\frac{4a^2y^2}{2}-\frac{4ay^3}{3}+\frac{y^4}{4}\right]_a^{2a}$$

$$=\frac{a}{2}\left[\frac{a^3}{3}\right]+\frac{1}{2}\left[\frac{16a^4}{2}-\frac{32a^4}{3}+\frac{16a^4}{4}-\left\{\frac{4a^4}{2}-\frac{4a^4}{3}+\frac{1a^4}{4}\right\}\right]$$

$$=a^4\left[\frac{1}{6}+\frac{1}{2}\left(\frac{96-128+48}{12}-\frac{24-16+3}{12}\right)\right]$$

$$=a^4\left[\frac{1}{6}+\frac{5}{24}\right]=\frac{3}{8}a^4$$

Change the order of integration in $\int_{0}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,dy\,dx$ and then evaluate it.

Solution:

The limits are

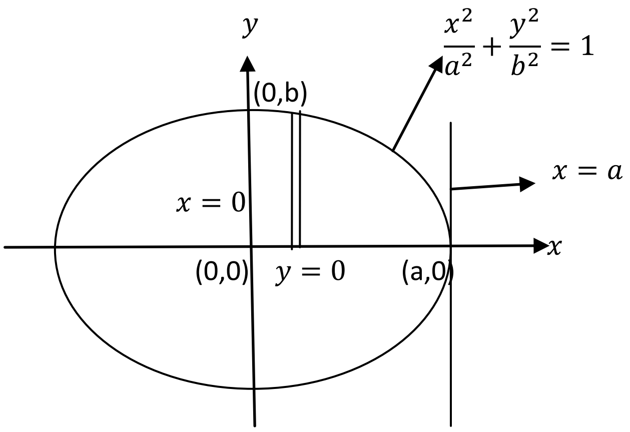
$x=0\ to\ x=a$

$y=0\ to\ y=\frac{b}{a}\sqrt{a^2-x^2}\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ is\ an\ ellipse$

Change the order of integration

The limits are

$x=0\ to\ x=\frac{a}{b}\sqrt{b^2-y^2}, y=0\ to\ y=b$



$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 dydx = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} x^2 dxdy$$

$$= \int_0^b \left[\frac{x^3}{3}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy = \frac{a^3}{3b^3}\int_0^b \left[\sqrt{b^2-y^2}\right]^3 dy$$

$Put\ y = b\sin\theta \Rightarrow dy = b\cos\theta\ d\theta$

When $y = 0 \Rightarrow b\sin\theta = 0 \Rightarrow \theta = 0$

$y = b \Rightarrow b\sin\theta = b \Rightarrow \theta = \frac{\pi}{2}$

$$= \frac{a^3}{3b^3}\int_0^{\frac{\pi}{2}} \left[\sqrt{b^2 - b^2\sin^2\theta}\right]^3 b\cos\theta\, d\theta$$

$$= \frac{a^3b^4}{3b^3}\int_0^{\frac{\pi}{2}} \left[\sqrt{1-\sin^2\theta}\right]^3 \cos\theta\, d\theta = \frac{a^3b^4}{3b^3}\int_0^{\frac{\pi}{2}} \left[\sqrt{\cos^2\theta}\right]^3 \cos\theta\, d\theta$$

$$= \frac{a^3b^4}{3b^3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{a^3b^4}{3b^3}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{n-1}{n}.\frac{n-3}{n-2}\dots\frac{1}{2}.\frac{\pi}{2}\ when\ n\ is\ even\right]$$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 dydx = \frac{\pi a^3 b}{16}$$

Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy$ and hence evaluate it.

Solution:

The limits are

$x = y\ to\ x = a$

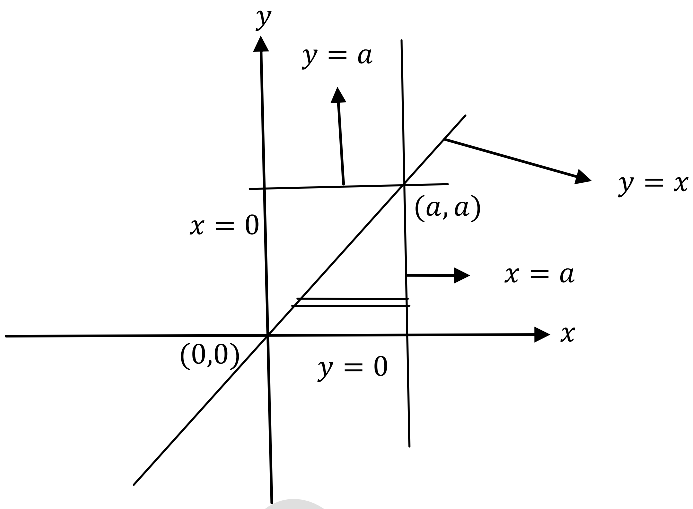
$y = 0\ to\ y = a$

Change the order of integration

The limits are

$x = 0 \; to \; x = a$

$y = 0 \; to \; y = x$



$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy = \int_0^a \int_0^x \frac{x}{x^2+y^2} dydx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx = \int_0^a \tan^{-1} 1 \, dx = \frac{\pi}{4}\int_0^a dx = \frac{\pi}{4}[x]_0^a = \frac{\pi}{4}a.$$

By changing the order of integration evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dydx$.

Solution:

The limits are

$x = 0 \; to \; x = 1$

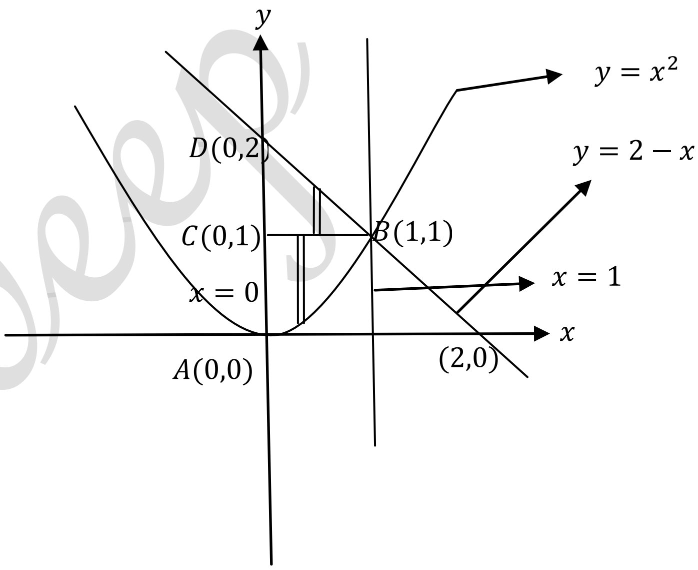
$y = x^2 \; to \; y = 2 - x$

To find the point of intersection

$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

$\therefore x = 1 \Rightarrow y = 1$



Change the order of integration

The area of integration is divided into two regions ABC and BCD.

The limits of the region ABC are

$x = 0 \; to \; x = \sqrt{y}$

$y = 0 \; to \; y = 1$

The limits of the region BCD are

$x = 0 \; to \; x = 2 - y$

$y = 1 \; to \; y = 2$

$$\int_0^1 \int_{x^2}^{2-x} xy\,dydx = \int_0^1 \int_0^{\sqrt{y}} xy\,dxdy + \int_1^2 \int_0^{2-y} xy\,dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy + \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$= \frac{1}{2}\int_0^1 y(y)\,dy + \frac{1}{2}\int_1^2 [2-y]^2 y\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 [4y - 4y^2 + y^3]\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{2}\left[\frac{16}{2} - \frac{32}{3} + \frac{16}{4} - \left\{\frac{4}{2} - \frac{4}{3} + \frac{1}{4}\right\}\right] = \frac{1}{6} + \frac{1}{2}\left(\frac{96-128+48}{12} - \frac{24-16+3}{12}\right)$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

Change of Variables

$Express \int_0^\infty \int_0^\infty f(x,y)dxdy\ in\ polar\ co-ordinates.$

Solution:

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty g(r,\theta)rdrd\theta$$

Transform the double integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$ into polar co – ordinates

and then evaluate it.

Solution:

The limits are

$x = 0 \ to \ x = a$

$y = \sqrt{ax - x^2} \ to \ y = \sqrt{a^2 - x^2}$

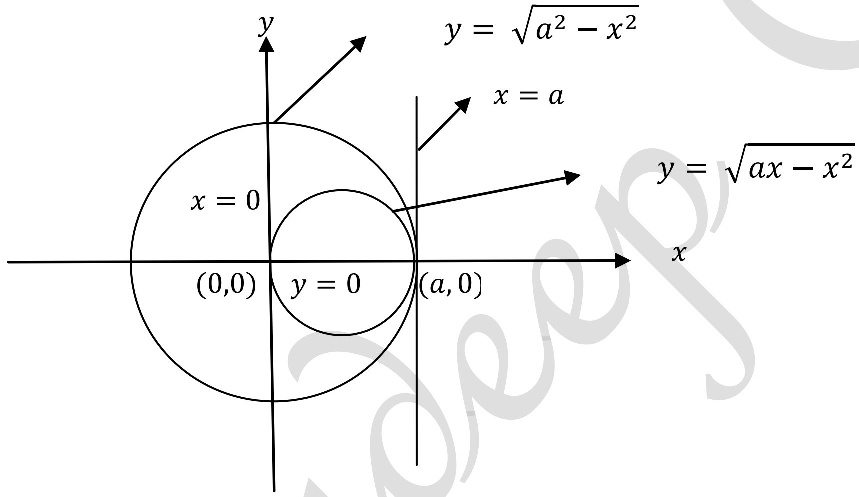
$y = \sqrt{ax - x^2} \Rightarrow y^2 = ax - x^2 \dots (1)$

$\Rightarrow y^2 = -\left(x^2 - ax + \frac{a^2}{4} - \left(\frac{a}{2}\right)^2\right)$

$\Rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \ is \ a \ circle \ with \ centre \ at \ \left(\frac{a}{2}, 0\right) \ and \ radius \frac{a}{2}$

$y = \sqrt{a^2 - x^2} \Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2 + y^2 = a^2 \dots (2) \ is \ a \ circle \ with \ centre \ at \ (0, 0) \ and \ radius \ a$



Put $x = r\cos\theta, y = r\sin\theta \ in \ (1) \ \& \ (2), we \ get$

$$y^2 = ax - x^2 \Rightarrow r^2\sin^2\theta = ar\cos\theta - r^2\cos^2\theta$$

$$\Rightarrow r^2\sin^2\theta + r^2\cos^2\theta = ar\cos\theta \Rightarrow r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$$

$x^2 + y^2 = a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$

$dxdy = rdrd\theta$

The limits in polar co-ordinates are

$r = a\cos\theta \ \ to \ r = a$

$\theta = 0 \ to \ \theta = \frac{\pi}{2}$

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}} = \int_0^{\frac{\pi}{2}} \int_{a\cos\theta}^{a} \frac{rdrd\theta}{\sqrt{a^2-r^2}} = \int_0^{\frac{\pi}{2}} \left[-\sqrt{a^2-r^2}\right]_{a\cos\theta}^{a} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[0+\sqrt{a^2-a^2\cos^2\theta}\right] d\theta = a\int_0^{\frac{\pi}{2}} [\sin\theta]\, d\theta = a[-\cos\theta]_0^{\frac{\pi}{2}} = a.$$

By changing to polar co − ordinates, evaluate $\int_0^\infty \int_0^\infty e^{x^2+y^2} dxdy$.

Solution:

The limits are

$x = 0\ to\ x = \infty$

$y = 0\ to\ y = \infty$

Put $x = r\cos\theta\ \ and\ y = r\sin\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$dxdy = rdrd\theta$

The limits are

$r = 0\ to\ r = \infty$

$\theta = 0\ to\ \theta = \frac{\pi}{2}$

$$\int_0^\infty \int_0^\infty e^{x^2+y^2} dxdy \ = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta$$

$Put\ z = r^2 \Rightarrow dz = 2r\ dr \Rightarrow \frac{dz}{2} = rdr$

$when\ r = 0 \Rightarrow z = 0, r = \infty \Rightarrow z = \infty$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-z}\frac{dz}{2} d\theta \ = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ [-e^{-z}]_0^\infty d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

By transforming into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over that annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2, (b>a)$.

Solution:

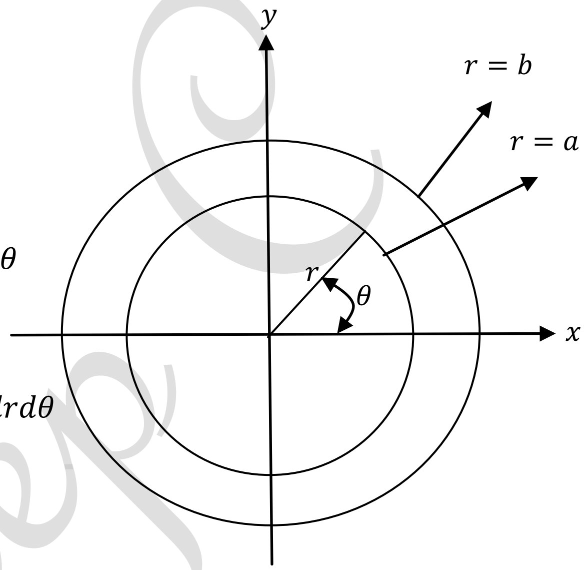
Put $x = r\cos\theta$ and $y = r\sin\theta$

$x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$

$x^2+y^2=b^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = b^2 \Rightarrow r^2 = b^2 \Rightarrow r = b$

$dxdy = rdrd\theta$

The limits are



$r = a$ to $r = b$

$\theta = 0$ to $\theta = 2\pi$

$$\iint \frac{x^2y^2}{x^2+y^2}dxdy = \int_0^{2\pi}\int_a^b \frac{r^4\cos^2\theta\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} rdrd\theta$$

$$= \int_0^{2\pi}\int_a^b \frac{r^4\cos^2\theta\sin^2\theta}{r^2} rdrd\theta = \int_0^{2\pi}\int_a^b \frac{r^3\sin^2 2\theta}{4} drd\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left[\frac{r^4}{4}\right]_a^b \sin^2 2\theta\, d\theta = \frac{b^4-a^4}{16}\int_0^{2\pi}\sin^2 2\theta\, d\theta$$

$$= \frac{b^4-a^4}{32}\int_0^{2\pi}(1-\cos 4\theta)d\theta = \frac{b^4-a^4}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = \frac{b^4-a^4}{32}[2\pi]$$

$$\iint \frac{x^2y^2}{x^2+y^2}dxdy = \frac{\pi}{16}(b^4-a^4)$$

Volume Integral

$$Evaluate \int_0^1\int_0^y\int_0^{x+y} dzdydx$$

$$\int_0^1\int_0^y\int_0^{x+y} dzdxdy = \int_0^1\int_0^y [z]_0^{x+y}\,dxdy = \int_0^1\int_0^y [x+y]\,dxdy = \int_0^1\left[\frac{x^2}{2}+xy\right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^2}{2} + y^2\right] dy = \left[\frac{y^3}{6} + \frac{y^3}{3}\right]_0^1 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Evaluate $\int_0^1 \int_0^1 \int_0^1 (x + y + z) dzdydx$

Solution:

$$\int_0^1 \int_0^1 \int_0^1 (x + y + z) dzdydx = \int_0^1 \int_0^1 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dxdy = \int_0^1 \int_0^1 \left[x + y + \frac{1}{2}\right] dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy + \frac{1}{2}x\right]_0^1 dy = \int_0^1 \left[\frac{1}{2} + y + \frac{1}{2}\right] dy = \int_0^1 [1 + y] dy = \left[y + \frac{y^2}{2}\right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

Evaluate $\int_1^2 \int_2^3 \int_1^3 (x - y + z) dzdydx$

Solution:

$$\int_1^2 \int_2^3 \int_1^3 (x - y + z) dzdydx = \int_1^2 \int_2^3 \left[xz - yz + \frac{z^2}{2}\right]_1^3 dxdy$$

$$= \int_1^2 \int_2^3 \left\{\left[3x - 3y + \frac{9}{2}\right] - \left[x - y + \frac{1}{2}\right]\right\} dxdy = \int_1^2 \int_2^3 [2x - 2y + 4]\, dydx$$

$$= \int_1^2 \left[2xy - 2\frac{y^2}{2} + 4y\right]_2^3 dx = \int_1^2 \{[6x - 9 + 12] - [4x - 4 + 8]\} dx = \int_1^2 [2x - 1] dx$$

$$= \left[2\frac{x^2}{2} - x\right]_1^2 = 4 - 2 - 1 + 1 = 2$$

Evaluate $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^a r^2 \, sin\, \theta \;\; drd\theta d\phi$

Solution:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^a r^2 \, sin\, \theta \;\; drd\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{4}} \sin\theta \, d\theta \int_0^a r^2 dr = [\phi]_0^{2\pi} [-\cos\theta]_0^{\frac{\pi}{4}} \left[\frac{r^3}{3}\right]_0^a$$

$$= 2\pi\left[-\cos\frac{\pi}{4} + \cos 0\right]\left[\frac{a^3}{3}\right] = \frac{2\pi a^3}{3}\left[1 - \frac{1}{\sqrt{2}}\right].$$

Find the volume of the ellipsoid $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$.

Solution:

To find the limits:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots (1)$$

$$\frac{z^2}{c^2} = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \Rightarrow z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \Rightarrow z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Put $z = 0$ in (1),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

Put $z = 0, y = 0$ in (1),

$$\frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$$

$The\ limits\ of\ x\ are\ x = -a\ to\ x = a$

$The\ limits\ of y\ are\ y = -b\sqrt{1 - \frac{x^2}{a^2}}\ to\ y = b\sqrt{1 - \frac{x^2}{a^2}}$

$The\ limits\ of y\ are\ z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ to\ z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

$$Volume\ = \iiint dxdydz$$

$$= \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dydx = \frac{8c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\, dydx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right)_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{8c}{b}\int_0^a \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\frac{b\sqrt{1-\frac{x^2}{a^2}}}{b\sqrt{1-\frac{x^2}{a^2}}}\, dx = 4bc\int_0^a \left(1-\frac{x^2}{a^2}\right)\sin^{-1}(1)\, dx$$

$$= 4bc\int_0^a \left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\, dydx = 2\pi bc\int_0^a \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2\pi bc\left(x-\frac{x^3}{3a^2}\right)_0^a = 2\pi bc\left(a-\frac{a^3}{3a^2}\right) = 2\pi bc\left(a-\frac{a}{3}\right) = \frac{4\pi abc}{3}$$

Find the volume of the sphere $\boldsymbol{x^2+y^2+z^2=a^2}$.

Solution:

To find the limits:

$x^2+y^2+z^2=a^2 \ldots (1)$

$z^2=a^2-x^2-y^2 \Rightarrow z=\pm\sqrt{a^2-x^2-y^2}$

Put $z=0$ in (1),

$x^2+y^2=a^2 \Rightarrow y^2=a^2-x^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$

Put $z=0, y=0$ in (1),

$x^2=a^2 \Rightarrow x=\pm a$

The limits of x are $x=-a$ *to* $x=a$

$The\ limits\ of\ y\ are\ y = -\sqrt{a^2 - x^2}\ to\ y = \sqrt{a^2 - x^2}$

$The\ limits\ of\ z\ are\ z = -\sqrt{a^2 - x^2 - y^2}\ to\ z = \sqrt{a^2 - x^2 - y^2}$

$Volume\ = \iiint dxdydz$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} dzdydx = 8\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8\int_{0}^{a} \left(\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_{0}^{a} \frac{a^2 - x^2}{2}\sin^{-1}\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}}\ dx = 4\int_{0}^{a} (a^2 - x^2)\sin^{-1}(1)\ \ dx$$

$$= 4\int_{0}^{a} (a^2 - x^2)\frac{\pi}{2}\ \ dydx = 2\pi\int_{0}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi\left(a^2x - \frac{x^3}{3}\right)_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

Find the volume of the tetrahedron bounded by the plane $x + y + z = 1$

and the coordinate planes.

Solution:

To find the limits:

$x + y + z = 1 \ldots (1)$

$z = 1 - x - y$

Put $z = 0$ in (1),

$x + y = 1 \Rightarrow y = 1 - x$

Put $z = 0, y = 0$ in (1),

$x = 1$

$The\ limits\ of\ x\ are\ x = 0\ to\ x = 1$

$The\ limits\ of y\ are\ y = 0\ to\ y = 1 - x$

$The\ limits\ of y\ are\ z = 0\ to\ z = 1 - x - y$

$$Volume\ = \iiint dxdydz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdydx = \int_0^1 \int_0^{1-x} (z)_0^{1-x-y} dydx = \int_0^1 \int_0^{1-x} (1 - x - y)dydx$$

$$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)_0^{1-x} dx = \int_0^1 \left(1 - x - x(1-x) - \frac{(1-x)^2}{2}\right) dx$$

$$= \int_0^1 \left(1 - x - x + x^2 - \frac{1 - 2x + x^2}{2}\right) dx$$

$$= \int_0^1 \left(\frac{1 - 2x + x^2}{2}\right) dx = \frac{1}{2}\left(x - \frac{2x^2}{2} + \frac{x^3}{3}\right)_0^1 = \frac{1}{2}\left(1 - 1 + \frac{1}{3}\right) = \frac{1}{6}$$

Evaluate $\iiint_V (x + y + z)dxdydz$ where the region V bounded by $x + y + z = a$

$(a > 0), x = 0, y = 0, z = 0$.

Solution:

To find the limits:

$x + y + z = a \ldots (1)$

$z = a - x - y$

Put $z = 0$ in (1),

$x + y = a \Rightarrow y = a - x$

Put $z = 0, y = 0$ in (1),

$x = a$

$The\ limits\ of\ x\ are\ x = 0\ to\ x = a$

$The\ limits\ of y\ are\ y = 0\ to\ y = a - x$

$The\ limits\ of y\ are\ z = 0\ to\ z = a - x - y$

$$\iiint_V (x+y+z)dxdydz = \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)dzdydx$$

$$= \int_0^a \int_0^{a-x} \left(xz + yz + \frac{z^2}{2} \right)_0^{a-x-y} dydx$$

$$= \int_0^a \int_0^{a-x} \left(x(a-x-y) + y(a-x-y) + \frac{(a-x-y)^2}{2} \right) dydx$$

$$= \int_0^a \int_0^{a-x} \left(ax - x^2 - xy + ay - xy - y^2 + \frac{a^2 + x^2 + y^2 - 2ax - 2ay + 2xy}{2} \right) dydx$$

$$= \int_0^a \int_0^{a-x} \left(\frac{a^2 - x^2 - y^2 - 2xy}{2} \right) dydx = \frac{1}{2} \int_0^a \int_0^{a-x} (a^2 - x^2 - y^2 - 2xy) dydx$$

$$= \frac{1}{2} \int_0^a \left(a^2 y - x^2 y - \frac{y^3}{3} - \frac{2xy^2}{2} \right)_0^{a-x} dx$$

$$= \frac{1}{2} \int_0^a \left(a^2(a-x) - x^2(a-x) - \frac{(a-x)^3}{3} - x(a-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^a \left(a^3 - a^2 x - x^2 a + x^3 - \frac{(a-x)^3}{3} - a^2 x - x^3 + 2ax^2 \right) dx$$

$$= \frac{1}{2} \int_0^a \left(a^3 - 2a^2 x - \frac{(a-x)^3}{3} + ax^2 \right) dx = \frac{1}{2} \left(a^3 x - \frac{2a^2 x^2}{2} + \frac{(a-x)^4}{12} + \frac{ax^3}{3} \right)_0^a$$

$$= \frac{1}{2} \left(a^4 - a^4 + \frac{a^4}{3} - \frac{a^4}{12} \right) = \frac{a^4}{8}$$

Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution:

To find the limits:

$y+z=4\ldots(1)$

$z=4-y$

$x^2+y^2=4\ldots(2)$

$y^2=4-x^2\Rightarrow y=\pm\sqrt{4-x^2}$

Put $y=0$ in (1),

$x^2=4\Rightarrow x=\pm 2$

$The\ limits\ of\ x\ are\ x=-2\ to\ x=2$

$The\ limits\ of y\ are\ y=-\sqrt{4-x^2}\ to\ y=\sqrt{4-x^2}$

$The\ limits\ of y\ are\ z=0\ to\ z=4-y$

$Volume\ =\ \iiint dxdydz$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dzdydx=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(z)_0^{4-y}dydx=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)dydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}4\,dydx-\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y\,dydx$$

$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}4\,dydx-0\ \ [Since\ y\ is\ odd\ function]$$

$$=8\int_{-2}^{2}(y)_0^{\sqrt{4-x^2}}dx=8\int_{-2}^{2}\sqrt{4-x^2}\,dx=16\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$=16\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$$

$$=16(2\sin^{-1}(1))=\frac{32\pi}{2}=16\pi$$

Evaluate $\iiint_V \frac{dzdydx}{(x+y+z+1)^3}$**, where** V **is the region bounded by** $x=0,y=0,$

$z=0$ **and** $x+y+z=1.$

Solution:

To find the limits:

$$x + y + z = 1 \ldots (1)$$

$z = 1 - x - y$

Put $z = 0$ in (1), we get

$y = 1 - x$

Put $z = 0, y = 0$ in (1), we get

$x = 1$

The limits are

$x = 0$ to $x = 1$

$y = 0$ to $y = 1 - x$

$z = 0$ to $z = 1 - x - y$

$$\iiint_V \frac{dzdydx}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dzdydx}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{dydx}{-2(x+y+z+1)^2}\right]_0^{1-x-y}$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{1}{-2(x+y+1-x-y+1)^2} - \frac{1}{-2(x+y+1)^2}\right] dydx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{1}{-2(2)^2} - \frac{1}{-2(x+y+1)^2}\right] dydx = \int_0^1 \int_0^{1-x} \left[\frac{1}{-8} + \frac{1}{2(x+y+1)^2}\right] dydx$$

$$= \int_0^1 \left[\frac{1}{-8}(y)_0^{1-x} + \frac{1}{2}\left[\frac{-1}{x+y+1}\right]_0^{1-x}\right] dx$$

$$= \int_0^1 \left[\frac{1}{-8}(1-x) + \frac{1}{2}\left[\frac{-1}{x+1-x+1} - \frac{-1}{x+0+1}\right]\right] dx$$

$$= \int_0^1 \left[\frac{1}{-8}(1-x) + \frac{1}{2}\left[\frac{-1}{2} + \frac{1}{x+1}\right]\right] dx = \left[\frac{(1-x)^2}{16} - \frac{1}{4}x + \frac{1}{2}\log(x+1)\right]_0^1$$

$$= -\frac{1}{16} - \frac{1}{4} + \frac{1}{2}\log 2 - \log 1 = \frac{1}{2}\log 2 - \frac{5}{16}$$

Evaluate $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ *for all positive values of* x, y, z *for which the integral is real.*

Solution:

The integral is real for $x^2 + y^2 + z^2 < 1 \ldots (1)$ $for\ x > 0, y > 0, z > 0$

To find the limits

$z^2 < 1 - x^2 - y^2 \Rightarrow z < \pm\sqrt{1 - x^2 - y^2}$

Put $z = 0$ in (1)

$y^2 < 1 - x^2 \Rightarrow y < \pm\sqrt{1 - x^2}$

Put $z = 0, y = 0$ in (1)

$x^2 < 1 \Rightarrow x < \pm 1$

The limits are

$x = 0$ to $x = 1$

$y = 0$ to $y = \sqrt{1 - x^2}$

$z = 0$ to $z = \sqrt{1 - x^2 - y^2}$

$$\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\right) - \sin^{-1} 0\right] dydx = \int_0^1 \int_0^{\sqrt{1-x^2}} [\sin^{-1} 1\,] dydx$$

$$= \frac{\pi}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} dydx = \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-1} + \frac{1}{2}\sin^{-1} 1 - \left[0 + \frac{1}{2}\sin^{-1} 0\right]\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\frac{\pi}{2}\right] = \frac{\pi^2}{8}$$

$$Evaluate \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx$$

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x}e^{y}e^{z}dzdydx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}e^{y}[e^{z}]_0^{x+\log y}dydx = \int_0^{\log 2}\int_0^{x} e^{x}e^{y}\left[e^{x+\log y} - e^{0}\right]dydx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}e^{y}[ye^{x} - 1]dydx = \int_0^{\log 2}\int_0^{x}[ye^{2x}e^{y} - e^{x}e^{y}]dydx$$

$$= \int_0^{\log 2} [e^{2x}\{ye^{y} - e^{y}\} - e^{x}e^{y}]_0^{x}dx = \int_0^{\log 2} [e^{2x}\{xe^{x} - e^{x}\} - e^{x}e^{x}] - [e^{2x}(-1) - e^{x}]dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{3x} - e^{2x} + e^{2x} + e^{x}]dx = \int_0^{\log 2} [xe^{3x} - e^{3x} + e^{x}]dx$$

$$= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{3^2} - \frac{e^{3x}}{3} + e^{x}\right]_0^{\log 2}$$

$$= \frac{\log 2\, e^{3\log 2}}{3} - \frac{e^{3\log 2}}{3^2} - \frac{e^{3\log 2}}{3} + e^{\log 2} - \left[0 - \frac{1}{9} - \frac{1}{3} + 1\right]$$

$$= 2^3\frac{\log 2}{3} - \frac{2^3}{3^2} - \frac{2^3}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 \qquad \{\because e^{a\log b} = b^{a}\}$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

Evaluate $\iiint x^2yzdxdydz$ *taken over the tetrahedron bounded by the planes*

$x = 0, y = 0, z = 0$ *and* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Solution:

To find the limits:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\ldots(1)$$

$z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Put $z=0$ in (1), we get

$y=b\left(1-\frac{x}{a}\right)$

Put $z=0, y=0$ in (1), we get

$x=a$

The limits are

$x=0\ to\ x=a$

$y=0\ to\ y=b\left(1-\frac{x}{a}\right)$

$z=0\ to\ z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$$\iiint x^2yz dxdydz=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2yz dzdydx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} x^2y\left[\frac{z^2}{2}\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} x^2y\frac{c^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2}dydx=\frac{c^2}{2}\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} x^2y\left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dydx$$

$$=\frac{c^2}{2}\int_0^a x^2\left[y\left\{\frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^3}{3\left(-\frac{1}{b}\right)}\right\}-\left\{\frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^4}{12\left(-\frac{1}{b}\right)^2}\right\}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{c^2}{2}\int_0^a x^2 \left[b\left(1-\frac{x}{a}\right)\left\{\frac{\left(1-\frac{x}{a}-\frac{b\left(1-\frac{x}{a}\right)}{b}\right)^3}{3\left(-\frac{1}{b}\right)}\right\} - \left\{\frac{\left(1-\frac{x}{a}-\frac{b\left(1-\frac{x}{a}\right)}{b}\right)^4}{12\left(-\frac{1}{b}\right)^2}\right\} + \frac{\left(1-\frac{x}{a}\right)^4}{12\left(-\frac{1}{b}\right)^2}\right] dx$$

$$= \frac{c^2}{2}\int_0^a x^2 \left[b^2 \frac{\left(1-\frac{x}{a}\right)^4}{12}\right] dx = \frac{b^2c^2}{24}\int_0^a x^2\left(1-\frac{x}{a}\right)^4 dx$$

$$= \frac{b^2c^2}{24}\left[x^2\left\{\frac{\left(1-\frac{x}{a}\right)^5}{5\left(-\frac{1}{a}\right)}\right\} - 2x\left\{\frac{\left(1-\frac{x}{a}\right)^6}{30\left(-\frac{1}{a}\right)^2}\right\} + 2\left\{\frac{\left(1-\frac{x}{a}\right)^7}{210\left(-\frac{1}{a}\right)^3}\right\}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[a^2\left\{\frac{\left(1-\frac{a}{a}\right)^5}{5\left(-\frac{1}{a}\right)}\right\} - 2a\left\{\frac{\left(1-\frac{a}{a}\right)^6}{30\left(-\frac{1}{a}\right)^2}\right\} + 2\left\{\frac{\left(1-\frac{a}{a}\right)^7}{210\left(-\frac{1}{a}\right)^3}\right\} - 2\left\{\frac{1}{210\left(-\frac{1}{a}\right)^3}\right\}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{105}\right] = \frac{a^3b^2c^2}{2520}$$

A circular hole of radius b is made centrally through a sphere of radius a. Find the volume of the remaining sphere.

Solution:

$$\text{Volume of the remaining sphere } V = \begin{cases} (V_1) & \text{Volume of the sphere of radius } 'a' \; - \\ (V_2) & \text{Volume of the circular hole of radius } 'b' \end{cases}$$

Volume of the circular hole of radius $'b'$ is the volume of the cylinder of radius $'b'$ and height $2a$.

To find the limits of V_1:

$x^2 + y^2 + z^2 = a^2 \ldots (1)$

$z^2 = a^2 - x^2 - y^2 \Rightarrow z = \pm\sqrt{a^2 - x^2 - y^2}$

Put $z = 0$ in (1),

$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$

Put $z = 0, y = 0$ in (1),

$x^2 = a^2 \Rightarrow x = \pm a$

$The\ limits\ of\ x\ are\ x = -a\ to\ x = a$

$The\ limits\ of y\ are\ y = -\sqrt{a^2 - x^2}\ to\ y = \sqrt{a^2 - x^2}$

$The\ limits\ of y\ are\ z = -\sqrt{a^2 - x^2 - y^2}\ to\ z = \sqrt{a^2 - x^2 - y^2}$

$$V_1 = \iiint dxdydz$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8\int_0^a \left(\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \frac{a^2 - x^2}{2}\sin^{-1}\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}}\ dx = 4\int_0^a (a^2 - x^2)\sin^{-1}(1)\ dx$$

$$= 4\int_0^a (a^2 - x^2)\frac{\pi}{2}\ dydx = 2\pi\int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi\left(a^2x - \frac{x^3}{3}\right)_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

To find the limits of V_2:

The equation of circle of radius b is

$$x^2 + y^2 = b^2 \ldots (2)$$

$$y^2 = b^2 - x^2 \Rightarrow y = \pm\sqrt{b^2 - x^2}$$

Put $y = 0$ in (2),

$$x^2 = b^2 \Rightarrow x = \pm b$$

$$The\ limits\ of\ x\ are\ x = -b\ to\ x = b$$

$$The\ limits\ of y\ are\ y = -\sqrt{b^2 - x^2}\ to\ y = \sqrt{b^2 - x^2}$$

$$The\ limits\ of z\ are\ z = -a\ to\ z = a$$

$$V_2 = \iiint dxdydz$$

$$= \int_{-b}^{b} \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \int_{-a}^{a} dzdydx$$

$$= 8\int_{0}^{b} \int_{0}^{\sqrt{b^2-x^2}} \int_{0}^{a} dzdydx = 8\int_{0}^{b} \int_{0}^{\sqrt{b^2-x^2}} (z)_0^a dydx$$

$$= 8a\int_{0}^{b} \int_{0}^{\sqrt{b^2-x^2}} dydx = 8a\int_{0}^{b} (y)_0^{\sqrt{b^2-x^2}} dx$$

$$= 8a\int_{0}^{a} \sqrt{b^2 - x^2}\, dx = 8a\left(\frac{x}{2}\sqrt{b^2 - x^2} + \frac{b^2}{2}\sin^{-1}\frac{x}{b}\right)_0^b$$

$$V_2 = 8a\frac{b^2}{2}\sin^{-1} 1 = \frac{4ab^2\pi}{2} = 2\pi ab^2$$

$$V = \frac{4}{3}\pi a^3 - 2\pi ab^2$$