Differential calculus

Radius of curvature in Cartesian coordinates

$$Radius\ of\ curvature\ \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$Where\ y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$$

Find the radius of curvature of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ldots (1)$$

Differentiating (1) with respect to x, we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\frac{d^2y}{dx^2} = -\frac{\left(x^{\frac{1}{3}}\frac{1}{3}\,y^{-\frac{2}{3}}\frac{dy}{dx} - y^{\frac{1}{3}}\frac{1}{3}\,x^{-\frac{2}{3}}\right)}{x^{\frac{2}{3}}}$$

$$= -\frac{\left(x^{\frac{1}{3}}\frac{1}{3}\,y^{-\frac{2}{3}}\left(-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right) - y^{\frac{1}{3}}\frac{1}{3}\,x^{-\frac{2}{3}}\right)}{x^{\frac{2}{3}}} = -\frac{\left(-\frac{1}{3}\,y^{-\frac{1}{3}} - y^{\frac{1}{3}}\frac{1}{3}\,x^{-\frac{2}{3}}\right)}{x^{\frac{2}{3}}}$$

$$= -\frac{\left(-\frac{1}{3y^{\frac{1}{3}}} - \frac{y^{\frac{1}{3}}}{3x^{\frac{2}{3}}}\right)}{x^{\frac{2}{3}}} = \frac{\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)}{3x^{\frac{2}{3}}x^{\frac{2}{3}}y^{\frac{1}{3}}}$$

$$\frac{d^2y}{dx^2} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}\ \ (from\ (1))$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\left(-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^2\right)^{\frac{3}{2}}}{\frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}}$$

$$= 3x^{\frac{4}{3}}y^{\frac{1}{3}}\frac{\left(1+\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}}}{a^{\frac{2}{3}}} = 3x^{\frac{4}{3}}y^{\frac{1}{3}}\frac{\left(\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}}}{a^{\frac{2}{3}}} = 3x^{\frac{4}{3}}y^{\frac{1}{3}}\frac{\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}}}{xa^{\frac{2}{3}}}\ (from\ (1))$$

$$= 3x^{\frac{4}{3}}y^{\frac{1}{3}}\frac{a}{xa^{\frac{2}{3}}} = 3x^{\frac{1}{3}}y^{\frac{1}{3}}a^{\frac{1}{3}}$$

$$\rho = 3(axy)^{\frac{1}{3}}$$

Find the radius of curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ on the curve $x^3+y^3=3axy$

$x^3+y^3=3axy \ldots (1)$

Differentiating (1) with respect to x, we get

$$3x^2+3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right)$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay-3x^2 \Rightarrow (3y^2-3ax)\frac{dy}{dx} = 3ay-3x^2$$

$$\frac{dy}{dx} = \frac{3ay-3x^2}{3y^2-3ax}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{3a\left(\frac{3a}{2}\right)-3\left(\frac{3a}{2}\right)^2}{3\left(\frac{3a}{2}\right)^2-3a\left(\frac{3a}{2}\right)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{(3y^2-3ax)\left(3a\frac{dy}{dx}-6x\right)-(3ay-3x^2)\left(6y\frac{dy}{dx}-3a\right)}{(3y^2-3ax)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)}$$

$$= \frac{\left(3\left(\frac{3a}{2}\right)^2 - 3a\left(\frac{3a}{2}\right)\right)\left(3a(-1) - 6\left(\frac{3a}{2}\right)\right) - \left(3a\left(\frac{3a}{2}\right) - 3\left(\frac{3a}{2}\right)^2\right)\left(6\left(\frac{3a}{2}\right)(-1) - 3a\right)}{\left(3\left(\frac{3a}{2}\right)^2 - 3a\left(\frac{3a}{2}\right)\right)^2}$$

$$= \frac{\left(\frac{27a^2}{4} - \frac{9a^2}{2}\right)(-3a - 9a) - \left(\frac{9a^2}{2} - \frac{27a^2}{4}\right)(-9a - 3a)}{\left(\frac{27a^2}{4} - \frac{9a^2}{2}\right)^2}$$

$$= \frac{\left(\frac{27a^2}{4} - \frac{9a^2}{2}\right)(-12a)(1 - (-1))}{\left(\frac{27a^2}{4} - \frac{9a^2}{2}\right)^2} = \frac{-24a}{\left(\frac{27a^2}{4} - \frac{18a^2}{4}\right)} = \frac{-24a}{\left(\frac{9a^2}{4}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{-32}{3a}$$

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + (-1)^2)^{\frac{3}{2}}}{\left(\frac{-32}{3a}\right)} = \frac{3a2^{\frac{3}{2}}}{-32} = \frac{3a2\sqrt{2}}{-32} = \frac{3a\sqrt{2}}{-16}$$

$$\rho = \frac{3a\sqrt{2}}{16} \ (Since\ radius\ cannot\ be\ negative)$$

Find the radius of curvature of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ ... (1)

Differentiating (1) with respect to x, we get

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = -\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{\left(\frac{1}{4}\right)^{\frac{1}{2}}} = -1$$

$$\frac{d^2y}{dx^2} = -\frac{\left(x^{\frac{1}{2}}\frac{1}{2}\,y^{-\frac{1}{2}}\frac{dy}{dx} - y^{\frac{1}{2}}\frac{1}{2}\,x^{-\frac{1}{2}}\right)}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = -\frac{\left(\left(\frac{1}{4}\right)^{\frac{1}{2}}\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}}(-1) - \left(\frac{1}{4}\right)^{\frac{1}{2}}\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{1}{2}}\right)}{\left(\frac{1}{4}\right)} = -\frac{\left(-\frac{1}{2}-\frac{1}{2}\right)}{\left(\frac{1}{4}\right)} = 4$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+(-1)^2)^{\frac{3}{2}}}{4} = \frac{2^{\frac{3}{2}}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

Find the radius of curvature of $xy = c^2$ at (x, y)

$xy = c^2 \ldots (1)$

Differentiating (1) with respect to x, we get

$$y + x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = -\frac{\left(x\left(-\frac{y}{x}\right) - y\right)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1+\left(-\frac{y}{x}\right)^2\right)^{\frac{3}{2}}}{\left(\frac{2y}{x^2}\right)} = \frac{\left(\frac{x^2+y^2}{x^2}\right)^{\frac{3}{2}}}{\left(\frac{2y}{x^2}\right)} = \frac{(x^2+y^2)^{\frac{3}{2}}}{x^3\left(\frac{2y}{x^2}\right)} = \frac{(x^2+y^2)^{\frac{3}{2}}}{2xy}$$

$$\rho = \frac{(x^2 + y^2)^{\frac{3}{2}}}{2c^2} \quad from\ (1)$$

Find the radius of curvature of $y^2 = 4ax$ at $(at^2, 2at)$

$y^2 = 4ax \ldots (1)$

Differentiating (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(at^2, 2at)} = -\frac{2a}{(2at)^2}\frac{1}{t} = -\frac{1}{2at^3}$$

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1 + \left(\frac{1}{t}\right)^2\right)^{\frac{3}{2}}}{-\frac{1}{2at^3}} = \frac{\left(\frac{t^2+1}{t^2}\right)^{\frac{3}{2}}}{-\frac{1}{2at^3}} = \frac{(t^2+1)^{\frac{3}{2}}}{t^3\left(-\frac{1}{2at^3}\right)}$$

$$\rho = 2a(t^2 + 1)^{\frac{3}{2}} \quad (Since\ the\ radius\ of\ curvature\ is\ non-negative)$$

Find the radius of curvature of $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ at (x, y)

$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1 \ldots (1)$

Differentiating (1) with respect to x, we get

$$\frac{1}{2\sqrt{a}}x^{-\frac{1}{2}} + \frac{1}{2\sqrt{b}}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{b}}y^{-\frac{1}{2}}\frac{dy}{dx} = -\frac{1}{2\sqrt{a}}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{2\sqrt{b}x^{-\frac{1}{2}}}{2\sqrt{a}y^{-\frac{1}{2}}} = -\frac{\sqrt{b}y^{\frac{1}{2}}}{\sqrt{a}x^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(x^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} - y^{\frac{1}{2}}\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(x^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}\left(-\frac{\sqrt{b}y^{\frac{1}{2}}}{\sqrt{a}x^{\frac{1}{2}}}\right) - y^{\frac{1}{2}}\frac{1}{2}x^{-\frac{1}{2}}\right)}{x}\right] = -\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\frac{1}{2}\left(-\frac{\sqrt{b}}{\sqrt{a}}\right) - \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}\right)}{x}\right]$$

$$= \frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\sqrt{bx} + \sqrt{ay}\right)}{2\sqrt{ax}x}\right]$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1 + \left(-\frac{\sqrt{b}y^{\frac{1}{2}}}{\sqrt{a}x^{\frac{1}{2}}}\right)^2\right)^{\frac{3}{2}}}{\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\sqrt{bx} + \sqrt{ay}\right)}{2\sqrt{ax}x}\right]} = \frac{\left(1 + \left(\frac{by}{ax}\right)\right)^{\frac{3}{2}}}{\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\sqrt{bx} + \sqrt{ay}\right)}{2\sqrt{ax}x}\right]}$$

$$= \frac{(ax+by)^{\frac{3}{2}}}{(ax)^{\frac{3}{2}}\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\sqrt{bx} + \sqrt{ay}\right)}{2\sqrt{ax}x}\right]}$$

$$\rho = \frac{(ax+by)^{\frac{3}{2}}}{(ax)^{\frac{3}{2}}\frac{\sqrt{b}}{\sqrt{a}}\left[\frac{\left(\sqrt{b}\sqrt{a}\right)}{2\sqrt{ax}x}\right]} \ (from\ (1))$$

$$\rho = \frac{2(ax+by)^{\frac{3}{2}}}{ab}$$

Radius of curvature in Parametric form

IF $x = f(t), y = g(t)$

$$Radius\ of\ curvature\ \rho = \frac{((f')^2 + (g')^2)^{\frac{3}{2}}}{f'g'' - g'f''}$$

$$Where\ f' = \frac{dx}{dt}, f'' = \frac{d^2x}{dt^2}, g' = \frac{dy}{dt}, g'' = \frac{d^2y}{dt^2}$$

Find the radius of curvature of $\boldsymbol{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$

The parametric form is $x = a\cos^3\theta,\ \ y = \mathrm{a}\sin^3\theta \ldots (1)$

$$f' = \frac{dx}{d\theta} = 3\,\mathrm{a}\cos^2\theta(-\sin\theta) = -3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$f'' = \frac{d^2x}{d\theta^2} = -3\,\mathrm{a}\cos^2\theta\cos\theta - 6\,\mathrm{a}\cos\theta(-\sin\theta)\sin\theta = -3acos^3\theta + 6\,\mathrm{a}\cos\theta\sin^2\theta$$

$$g' = \frac{dy}{d\theta} = 3\ \mathrm{a}\sin^2\theta\cos\theta$$

$$g'' = \frac{d^2y}{d\theta^2} = 3\,\mathrm{a}\sin^2\theta(-\sin\theta) + 6\,\mathrm{a}\sin\theta\cos\theta\cos\theta = -3a\sin^3\theta + 6\,\mathrm{a}\sin\theta\cos^2\theta$$

$$Radius\ of\ curvature\ \rho = \frac{((f')^2 + (g')^2)^{\frac{3}{2}}}{f'g'' - g'f''}$$

$$= \frac{((-3\,\mathrm{a}\cos^2\theta\sin\theta)^2 + (3\ \mathrm{a}\sin^2\theta\cos\theta)^2)^{\frac{3}{2}}}{(-3\,\mathrm{a}\cos^2\theta\sin\theta)(-3a\sin^3\theta + 6\,\mathrm{a}\sin\theta\cos^2\theta) - (3\ \mathrm{a}\sin^2\theta\cos\theta)(-3acos^3\theta + 6\,\mathrm{a}\cos\theta\sin^2\theta)}$$

$$= \frac{(3^2\ \mathrm{a}^2\cos^4\theta\sin^2\theta + 3^2\ \mathrm{a}^2\cos^2\theta\sin^4\theta)^{\frac{3}{2}}}{(9a^2\cos^2\theta\sin^2\theta)(\sin^2\theta - 2\cos^2\theta + \cos^2\theta - 2\sin^2\theta)}$$

$$= \frac{(3^2a^2\cos^2\theta\sin^2\theta)^{\frac{3}{2}}(\cos^2\theta + \sin^2\theta)^{\frac{3}{2}}}{(9a^2\cos^2\theta\sin^2\theta)(-\sin^2\theta - \cos^2\theta)}$$

$$= \frac{(3^3a^3\cos^3\theta\sin^3\theta)(\cos^2\theta + \sin^2\theta)^{\frac{3}{2}}}{-(9a^2\cos^2\theta\sin^2\theta)(\sin^2\theta + \cos^2\theta)}$$

$$\rho = -3\,\mathrm{a}\cos\theta\sin\theta$$

$$\rho = 3\,\mathrm{a}\cos\theta\sin\theta \quad (Since\ radius\ cannot\ be\ negative)$$

$$\rho = 3a\left(\frac{x}{a}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{1}{3}} = 3(axy)^{\frac{1}{3}}$$

Find the radius of curvature of $\boldsymbol{y^2 = 4ax}$

The parametric form is $x = at^2, y = 2at \ldots (1)$

$$f' = \frac{dx}{dt} = 2at, f'' = \frac{d^2x}{dt^2} = 2a, g' = \frac{dy}{dt} = 2a, g'' = \frac{d^2y}{dt^2} = 0$$

$$Radius\ of\ curvature\ \rho = \frac{((f')^2 + (g')^2)^{\frac{3}{2}}}{f'g'' - g'f''}$$

$$= \frac{((2at)^2 + (2a)^2)^{\frac{3}{2}}}{(2at)(0) - (2a)(2a)} = \frac{(2a)^3(t^2+1)^{\frac{3}{2}}}{-(2a)^2}$$

$$\rho = 2a(t^2+1)^{\frac{3}{2}}\ (Since\ radius\ cannot\ be\ negative)$$

Find the radius of curvature of $x^2 = 4ay$

The parametric form is $x = 2at, y = at^2$ ... (1)

$$f' = \frac{dx}{dt} = 2a, f'' = \frac{d^2x}{dt^2} = 0, g' = \frac{dy}{dt} = 2at, g'' = \frac{d^2y}{dt^2} = 2a$$

$$Radius\ of\ curvature\ \rho = \frac{((f')^2 + (g')^2)^{\frac{3}{2}}}{f'g'' - g'f''}$$

$$= \frac{((2a)^2 + (2at)^2)^{\frac{3}{2}}}{(2a)(2a) - (2at)(0)} = \frac{(2a)^3(1+t^2)^{\frac{3}{2}}}{(2a)^2}$$

$$\rho = 2a(1+t^2)^{\frac{3}{2}}$$

Centre of Curvature in Cartesian form:

$$\bar{X} = x - \frac{y_1(1+(y_1)^2)}{y_2}$$

$$\bar{Y} = y + \frac{(1+(y_1)^2)}{y_2}$$

Circle of curvature

$$(x-\bar{X})^2 + (y-\bar{Y})^2 = \rho^2$$

Find the circle of the curve $xy = c^2 at\ (c, c)$

$$xy = c^2 \ldots (1)$$

Differentiating (1) with respect to x, we get

$$xy_1 + y = 0 \Rightarrow y_1 = -\frac{y}{x}$$

$$(y_1)_{(c,c)} = -\frac{c}{c} = -1$$

$$y_2 = -\frac{xy_1 - y}{x^2}$$

$$(y_2)_{(c,c)} = -\frac{c(-1) - c}{c^2} = \frac{2c}{c^2} = \frac{2}{c}$$

$$\bar{X} = x - \frac{y_1(1 + (y_1)^2)}{y_2}$$

$$\bar{X} = c - \frac{-1(1 + (-1)^2)}{\left(\frac{2}{c}\right)} = c + \frac{2c}{2} = 2c$$

$$\bar{Y} = y + \frac{(1 + (y_1)^2)}{y_2}$$

$$\bar{Y} = c + \frac{(1 + (-1)^2)}{\left(\frac{2}{c}\right)} = c + \frac{2c}{2} = 2c$$

The center of curvature is $(2c, 2c)$

$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1 + (-1)^2)^{\frac{3}{2}}}{\left(\frac{2}{c}\right)} = \frac{2^{\frac{3}{2}}c}{2} = \sqrt{2}c$$

The circle of curvature is given by

$$(x - \bar{X})^2 + (y - \bar{Y})^2 = \rho^2$$

$$(x - 2c)^2 + (y - 2c)^2 = 2c^2$$

Centre of Curvature in parametric form:

IF $x = f(t), y = g(t)$

$$\bar{X} = x - \frac{\dot{y}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$where\ \dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \dot{y} = \frac{dy}{dt}, \ddot{y} = \frac{d^2y}{dt^2}$

Find the equation of the evolute of the curve $\boldsymbol{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$

Solution:

The parametric form is $x = a\cos^3\theta, y = \mathrm{a}\sin^3\theta$ ... (1)

$$\dot{x} = \frac{dx}{d\theta} = 3\,\mathrm{a}\cos^2\theta(-\sin\theta) = -3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -3\,\mathrm{a}\cos^2\theta\cos\theta - 6\,\mathrm{a}\cos\theta(-\sin\theta)\sin\theta = -3acos^3\theta + 6\,\mathrm{a}\cos\theta\sin^2\theta$$

$$\dot{y} = \frac{dy}{d\theta} = 3\,\mathrm{a}\sin^2\theta\cos\theta$$

$$\ddot{y} = \frac{d^2y}{d\theta^2} = 3\,\mathrm{a}\sin^2\theta(-\sin\theta) + 6\,\mathrm{a}\sin\theta\cos\theta\cos\theta = -3a\sin^3\theta + 6\,\mathrm{a}\sin\theta\cos^2\theta$$

$$\bar{X} = x - \frac{\dot{y}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= a\cos^3\theta - \frac{3\,\mathrm{a}\sin^2\theta\cos\theta\,((-3\,\mathrm{a}\cos^2\theta\sin\theta)^2 + (3\,\mathrm{a}\sin^2\theta\cos\theta)^2)}{(-3\,\mathrm{a}\cos^2\theta\sin\theta)(-3a\sin^3\theta + 6\,\mathrm{a}\sin\theta\cos^2\theta) - (3\,\mathrm{a}\sin^2\theta\cos\theta)(-3acos^3\theta + 6\,\mathrm{a}\cos\theta\sin^2\theta)}$$

$$= a\cos^3\theta - \frac{3\,\mathrm{a}\sin^2\theta\cos\theta\,(3^2\,\mathrm{a}^2\cos^4\theta\sin^2\theta + 3^2\,\mathrm{a}^2\cos^2\theta\sin^4\theta)}{(9a^2\cos^2\theta\sin^2\theta)(\sin^2\theta - 2\cos^2\theta + \cos^2\theta - 2\sin^2\theta)}$$

$$= a\cos^3\theta - \frac{(3\,\mathrm{a}\sin^2\theta\cos\theta)3^2a^2\cos^2\theta\sin^2\theta\,(\cos^2\theta + \sin^2\theta)}{(9a^2\cos^2\theta\sin^2\theta)(-\sin^2\theta - \cos^2\theta)}$$

$$= a\cos^3\theta - \frac{(3\,\mathrm{a}\sin^2\theta\cos\theta)}{-(\sin^2\theta + \cos^2\theta)}$$

$$\bar{X} = a\cos^3\theta + 3\,\mathrm{a}\sin^2\theta\cos\theta$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \mathrm{a}\sin^3\theta + \frac{-3\,\mathrm{a}\cos^2\theta\sin\theta\,((-3\,\mathrm{a}\cos^2\theta\sin\theta)^2 + (3\,\mathrm{a}\sin^2\theta\cos\theta)^2)}{(-3\,\mathrm{a}\cos^2\theta\sin\theta)(-3a\sin^3\theta + 6\,\mathrm{a}\sin\theta\cos^2\theta) - (3\,\mathrm{a}\sin^2\theta\cos\theta)(-3acos^3\theta + 6\,\mathrm{a}\cos\theta\sin^2\theta)}$$

$$= \mathrm{a}\sin^3\theta + \frac{-3\,\mathrm{a}\cos^2\theta\sin\theta\,(3^2\,\mathrm{a}^2\cos^4\theta\sin^2\theta + 3^2\,\mathrm{a}^2\cos^2\theta\sin^4\theta)}{(9a^2\cos^2\theta\sin^2\theta)(\sin^2\theta - 2\cos^2\theta + \cos^2\theta - 2\sin^2\theta)}$$

$$= a\sin^3\theta + \frac{(-3\,a\cos^2\theta\sin\theta)3^2a^2\cos^2\theta\sin^2\theta\,(\cos^2\theta+\sin^2\theta)}{(9a^2\cos^2\theta\sin^2\theta)(-\sin^2\theta-\cos^2\theta)}$$

$$= a\sin^3\theta + \frac{(-3\,a\cos^2\theta\sin\theta)}{-(\sin^2\theta+\cos^2\theta)}$$

$$\bar{Y} = a\sin^3\theta + 3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$\bar{X}+\bar{Y} = a\cos^3\theta + 3\,\mathrm{a}\sin^2\theta\cos\theta + a\sin^3\theta + 3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$\bar{X}+\bar{Y} = a(\cos\theta+\sin\theta)^3 \Rightarrow (\bar{X}+\bar{Y})^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta+\sin\theta)\ \ldots(2)$$

$$\bar{X}-\bar{Y} = a\cos^3\theta + 3\,\mathrm{a}\sin^2\theta\cos\theta - a\sin^3\theta - 3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$\bar{X}-\bar{Y} = a(\cos\theta-\sin\theta)^3 \Rightarrow (\bar{X}-\bar{Y})^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta-\sin\theta)\ \ldots(3)$$

$(2)^2 + (3)^2$ *gives*

$$(\bar{X}+\bar{Y})^{\frac{2}{3}} + (\bar{X}-\bar{Y})^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos\theta+\sin\theta)^2 + a^{\frac{2}{3}}(\cos\theta-\sin\theta)^2$$

$$= a^{\frac{2}{3}}(\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta)$$

$$(\bar{X}+\bar{Y})^{\frac{2}{3}} + (\bar{X}-\bar{Y})^{\frac{2}{3}} = 2a^{\frac{2}{3}}\ \ldots(4)$$

The locus of (4) *is* the evolute of the given curve

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

Find the equation of the evolute of the curve $y^2 = 4ax$.

Solution:

The parametric form is $x = at^2, y = 2at\ \ \ldots(1)$

$$\dot{x} = \frac{dx}{dt} = 2at, \ddot{x} = \frac{d^2x}{dt^2} = 2a, \dot{y} = \frac{dy}{dt} = 2a, \ddot{y} = \frac{d^2y}{dt^2} = 0$$

$$\bar{X} = x - \frac{\dot{y}((\dot{x})^2+(\dot{y})^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\bar{X} = at^2 - \frac{2a((2at)^2+(2a)^2)}{(2at)(0)-(2a)(2a)} = at^2 + \frac{\left((2a)^3(t^2+1)\right)}{(2a)^2}$$

$$\bar{X} = at^2 + 2a(t^2+1) = at^2 + 2at^2 + 2a = 3at^2 + 2a$$

$$\bar{X} - 2a = 3at^2 \Rightarrow \frac{\bar{X} - 2a}{3a} = t^2 \Rightarrow t = \left(\frac{\bar{X} - 2a}{3a}\right)^{\frac{1}{2}} \dots (2)$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\bar{Y} = 2at + \frac{2at((2at)^2 + (2a)^2)}{(2at)(0) - (2a)(2a)} = 2at - \frac{\left((2a)^3 t(t^2 + 1)\right)}{(2a)^2}$$

$$\bar{Y} = 2at - 2at(t^2 + 1) = 2at - 2at^3 - 2at = -2at^3$$

$$\bar{Y} = -2at^3 \Rightarrow \frac{\bar{Y}}{-2a} = t^3 \Rightarrow t = \left(\frac{\bar{Y}}{-2a}\right)^{\frac{1}{3}} \dots (3)$$

The center of curvature is $(3at^2 + 2a, -2at^3)$

From (2) and (3), we get

$$\left(\frac{\bar{Y}}{-2a}\right)^{\frac{1}{3}} = \left(\frac{\bar{X} - 2a}{3a}\right)^{\frac{1}{2}}$$

Raising the powers with 6 on both sides, we get

$$\left(\frac{\bar{Y}}{-2a}\right)^2 = \left(\frac{\bar{X} - 2a}{3a}\right)^3 \Rightarrow \frac{\bar{Y}^2}{4a^2} = \frac{(\bar{X} - 2a)^3}{27a^3}$$

$$27a\bar{Y}^2 = 4\ (\bar{X} - 2a)^3$$

The locus of (4) is the evolute of the given curve

$$27ay^2 = 4\ (x - 2a)^3$$

Find the equation of the evolute of the curve $x^2 = 4ay$.

Solution:

The parametric form is $x = 2at, y = at^2$ $\dots (1)$

$$\dot{x} = \frac{dx}{dt} = 2a, \ddot{x} = \frac{d^2x}{dt^2} = 0, \dot{y} = \frac{dy}{dt} = 2at, \ddot{y} = \frac{d^2y}{dt^2} = 2a$$

$$\bar{X} = x - \frac{\dot{y}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\bar{X} = 2at - \frac{2at((2a)^2 + (2at)^2)}{(2a)(2a) - (2at)(0)} = 2at - \frac{((2a)^3 t(1+t^2))}{(2a)^2}$$

$$\bar{X} = 2at - 2at(t^2+1) = 2at - 2at^3 - 2at = -2at^3$$

$$\bar{X} = -2at^3 \Rightarrow \frac{\bar{X}}{-2a} = t^3 \Rightarrow t = \left(\frac{\bar{X}}{-2a}\right)^{\frac{1}{3}} \dots (2)$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\bar{Y} = at^2 + \frac{2a((2at)^2 + (2a)^2)}{(2a)(2a) - (2at)(0)} = 2at + \frac{((2a)^3(t^2+1))}{(2a)^2}$$

$$\bar{Y} = at^2 + 2a(t^2+1) = at^2 + 2at^2 + 2a = 3at^2 + 2a$$

$$\bar{Y} - 2a = 3at^2 \Rightarrow \frac{\bar{Y} - 2a}{3a} = t^2 \Rightarrow t = \left(\frac{\bar{Y} - 2a}{3a}\right)^{\frac{1}{2}} \dots (3)$$

The center of curvature is $(-2at^3, 3at^2 + 2a\ )$

From (2) and (3), we get

$$\left(\frac{\bar{X}}{-2a}\right)^{\frac{1}{3}} = \left(\frac{\bar{Y} - 2a}{3a}\right)^{\frac{1}{2}}$$

Raising the powers with 6 on both sides, we get

$$\left(\frac{\bar{X}}{-2a}\right)^{2} = \left(\frac{\bar{Y} - 2a}{3a}\right)^{3} \Rightarrow \frac{\bar{X}^2}{4a^2} = \frac{(\bar{Y} - 2a)^3}{27a^3}$$

$$27a\bar{X}^2 = 4\ \ (\bar{Y} - 2a)^3$$

The locus of (4) is the evolute of the given curve

$$27ax^2 = 4\ \ (y - 2a)^3$$

Find the equation of the evolute of the curve $xy = c^2$.

Solution:

The parametric form is $x = ct, y = \frac{c}{t}$ … (1)

$$\dot{x}=\frac{dx}{dt}=c,\ddot{x}=\frac{d^2x}{dt^2}=0,\dot{y}=\frac{dy}{dt}=-\frac{c}{t^2},\ddot{y}=\frac{d^2y}{dt^2}=\frac{2c}{t^3}$$

$$\bar{X}=x-\frac{\dot{y}((\dot{x})^2+(\dot{y})^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\bar{X}=ct-\frac{\left(-\frac{c}{t^2}\right)\left((c)^2+\left(-\frac{c}{t^2}\right)^2\right)}{(c)\left(\frac{2c}{t^3}\right)-\left(-\frac{c}{t^2}\right)(0)}=ct+\frac{\left(\frac{c^3}{t^2}\right)\left(1+\frac{1}{t^4}\right)}{(c)\left(\frac{2c}{t^3}\right)}$$

$$\bar{X}=ct+\frac{ct\left(1+\frac{1}{t^4}\right)}{2}=\frac{2ct+ct+\frac{c}{t^3}}{2}=\frac{3ct+\frac{c}{t^3}}{2}$$

$$\bar{X}=\frac{c}{2}\left(3t+\frac{1}{t^3}\right)$$

$$\bar{Y}=y+\frac{\dot{x}((\dot{x})^2+(\dot{y})^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\bar{Y}=\frac{c}{t}+\frac{(c)\left((c)^2+\left(-\frac{c}{t^2}\right)^2\right)}{(c)\left(\frac{2c}{t^3}\right)-\left(-\frac{c}{t^2}\right)(0)}=\frac{c}{t}+\frac{ct^3\left(1+\frac{1}{t^4}\right)}{2}$$

$$\bar{Y}=\frac{2c+ct^4+c}{2t}=\frac{ct^4+3c}{2t}=\frac{c}{2}\left(t^3+\frac{3}{t}\right)$$

The center of curvature is $\left(\frac{c}{2}\left(3t+\frac{1}{t^3}\right),\frac{c}{2}\left(t^3+\frac{3}{t}\right)\right)$

$$\bar{X}+\bar{Y}=\frac{c}{2}\left(3t+\frac{1}{t^3}+t^3+\frac{3}{t}\right)=\frac{c}{2}\left(t^3+3t+\frac{3}{t}+\frac{1}{t^3}\right)=\frac{c}{2}\left(t+\frac{1}{t}\right)^3$$

$$(\bar{X}+\bar{Y})^{\frac{1}{3}}=\left(\frac{c}{2}\right)^{\frac{1}{3}}\left(t+\frac{1}{t}\right)\ldots(2)$$

$$\bar{X}-\bar{Y}=\frac{c}{2}\left(3t+\frac{1}{t^3}-t^3-\frac{3}{t}\right)=-\frac{c}{2}\left(t^3-3t+\frac{3}{t}-\frac{1}{t^3}\right)=-\frac{c}{2}\left(t-\frac{1}{t}\right)^3$$

$$(\bar{X}-\bar{Y})^{\frac{1}{3}}=\left(-\frac{c}{2}\right)^{\frac{1}{3}}\left(t-\frac{1}{t}\right)\ldots(3)$$

$(2)^2-(3)^2\ gives$

$$(\bar{X}+\bar{Y})^{\frac{2}{3}}-(\bar{X}-\bar{Y})^{\frac{2}{3}}=\left(\frac{c}{2}\right)^{\frac{2}{3}}\left(t+\frac{1}{t}\right)^{2}-\left(-\frac{c}{2}\right)^{\frac{2}{3}}\left(t-\frac{1}{t}\right)^{2}$$

$$(\bar{X}+\bar{Y})^{\frac{2}{3}}-(\bar{X}-\bar{Y})^{\frac{2}{3}}=\left(\frac{c}{2}\right)^{\frac{2}{3}}\left[\left(t+\frac{1}{t}\right)^{2}-\left(t-\frac{1}{t}\right)^{2}\right]=\left(\frac{c}{2}\right)^{\frac{2}{3}}\left(t^2+2+\frac{1}{t^2}-\left(t^2-2+\frac{1}{t^2}\right)\right)$$

$$(\bar{X}+\bar{Y})^{\frac{2}{3}}-(\bar{X}-\bar{Y})^{\frac{2}{3}}=4\left(\frac{c}{2}\right)^{\frac{2}{3}}$$

$$(\bar{X}+\bar{Y})^{\frac{2}{3}}-(\bar{X}-\bar{Y})^{\frac{2}{3}}=(4c)^{\frac{2}{3}}$$

The locus of (4) is the evolute of the given curve

$$(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}$$

Find the equation of the evolute of the curve $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$

Solution:

The parametric form is $x=a\,cos\,\theta\,,y=b\,sin\,\theta$ ... (1)

$$\dot{x}=\frac{dx}{d\theta}=-a\sin\theta\,,\ddot{x}=\frac{d^2x}{d\theta^2}=-a\cos\theta\,,\dot{y}=\frac{dy}{d\theta}=b\cos\theta\,,\ddot{y}=\frac{d^2y}{d\theta^2}=-b\sin\theta$$

$$\bar{X}=x-\frac{\dot{y}((\dot{x})^2+(\dot{y})^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$=a\,cos\,\theta-\frac{b\cos\theta\,((-a\sin\theta)^2+(b\cos\theta)^2)}{(-a\sin\theta)(-b\sin\theta)-(b\cos\theta)(-a\cos\theta)}$$

$$=a\,cos\,\theta-\frac{b\cos\theta\,(a^2\sin^2\theta+b^2\cos^2\theta)}{ab\sin^2\theta+ab\cos^2\theta}$$

$$=a\,cos\,\theta-\frac{b\cos\theta\,(a^2\sin^2\theta+b^2\cos^2\theta)}{ab\,(\sin^2\theta+\cos^2\theta)}$$

$$=a\,cos\,\theta-\frac{\cos\theta\,(a^2\sin^2\theta+b^2\cos^2\theta)}{a}=\frac{a^2\,cos\,\theta-\cos\theta\,(a^2(1-\cos^2\theta)+b^2\cos^2\theta)}{a}$$

$$\bar{X}=\frac{a^2\,cos\,\theta-a^2\cos\theta+(a^2\cos^3\theta-b^2\cos^3\theta)}{a}=\frac{(a^2-b^2)\cos^3\theta}{a}$$

$$a\bar{X} = (a^2 - b^2)\cos^3\theta \Rightarrow \frac{a\bar{X}}{a^2 - b^2} = \cos^3\theta \Rightarrow \left(\frac{a\bar{X}}{a^2 - b^2}\right)^{\frac{1}{3}} = \cos\theta \;...(2)$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= b\,sin\,\theta + \frac{-a\,sin\,\theta\,((-a\sin\theta)^2 + (b\cos\theta)^2)}{(-a\sin\theta)(-b\sin\theta) - (b\cos\theta)(-a\cos\theta)}$$

$$= b\,sin\,\theta - \frac{a\,sin\,\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)}{ab\sin^2\theta + ab\cos^2\theta}$$

$$= b\,sin\,\theta - \frac{a\,sin\,\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)}{ab\,(\sin^2\theta + \cos^2\theta)}$$

$$= b\,sin\,\theta - \frac{sin\,\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)}{b} = \frac{b^2\,sin\,\theta - sin\,\theta\,(a^2\sin^2\theta + b^2(1 - \sin^2\theta))}{b}$$

$$\bar{Y} = \frac{b^2\,sin\,\theta - a^2\sin^3\theta + b^2\,sin\,\theta + b^2\sin^3\theta}{b} = \frac{-(a^2 - b^2)\sin^3\theta}{b}$$

$$b\bar{Y} = -(a^2 - b^2)\sin^3\theta \Rightarrow \frac{-b\bar{Y}}{a^2 - b^2} = \sin^3\theta \Rightarrow \left(\frac{-b\bar{Y}}{a^2 - b^2}\right)^{\frac{1}{3}} = \sin\theta \;...(3)$$

$(2)^2 + (3)^2$ *gives*

$$\left(\frac{a\bar{X}}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{-b\bar{Y}}{a^2 - b^2}\right)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta = 1$$

$$(a\bar{X})^{\frac{2}{3}} + (b\bar{Y})^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}} \;...(4)$$

The locus of (4) is the evolute of the given curve

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

Find the equation of the evolute of the curve $x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta)$

Solution:

$$x = a(\theta - sin\,\theta), y = a(1 - cos\,\theta) \;\ldots (1)$$

$$\dot{x} = \frac{dx}{d\theta} = a(1 - cos\,\theta), \ddot{x} = \frac{d^2x}{d\theta^2} = a\sin\theta\,, \dot{y} = \frac{dy}{d\theta} = a\sin\theta\,, \ddot{y} = \frac{d^2y}{d\theta^2} = a\,cos\,\theta$$

$$\bar{X} = x - \frac{\dot{y}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= a(\theta - sin\,\theta) - \frac{a\sin\theta\left(\left(a(1 - cos\,\theta)\right)^2 + (a\sin\theta)^2\right)}{a(1 - cos\,\theta)(a\,cos\,\theta) - (a\sin\theta)(a\sin\theta)}$$

$$= a(\theta - sin\,\theta) - \frac{a\sin\theta\,(a^2 + a^2\cos^2\theta - 2a^2\,cos\,\theta + a^2\sin^2\theta)}{a^2\,cos\,\theta - a^2\cos^2\theta - a^2\sin^2\theta}$$

$$= a\,\theta - a\,sin\,\theta - \frac{a\sin\theta\,(a^2 + a^2 - 2a^2\,cos\,\theta)}{a^2\,cos\,\theta - a^2} = a\,\theta - a\,sin\,\theta - \frac{a\sin\theta\,(2a^2 - 2a^2\,cos\,\theta)}{a^2\,cos\,\theta - a^2}$$

$$= a\,\theta - a\sin\theta + \frac{2a\sin\theta\,(a^2\,cos\,\theta - a^2)}{a^2\,cos\,\theta - a^2} = a\,\theta - a\,sin\,\theta + 2a\sin\theta$$

$$\bar{X} = a\theta + a\sin\theta \ldots (2)$$

$$\bar{Y} = y + \frac{\dot{x}((\dot{x})^2 + (\dot{y})^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= a(1 - cos\,\theta) + \frac{a(1 - cos\,\theta)\left(\left(a(1 - cos\,\theta)\right)^2 + (a\sin\theta)^2\right)}{a(1 - cos\,\theta)(a\,cos\,\theta) - (a\sin\theta)(a\sin\theta)}$$

$$= a(1 - cos\,\theta) + \frac{a(1 - cos\,\theta)(a^2 + a^2\cos^2\theta - 2a^2\,cos\,\theta + a^2\sin^2\theta)}{a^2\,cos\,\theta - a^2\cos^2\theta - a^2\sin^2\theta}$$

$$= a(1 - cos\,\theta) + \frac{a(1 - cos\,\theta)(a^2 + a^2 - 2a^2\,cos\,\theta)}{a^2\,cos\,\theta - a^2}$$

$$= a(1 - cos\,\theta) + \frac{a(1 - cos\,\theta)(2a^2 - 2a^2\,cos\,\theta)}{a^2\,cos\,\theta - a^2}$$

$$= a(1 - cos\,\theta) - \frac{2a(1 - cos\,\theta)(a^2\,cos\,\theta - a^2)}{a^2\,cos\,\theta - a^2} = a(1 - cos\,\theta) - 2a(1 - cos\,\theta)$$

$$\bar{Y} = -a(1 - cos\,\theta) \ldots (3)$$

The locus of (2) *and* (3) *is* the evolute of the given curve

$$x = a\theta + a\sin\theta, y = -a(1 - cos\,\theta)$$

Envelope

Find the envelope of the straight line $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ where $'\theta'$ is the parameter.

Solution:

$$\frac{ax}{cos\ \theta} - \frac{by}{sin\ \theta} = a^2 - b^2 \ldots (1)$$

$$ax\ sec\ \theta - by\ cosec\ \theta = a^2 - b^2 \ldots (2)$$

Differentiating (2) with respect to θ, we get

$$ax\ sec\theta \tan\ \theta + by\ cosec\ \theta \cot\theta = 0 \Rightarrow ax\ sec\theta \tan\ \theta = -by\ cosec\ \theta \cot\theta$$

$$\frac{sec\theta \tan\ \theta}{cosec\ \theta \cot\theta} = -\frac{by}{ax} \Rightarrow \frac{\frac{1}{\cos\theta}\tan\ \theta}{\frac{1}{\sin\theta}\cot\theta} = -\frac{by}{ax} \Rightarrow \frac{\sin\theta}{\cos\theta}\tan\ \theta \tan\ \theta = -\frac{by}{ax}$$

$$\tan^3\ \theta = -\frac{by}{ax} \Rightarrow \tan\ \theta = \frac{(-by)^{\frac{1}{3}}}{(ax)^{\frac{1}{3}}} = \frac{opp}{adj}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{opp}{\sqrt{opp^2 + adj^2}} = \frac{(-by)^{\frac{1}{3}}}{\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}} \ldots (3)$$

$$\cos\theta = \frac{adj}{hyp} = \frac{adj}{\sqrt{opp^2 + adj^2}} = \frac{(ax)^{\frac{1}{3}}}{\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}} \ldots (4)$$

Substituting (3) and (4) in (1), we get

$$\frac{ax}{\frac{(ax)^{\frac{1}{3}}}{\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}} - \frac{by}{\frac{(-by)^{\frac{1}{3}}}{\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}} = a^2 - b^2$$

$$(ax)^{\frac{2}{3}}\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}} + (-by)^{\frac{2}{3}}\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}} = a^2 - b^2$$

$$\left((ax)^{\frac{2}{3}} + (-by)^{\frac{2}{3}}\right)\sqrt{(-by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}} = a^2 - b^2$$

$$\left((ax)^{\frac{2}{3}} + (-by)^{\frac{2}{3}}\right)^{\frac{3}{2}} = a^2 - b^2 \Rightarrow (ax)^{\frac{2}{3}} + (-by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\ which\ gives$$

$$the\ envelope\ of\ family\ of\ given\ curve$$

Find the envelope of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters $'a'$ and $'b'$ are related by the equation $a + b = c$.

Solution:

$\frac{x}{a} + \frac{y}{b} = 1 \dots (1)$

$$a + b = c \Rightarrow\ b = c - a \dots (2)$$

Substituting (2) in (1), we get

$\frac{x}{a} + \frac{y}{c - a} = 1$

$(c - a)x + ay = a(c - a) \Rightarrow a^2 + (y - x - c)a + cx = 0$

The above equation is quadratic in a.

Here $A = 1, B = (y - x - c), C = cx$

$B^2 - 4AC = 0$

$(y - x - c)^2 - 4cx = 0 \Rightarrow\ (y - x - c)^2 = 4cx \Rightarrow\ (y - x - c) = \sqrt{4cx}$

$y = x + c + 2\sqrt{x}\sqrt{c}\ \Rightarrow\ y = \left(\sqrt{x} + \sqrt{c}\right)^2 \Rightarrow \pm\sqrt{y} = \sqrt{x} + \sqrt{c}$

$\sqrt{x} + \sqrt{y} = \sqrt{c}$ which gives the envelope of the family of curves.

Find the envelope of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters $'a'$ and

$'b'$ are related by the equation $ab = c^2$.

Solution:

$\frac{x}{a} + \frac{y}{b} = 1 \ldots (1)$

$$ab = c^2 \Rightarrow b = \frac{c^2}{a} \ldots (2)$$

Substituting (2) in (1), we get

$\frac{x}{a} + \frac{y}{\left(\frac{c^2}{a}\right)} = 1$

$c^2x + a^2y = ac^2 \Rightarrow a^2y - ac^2 + c^2x = 0$

The above equation is quadratic in a.

Here $A = y, B = -c^2, C = c^2x$

$B^2 - 4AC = 0$

$(-c^2)^2 - 4c^2xy = 0 \Rightarrow c^4 = 4c^2xy \Rightarrow 4xy = c^2$

$4xy = c^2$ which gives the envelope of the family of curves.

Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters $'a'$ and $'b'$ are related by the equation $a^2 + b^2 = c^2$.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots (1)$$

$$a^2 + b^2 = c^2 \ldots (2)$$

Differentiating (1) and (2) with respect to a, we get

$$-\frac{2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0 \Rightarrow -\frac{2x^2}{a^3} = \frac{2y^2}{b^3}\frac{db}{da} \Rightarrow \frac{db}{da} = -\frac{b^3x^2}{y^2a^3} \ldots (3)$$

$$2a + 2b\frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{a}{b} \ldots (4)$$

From (3) and (4), we get

$$-\frac{b^3x^2}{y^2a^3} = -\frac{a}{b} \Rightarrow \frac{x^2}{a^4} = \frac{y^2}{b^4} \Rightarrow \frac{\frac{x^2}{a^2}}{a^2} = \frac{\frac{y^2}{b^2}}{b^2} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^2+b^2} = \frac{1}{c^2}\big(from\ (1)\ and\ (2)\big)$$

$$\frac{x^2}{a^4} = \frac{1}{c^2} \Rightarrow a^4 = c^2x^2 \Rightarrow a^2 = cx \ldots (5)$$

$$\frac{y^2}{b^4} = \frac{1}{c^2} \Rightarrow b^4 = c^2y^2 \Rightarrow b^2 = cy \ldots (6)$$

Substituting (5) and (6) in (1), we get

$$\frac{x^2}{cx} + \frac{y^2}{cy} = 1 \Rightarrow x + y = c$$

$x + y = c$ which gives the envelope of the family of curves.

Evolutes as the envelope of the normal

Find the equation of the evolute as the envelope of the normal of the curve

$\boldsymbol{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}}$

Solution:

The parametric form is $x = a\,cos^3\,\theta\,, y = a\,sin^3\theta\ \ \ldots (1)$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta\,, \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = \frac{-\sin\theta}{\cos\theta}$$

Equation of normal is given is by

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - a\,sin^3\theta = -\frac{1}{\frac{-\sin\theta}{\cos\theta}}(x - a\,cos^3\,\theta)$$

$$y - a\,sin^3\theta = \frac{\cos\theta}{\sin\theta}(x - a\,cos^3\,\theta)$$

$$y \sin\theta - a\sin^4\theta = x\cos\theta - a\cos^4\theta$$

$$y \sin\theta - a\sin^4\theta - x\cos\theta + a\cos^4\theta = 0$$

$$y \sin\theta - x\cos\theta + a\,(\cos^4\theta - \sin^4\theta) = 0$$

$$y \sin\theta - x\cos\theta + a\,(\cos^2\theta - \sin^2\theta)\,(\cos^2\theta + \sin^2\theta) = 0$$

$$y \sin\theta - x\cos\theta + a\,(\cos^2\theta - \sin^2\theta) = 0$$

$$y \sin\theta - x\cos\theta = -a\,(\cos^2\theta - \sin^2\theta) \ldots (2)$$

To find the envelope:

Differentiating (2) partially with respect to θ, we get

$$y\cos\theta + x\sin\theta = -a\,(-2\cos\theta\sin\theta - 2\cos\theta\sin\theta)$$

$$y\cos\theta + x\sin\theta = 4a\cos\theta\sin\theta \ldots (3)$$

$(2) \times \cos\theta - (3) \times \sin\theta$, we get

$$y\cos\theta\sin\theta - x\cos^2\theta - y\cos\theta\sin\theta - x\sin^2\theta$$
$$= -a\cos^3\theta + a\cos\theta\sin^2\theta - 4a\cos\theta\sin^2\theta$$

$$-x\,(\cos^2\theta + \sin^2\theta) = -a\cos^3\theta - 3a\cos\theta\sin^2\theta$$

$$x = a\cos^3\theta + 3a\cos\theta\sin^2\theta$$

$(2) \times \sin\theta + (3) \times \cos\theta$, we get

$$y\sin^2\theta - x\cos\theta\sin\theta + y\cos^2\theta + x\cos\theta\sin\theta$$
$$= -a\sin\theta\cos^2\theta + a\sin^3\theta + 4a\cos^2\theta\sin\theta$$

$$y\,(\sin^2\theta + \cos^2\theta) = a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$y = a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$x + y = a\cos^3\theta + 3\,\mathrm{a}\sin^2\theta\cos\theta + \mathrm{a}\sin^3\theta + 3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$x + y = a(\cos\theta + \sin\theta)^3 \Rightarrow (x+y)^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta + \sin\theta) \ldots (4)$$

$$x - y = a\cos^3\theta + 3\,\mathrm{a}\sin^2\theta\cos\theta - \mathrm{a}\sin^3\theta - 3\,\mathrm{a}\cos^2\theta\sin\theta$$

$$x - y = a(\cos\theta - \sin\theta)^3 \Rightarrow (\bar{X} - \bar{Y})^{\frac{1}{3}} = a^{\frac{1}{3}}(\cos\theta - \sin\theta) \ldots (5)$$

$(4)^2 + (5)^2$ *gives*

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos\theta + \sin\theta)^2 + a^{\frac{2}{3}}(\cos\theta - \sin\theta)^2$$

$$= a^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta)$$

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$$

Find the evolute of the parabola $y^2 = 4ax$ considering it as the envelope of its normals.

Solution:

The parametric form is $x = at^2, y = 2at$ ... (1)

$$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Equation of normal is given is by

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 2at = -\frac{1}{\left(\frac{1}{t}\right)}(x - at^2)$$

$y - 2at = -t(x - at^2) \Rightarrow y + xt = 2at + at^3$ ... (2)

To find the envelope:

Differentiating (2) partially with respect to t, we get

$$x = 2a + 3at^2$$

$$\Rightarrow \frac{x-2a}{3a} = t^2 \Rightarrow \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}} = t \text{ ... } (3)$$

Substituting $x = 2a + 3at^2$ in (2)

$$y + (2a + 3at^2)t = 2at + at^3$$

$$y = -2at - 3at^3 + 2at + at^3 = -2at^3$$

$$\Rightarrow \frac{y}{-2a} = t^3 \ \Rightarrow \left(\frac{y}{-2a}\right)^{\frac{1}{3}} = t \ldots (4)$$

From (3) and (4)

$$\left(\frac{y}{-2a}\right)^{\frac{1}{3}} = \left(\frac{x-2a}{3a}\right)^{\frac{1}{2}}$$

Raising the powers with 6 on both sides, we get

$$\left(\frac{y}{-2a}\right)^2 = \left(\frac{x-2a}{3a}\right)^3 \Rightarrow \frac{y^2}{4a^2} = \frac{(x-2a)^3}{27a^3}$$

$$27ay^2 = 4\ (x-2a)^3$$

Find the evolute of the parabola $x^2 = 4ay$ considering it as the envelope of its normals.

Solution:

The parametric form is $x = 2at, y = at^2 \ldots (1)$

$$\frac{dx}{dt} = 2a, \frac{dy}{dt} = 2at$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2at}{2a} = t$$

Equation of normal is given is by

$$y - y_1 = -\frac{1}{t}(x - x_1)$$

$y - at^2 = -\frac{1}{t}(x - 2at)$

$yt - at^3 = -(x - 2at) \Rightarrow yt - at^3 = -x + 2at \ldots (2)$

To find the envelope:

Differentiating (2) partially with respect to t, we get

$$y - 3at^2 = 2a \Rightarrow y = 3at^2 + 2a$$

$$\Rightarrow \frac{y-2a}{3a} = t^2 \Rightarrow \left(\frac{y-2a}{3a}\right)^{\frac{1}{2}} = t \ldots (3)$$

Substituting $y = 2a + 3at^2$ in (2)

$$(2a + 3at^2)t - at^3 = -x + 2at$$

$$2at + 3at^3 - at^3 = -x + 2at \Rightarrow 2at^3 = -x$$

$$\frac{x}{-2a} = t^3 \Rightarrow \left(\frac{x}{-2a}\right)^{\frac{1}{3}} = t \ldots (4)$$

From (3) and (4)

$$\left(\frac{x}{-2a}\right)^{\frac{1}{3}} = \left(\frac{y-2a}{3a}\right)^{\frac{1}{2}}$$

Raising the powers with 6 on both sides, we get

$$\left(\frac{x}{-2a}\right)^2 = \left(\frac{y-2a}{3a}\right)^3 \Rightarrow \frac{x^2}{4a^2} = \frac{(y-2a)^3}{27a^3}$$

$$27ax^2 = 4\ (y-2a)^3$$

Find the equation of the evolute as the envelope of the normal of the curve $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Solution:

The parametric form is $x = a\cos\theta\,, y = b\sin\theta\ \ldots (1)$

$$\frac{dx}{d\theta} = -a\sin\theta\,, \frac{dy}{d\theta} = b\cos\theta$$

$$m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$$

Equation of normal is given is by

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - b\,sin\,\theta = -\frac{1}{\frac{b\cos\theta}{-a\sin\theta}}(x - a\,cos\,\theta)$$

$$y - b\,sin\,\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\,cos\,\theta)$$

$$yb\cos\theta - b^2\cos\theta\,sin\,\theta = xa\sin\theta - a^2\cos\theta\,sin\,\theta$$

$$yb\cos\theta - xa\sin\theta + a^2\cos\theta\,sin\,\theta - b^2\cos\theta\,sin\,\theta = 0$$

$$yb\cos\theta - xa\sin\theta + (a^2 - b^2)\cos\theta\,sin\,\theta = 0 \ldots (2)$$

To find the envelope:

Differentiating (2) partially with respect to θ, we get

$$-yb\sin\theta - xa\cos\theta + (a^2 - b^2)(\cos^2\theta - \sin^2\theta) = 0 \ldots (3)$$

$(2) \times\ sin\,\theta +\ (3) \times\ \cos\theta$, we get

$$yb\cos\theta\sin\theta - xa\,sin^2\,\theta - y\,b\cos\theta\sin\theta - xa\,cos^2\,\theta$$
$$+ (a^2 - b^2)\cos\theta\sin^2\theta\ + (a^2 - b^2)\cos\theta\,(\cos^2\theta - \sin^2\theta)\ = 0$$

$$-ax\,(cos^2\,\theta + sin^2\,\theta) + (a^2 - b^2)\cos\theta\,(\sin^2\theta + \cos^2\theta - \sin^2\theta) = 0$$

$$-ax + (a^2 - b^2)\cos^3\theta = 0 \Rightarrow ax = (a^2 - b^2)\cos^3\theta \Rightarrow \frac{ax}{a^2 - b^2} = \cos^3\theta$$

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{1}{3}} = cos\,\theta \ldots (4)$$

$(2) \times\ cos\,\theta -\ (3) \times\ \sin\theta$, we get

$$yb\cos^2\theta - xa\cos\theta\sin\theta + yb\sin^2\theta + xa\cos\theta\sin\theta$$
$$+ (a^2 - b^2)\sin\theta\cos^2\theta - (a^2 - b^2)\sin\theta\,(\cos^2\theta - \sin^2\theta)\ = 0$$

$$yb + (a^2 - b^2)\sin\theta\,(\cos^2\theta\ - \cos^2\theta + \sin^2\theta) = 0$$

$$yb + (a^2 - b^2)\sin^3\theta = 0 \Rightarrow by = -(a^2 - b^2)\sin^3\theta \Rightarrow \frac{-yb}{a^2 - b^2} = \sin^3\theta$$

$$\left(\frac{-by}{a^2 - b^2}\right)^{\frac{1}{3}} = sin\,\theta \ldots (5)$$

$(4)^2 + (5)^2$ *gives*

$$\left(\frac{ax}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{-by}{a^2 - b^2}\right)^{\frac{2}{3}} = cos^2\,\theta + sin^2\,\theta = 1$$

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$