Tutorial - 1

1. Obtain the principal disjunctive normal form and principal conjunction form of the statement

$$\boldsymbol{P \vee \left(\sim P \to (Q \vee (\sim Q \to R))\right)}$$

Solution:

Let $S \Leftrightarrow \; P \vee \left(\sim P \to (Q \vee (\sim Q \to R))\right)$

$A: \sim P \to (Q \vee (\sim Q \to R))$

$\mathbf{P}$	$\boldsymbol{Q}$	$\boldsymbol{R}$	$\boldsymbol{\sim P}$	$\boldsymbol{\sim Q}$	$\boldsymbol{\sim Q \to R}$	$\boldsymbol{Q \vee (\sim Q \to R)}$	$\boldsymbol{A}$	$\mathbf{S}$	$\boldsymbol{Minterm}$	$\boldsymbol{Maxterm}$
T	T	T	F	F	T	T	T	T	$\boldsymbol{P \wedge Q \wedge R}$	
T	F	T	F	T	T	T	T	T	$\boldsymbol{P \wedge \sim Q \wedge R}$	
F	T	T	T	F	T	T	T	T	$\boldsymbol{\sim P \wedge Q \wedge R}$	
F	F	T	T	T	T	T	T	T	$\boldsymbol{\sim P \wedge \sim Q \wedge R}$	
T	T	F	F	F	T	T	T	T	$\boldsymbol{P \wedge Q \wedge \sim R}$	
T	F	F	F	T	F	F	T	T	$\boldsymbol{P \wedge \sim Q \wedge \sim R}$	
F	T	F	T	F	T	T	T	T	$\boldsymbol{\sim P \wedge Q \wedge \sim R}$	
F	F	F	T	T	F	F	F	F		$\mathbf{P \vee Q \vee R}$

$S \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge \sim Q \wedge R) \vee (P \wedge Q \wedge \sim R)$
$\vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R)$ is a PDNF

$S \Leftrightarrow \mathrm{P \vee Q \vee R}$ is a PCNF

2. Show that $\mathbf{P \vee (Q \wedge R)}$ and $\mathbf{(P \vee Q) \wedge (P \vee R)}$ are logically equivalent.

Solution:

To prove: $S: \left(\mathrm{P \vee (Q \wedge R\,)}\right) \leftrightarrow \left(\mathrm{(P \vee Q) \wedge (P \vee R)}\right)$ is a tautology.

$\mathbf{P}$	$\boldsymbol{Q}$	$\boldsymbol{R}$	$\mathbf{Q \wedge R}$	$\mathbf{P \vee (Q \wedge R\,)}$	$\mathbf{P \vee Q}$	$\mathbf{P \vee R}$	$\mathbf{(P \vee Q) \wedge (P \vee R)}$	$\boldsymbol{S}$
T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F	T
T	T	F	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	F	F	F	F	F	F	T

Since all the values in last column are true. $\left(\mathrm{P \vee (Q \wedge R\,)}\right) \leftrightarrow \left(\mathrm{(P \vee Q) \wedge (P \vee R)}\right)$ is a tautology.

$$\therefore \left(\mathrm{P \vee (Q \wedge R\,)}\right) \Leftrightarrow \left(\mathrm{(P \vee Q) \wedge (P \vee R)}\right)$$

3. Find the principal disjunctive normal form of the statement $\boldsymbol{\left(q \vee (p \wedge r)\right) \wedge \sim \left((p \vee r) \wedge q\right)}$.

Solution:

Let S $\Leftrightarrow \left(q \vee (p \wedge r)\right) \wedge \sim \left((p \vee r) \wedge q\right).$

$\Leftrightarrow \left(q \vee (p \wedge r)\right) \wedge (\sim (p \vee r) \vee \sim q)$

$\Leftrightarrow \left(q \vee (p \wedge r)\right) \wedge \left((\sim p \wedge \sim r) \vee \sim q\right)$

$\Leftrightarrow \left((q \vee p) \wedge (q \vee r)\right) \wedge \left((\sim p \vee \sim q) \wedge (\sim r \vee \sim q)\right)$

$\Leftrightarrow (q \vee p \vee F) \wedge (F \vee q \vee r) \wedge (\sim p \vee \sim q \vee F) \wedge (F \vee \sim r \vee \sim q)$
$\Leftrightarrow \big(q \vee p \vee (r \wedge \sim r)\big) \wedge \big((p \wedge \sim p) \vee q \vee r\big) \wedge \big(\sim p \vee \sim q \vee (r \wedge \sim r)\big) \wedge \big((p \wedge \sim p) \vee \sim r \vee \sim q\big)$
$\Leftrightarrow (q \vee p \vee r) \wedge (q \vee p \vee \sim r) \wedge (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$
$\wedge (\sim p \vee \sim q \vee \sim r) \wedge (p \vee \sim r \vee \sim q) \wedge (\sim p \vee \sim r \vee \sim q)$
$\Leftrightarrow (q \vee p \vee \sim r) \wedge (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$
$\wedge (\sim p \vee \sim q \vee \sim r) \wedge (p \vee \sim r \vee \sim q)$
$\Leftrightarrow (p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r)$
$\wedge (\sim p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee \sim r)$ *which is a PCNF*

$\sim S \Leftrightarrow$ *remaining max terms in* S
$\sim S \Leftrightarrow (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r)$
$\sim\sim S \Leftrightarrow \sim [(p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r)]$
$S \Leftrightarrow (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$ *which is PDNF.*
$S \Leftrightarrow \sim (p \vee \sim q \vee r) \vee \sim (\sim p \vee q \vee \sim r)$

Tutorial – 2

1. Show that the hypothesis, "It is not sunny this afternoon and it is colder than yesterday", "we will go swimming only if it is sunny", "If we do not go swimming, then we will take a canoe trip" and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset".

Solution:

Let S represents it is sunny this afternoon.

Let C represents it is colder than yesterday.

Let W represents we will go swimming.

Let T_r represents we will take a canoe trip.

Let H represents we will be home by sunset.

The inference is $\sim S \wedge C, W \to S, \sim W \to T_r, T_r \to H \Rightarrow H$

1. $\sim S \wedge C$	*Rule P*
2. $W \to S$	*Rule P*
3. $\sim W \to T_r$	*Rule P*
4. $T_r \to H$	*Rule P*
5. $\sim S$	*Rule T*, 1, $P \wedge Q \Rightarrow P$
6. $\sim W$	*Rule T*, 5,2, *Modus tollens*
7. T_r	*Rule T*, 6,3, *Modus phones*
8. H	*Rule T*, 7,4, *Modus phones*

2. Using indirect method of proof, derive $p \to \sim s$ from the premises $p \to (q \vee r), q \to \sim p$, $s \to \sim r$ and p.

Solution:

Let us assume $\sim (p \to \sim s\,)$ be the addition premise.

1.	$p \to (q \vee r)$	Rule P
2.	$q \to \sim p$	Rule P
3.	$s \to \sim r$	Rule P
4.	p	Rule P
5.	$\sim (p \to \sim s\,)$	Addition premise
6.	$q \vee r$	Rule T,1,4, Modus phones

7.	$\sim q$	Rule T,2,4, Modus tollens
8.	r	Rule T,6,7, Disjunctive Syllogism
9.	$\sim s$	Rule T,3,8, Modus tollens
10.	$\sim(\sim p \vee \sim s)$	Rule T,5, $a \to b \Rightarrow \sim a \vee b$
11.	$p \wedge s$	Rule T,10, Demorgan's law
12.	s	Rule T,11, $a \wedge b \Rightarrow b$
13.	$s \wedge \sim s$	Rule T,9,12, $a, b \Rightarrow a \wedge b$
14.	F	Rule T,13, $a \wedge \sim a \Rightarrow F$

which is a contradiction.

∴ Our assumption is wrong.

$p \to (q \vee r), q \to \sim p, s \to \sim r$ and $p \Rightarrow p \to \sim s$.

3. Determine the validity of the following argument:

If 7 is less than 4 then 7 is not a prime number, 7 is not less than 4. Therefore 7 is a prime number.

Solution:

Let L represents 7 is less than 4.

Let N represents 7 is a prime number

The inference is $L \to \sim N, \sim L \Rightarrow N$

1.	$\boldsymbol{L \to \sim N}$	Rule P
2.	$\boldsymbol{\sim L}$	Rule P

The argument is not valid, since $L \to \sim N, \sim L \not\Rightarrow N$

Tutorial – 3

1. By indirect method prove that $(x)\ (P(x) \to Q(x)), (\exists x)P(x)\ \Rightarrow\ (\exists x)Q(x)$

Solution:

Let us assume that $\neg(\exists x)Q(x)$ as additional premise

1. $\neg(\exists x)Q(x)$	*Additional premise*
2. $(x)\neg Q(x)$	*Rule T, 1, De Morgan's law*
3. $\neg Q(a)$	*Rule T, 2, US*
4. $(\exists x)P(x)$	*Rule P*
5. $P(a)$	*Rule T, 4, ES*
6. $P(a) \wedge \neg Q(a)$	*Rule T, 5,3 and conjuction*
7. $\neg(\neg P(a) \vee Q(a))$	*Rule T, 6, De Morgan's law*
8. $\neg(P(a) \to Q(a))$	*Rule T, 7, Equivalence*
9. $(x)(P(x) \to Q(x))$	*Rule P*
10. $P(a) \to Q(a)$	*Rule T, 9, US*
11. $\neg(P(a) \to Q(a)) \wedge P(a) \to Q(a)$	*Rule T, 8,10 and conjuction*
12. F	*Rule T, 11 and negation law*

2. Prove that $(x)\ (P(x) \vee Q(x))\ \Rightarrow (x)P(x) \vee\ (\exists x)Q(x)$

Proof:

Let us prove this by indirect method

Let us assume that $\neg((x)P(x) \vee\ (\exists x)Q(x))$ as additional premise

1.	$\boldsymbol{\neg((x)P(x) \vee\ (\exists x)Q(x))}$	***Additional premise***

2.	$\neg(x)\,P(x) \wedge \neg(\exists x)Q(x)$	$Rule\ T, 1, De\ Morgan's\ law$
3.	$\neg(x)\,P(x)$	$Rule\ T, 2$
4.	$(\exists x)\neg P(x)$	$Rule\ T, 3, De\ Morgan's\ law$
5.	$\neg P(a)$	$Rule\ ES, 4$
6.	$\neg(\exists x)Q(x)$	$Rule\ T, 2$
7.	$(x)\neg Q(x)$	$Rule\ T, 6, De\ Morgan's\ law$
8.	$\neg Q(a)$	$Rule\ US, 7$
9.	$\neg P(a) \wedge \neg Q(a)$	$Rule\ T, 5, 8, conjunction$
10.	$\neg\big(P(a) \vee Q(a)\big)$	$Rule\ T, 9, De\ Morgan's\ law$
11.	$(x)\,(P(x) \vee (Q(x))$	$Rule\ P$
12.	$P(a) \vee Q(a)$	$Rule\ US, 11$
13.	$\neg\big(P(a) \vee Q(a)\big) \wedge \big(P(a) \vee Q(a)\big)$	$Rule\ T, 11, 12, conjunction$
14.	F	$Rule\ T, 13$

3. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Proof:

Suppose $\sqrt{2}$ is rational.

$\therefore \sqrt{2} = \frac{p}{q}$ for $p, q \in Z, q \neq 0, p$ & q have no common divisor.

$$\frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$$

Since p^2 is an even integer, p is even integer.

$\therefore p = 2m$ for some integer m.

$$\therefore (2m)^2 = p^2 = 2q^2 \Rightarrow q^2 = 2m^2$$

Since q^2 is an even integer, q is even integer.

$\therefore q = 2k$ for some integer k.

Thus p and q are even. Hence they have a common factor 2.

which is a contradiction to our assumption.

$\therefore \sqrt{2}$ is irrational.

Tutorial – 4

1. Use mathematical induction to prove the inequality $n < 2^n$ for all positive integer n.

Proof:

$Let\ P(n): n < 2^n$ ... (1)

$P(1): 1 < 2^1$

$\Rightarrow 1 < 2$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$P(n+1): n+1 < 2^{n+1}$

$n < 2^n$ $(from\ (1))$

$$n + 1 < 2^n + 1$$

$$n + 1 < 2^n + 2^n \quad [\because 1 < 2^n]$$

$$n + 1 < 2.2^n$$
$$n + 1 < 2^{n+1}$$

$\therefore P(n+1)$ is true.
$\therefore$ By induction method,

$$P(n): n < 2^n \textit{ is true for all positive integers.}$$

2. Prove, by mathematical induction, that for all $n \geq 1, n^3 + 2n$ is a multiple of 3.

Solution:

$Let\ P(n): n \geq 1, n^3 + 2n$ is a multiple of 3. ... (1)

$P(1): 1^3 + 2(1) = 1 + 2 = 3$ is a multiple of 3.

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): (n+1)^3 + 2(n+1) \text{ is a multiple of } 3$$

$$(n+1)^3 + 2(n+1) = n^3 + 3n + 3n^2 + 1 + 2n + 2$$
$$= n^3 + 2n + 3n + 3n^2 + 3$$
$$= n^3 + 2n + 3(n^2 + n + 1)$$

From (1) $n^3 + 2n$ is a multiple of 3

$$\therefore (n+1)^3 + 2(n+1) \textit{ is a multiple of } 3$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$P(n): n \geq 1, n^3 + 2n$ is a multiple of 3, is true for all positive integer n.

3. What is the maximum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade if there are five possible grades A, B, C, D and F.

Solution:

By Pigeonhole principle, If there are n holes and k pigeons $n \leq k$ then there is at least one hole contains at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Here $n = 5$

$$\left\lfloor \frac{k-1}{5} \right\rfloor + 1 = 6$$

$$\frac{k-1}{5} = 5 \Rightarrow k - 1 = 25 \Rightarrow k = 26$$

The maximum number of students required in a discrete mathematics class is 26.

Tutorial – 5

1. Using method of generating function to solve the recurrence relation
$a_n + 3a_{n-1} - 4a_{n-2} = 0; n \geq 2$, given that $a_0 = 3$ and $a_1 = -2$.

Solution:

$$Let\ G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \quad \ldots (1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Given $a_n + 3a_{n-1} - 4a_{n-2} = 0$

Multiplying by x_n and summing from 2 to ∞, we have

$$\sum_{n=2}^{\infty} a_n x^n + 3\sum_{n=2}^{\infty} a_{n-1}x^n - 4\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n + 3x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 4x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$$(a_2x^2 + a_3x^3 + a_4x^4 + \cdots) + 3x(a_1x + a_2x^2 + \cdots) - 4x^2(a_0 + a_1x + a_2x^2 + \cdots) = 0$$

$$G(x) - a_0 - a_1x + 3xG(x) - 3xa_0 - 4x^2G(x) = 0 \qquad [from\ (1)]$$

$$G(x)(1 + 3x - 4x^2) - 3 + 2x - 9x = 0$$

$$G(x)(1 + 3x - 4x^2) = 3 + 7x$$

$$G(x) = \frac{3 + 7x}{(1 + 3x - 4x^2)} = \frac{3 + 7x}{(1 + 4x)(1 - x)}$$

$$\frac{3 + 7x}{(1 + 4x)(1 - x)} = \frac{A}{(1 + 4x)} + \frac{B}{(1 - x)}$$

$$3 + 7x = A(1 - x) + B(1 + 4x) \ldots (2)$$

Put $x = -\frac{1}{4}$ in (2)

$$3 + 7\left(-\frac{1}{4}\right) = A\left(1 + \frac{1}{4}\right) \Rightarrow \frac{5}{4}A = 3 - \frac{7}{4} \Rightarrow A = \frac{5}{5} = 1$$

Put $x = 1$ in (2)

$$3 + 7 = B(1 + 4) \Rightarrow 5B = 10 \Rightarrow B = 2$$

$$G(x) = \frac{1}{(1 + 4x)} + \frac{2}{(1 - x)}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-4)^n x^n + 2\sum_{n=0}^{\infty} x^n \qquad \left[\because \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n\right]$$

a_n = Coefficient of x^n in $G(x)$

$a_n = (-4)^n + 2$

2. Using the generating function, solve the difference equation

$$\mathbf{y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2}$$

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n$ … (1) where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - y_{n+1} - 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - \sum_{n=0}^{\infty} y_{n+1}x^n - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{1}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x)-y_1x-y_0)-\frac{1}{x}(G(x)-y_0)-6G(x)=0 \qquad [from\ (1)]$$
$$G(x)\left(\frac{1}{x^2}-\frac{1}{x}-6\right)-\frac{y_1}{x}-\frac{y_0}{x^2}+\frac{y_0}{x}=0$$
$$G(x)\left(\frac{1}{x^2}-\frac{1}{x}-6\right)-\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x}=0 \Rightarrow G(x)\left(\frac{6x^2-x+1}{x^2}\right)=\frac{2}{x^2}-\frac{1}{x}$$
$$G(x)\left(\frac{1-x-6x^2}{x^2}\right)=\frac{2-x}{x^2}$$
$$G(x)=\frac{2-x}{1-x-6x^2}=\frac{2-x}{(1-3x)(1+2x)}$$
$$\frac{2-x}{(1-3x)(1+2x)}=\frac{A}{(1-3x)}+\frac{B}{(1+2x)}$$
$$2-x=A(2x+1)+B(1-3x)\ \dots(2)$$
Put $x=-\frac{1}{2}$ in (2)
$$2-\left(-\frac{1}{2}\right)=B\left(1+\frac{3}{2}\right)\Rightarrow\frac{5}{2}B=\frac{5}{2}\Rightarrow B=1$$
Put $x=\frac{1}{3}$ in (2)
$$2-\left(\frac{1}{3}\right)=A\left(\frac{2}{3}+1\right)\Rightarrow\frac{5}{3}A=\frac{5}{3}\Rightarrow A=1$$
$$G(x)=\frac{1}{(1-3x)}+\frac{1}{(1+2x)}=\frac{1}{(1-3x)}+\frac{1}{(1-(-2x))}$$

$$\sum_{n=0}^{\infty}y_nx^n=\sum_{n=0}^{\infty}3^nx^n+\sum_{n=0}^{\infty}(-2)^nx^n \qquad \left[\because \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\right]$$
y_n = Coefficient of x^n in $G(x)$
$y_n=3^n+(-2)^n$

3. Find the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7.

Solution:

Let A,B,C and D represents the integer from 1 to 250 that are divisible by 2,3,5 and 7 respectively.
$$|A|=\left\lfloor\frac{250}{2}\right\rfloor=125,|B|=\left\lfloor\frac{250}{3}\right\rfloor=83,|C|=\left\lfloor\frac{250}{5}\right\rfloor=50,|D|=\left\lfloor\frac{250}{7}\right\rfloor=35$$
$$|A\cap B|=\left\lfloor\frac{250}{2\times 3}\right\rfloor=41,|A\cap C|=\left\lfloor\frac{250}{2\times 5}\right\rfloor=25,|A\cap D|=\left\lfloor\frac{250}{2\times 7}\right\rfloor=17,|B\cap C|=\left\lfloor\frac{250}{3\times 5}\right\rfloor=16$$
$$|B\cap D|=\left\lfloor\frac{250}{3\times 7}\right\rfloor=11,|C\cap D|=\left\lfloor\frac{250}{5\times 7}\right\rfloor=7,|A\cap B\cap C|=\left\lfloor\frac{250}{2\times 3\times 5}\right\rfloor=8$$
$$|A\cap B\cap D|=\left\lfloor\frac{250}{2\times 3\times 7}\right\rfloor=5,|A\cap C\cap D|=\left\lfloor\frac{250}{2\times 5\times 7}\right\rfloor=3,|B\cap C\cap D|=\left\lfloor\frac{250}{3\times 5\times 7}\right\rfloor=2$$
$$|A\cap B\cap C\cap D|=\left\lfloor\frac{250}{2\times 3\times 5\times 7}\right\rfloor=1$$
∴The number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7 is
$$|A\cup B\cup C\cup D|=|A|+|B|+|C|+|D|-|A\cap B|-|A\cap C|-|A\cap D|-|B\cap C|-|B\cap D|$$

$$-|C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$
$$|A \cup B \cup C \cup D| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$
$$|A \cup B \cup C \cup D| = 193$$

Tutorial – 6

1. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from mathematics department and four from the computer science department?

Solution:

The number of ways to select 3 mathematics faculty members from 9 faculty members is $9C_3$ ways.

The number of ways to select 4 computer Science faculty members from 11 faculty members is $11C_4$ ways.

The number of ways to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty member from mathematics department and four from the computer science department is $9C_3 . 11C_4$ ways.

$$9C_3 . 11C_4 = \frac{9 \times 8 \times 7}{3!} . \frac{11 \times 10 \times 9 \times 8}{4!} = 27720$$

2. How many positive integers n can be formed using the digits $3, 4, 4, 5, 5, 6, 7$ if n has to exceed 5000000?

Solution:

The positive integer n exceeds 5000000 if the first digit is either 5 or 6 or 7.If the first digit is 5 then the remaining six digits are 3,4,4,5,6,7.

Then the number of positive integers formed by six digits is

$$\frac{6!}{2!} = 360 \qquad [Since\ 4\ appears\ twice]$$

If the first digit is 6 then the remaining six digits are 3,4,4,5,5,7.

Then the number of positive integers formed by six digits is

$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$

If the first digit is 7 then the remaining six digits are 3,4,4,5,6,5.

Then the number of positive integers formed by six digits is

$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$

∴ The number of positive integers n can be formed using the digits 3,4,4,5,5,6,7 if n has to exceed 5000000 is $360 + 180 + 180 = 720$.

3. Find the number of distinct permutations that can be formed from all the letters of each word (1) RADAR (2) UNUSUAL.

Solution:

(1) The word RADAR contains 5 letters of which 2 A's and 2 R's are there.

$$The\ number\ of\ possible\ words\ = \frac{5!}{2!\,2!} = 30$$

Number of distinct permutation $= 30$.

(2) The word UNUSUAL contains 7 letters of which 3 U's are there.

$$The\ number\ of\ possible\ words\ = \frac{7!}{3!} = 840$$

Number of distinct permutation $= 840$.

4.What is the maximum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade if there are five possible grades $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{F}$.

Solution:

By Pigeonhole principle, If there are n holes and k pigeons $n \leq k$ then there is at least one hole contains at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Here $n = 5$

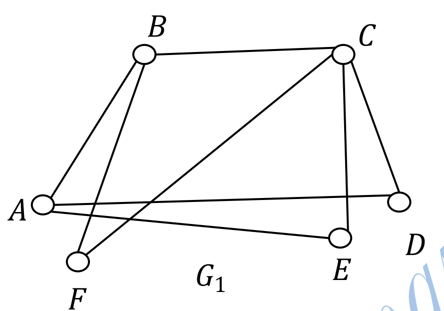
$$\left\lfloor \frac{k-1}{5} \right\rfloor + 1 = 6$$

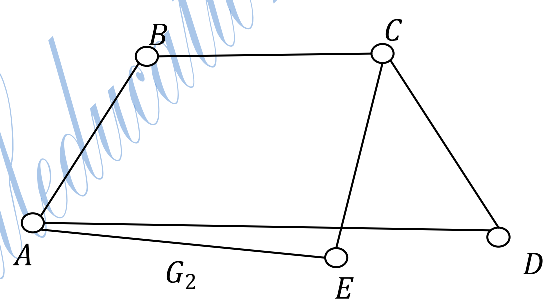
$$\frac{k-1}{5} = 5 \Rightarrow k - 1 = 25 \Rightarrow k = 26$$

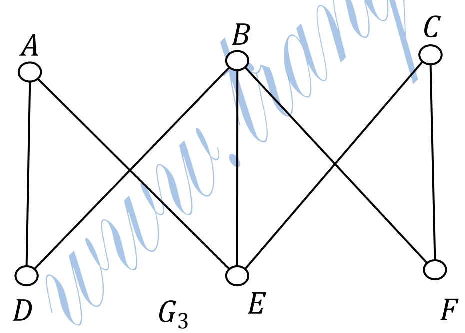
The maximum number of students required in a discrete mathematics class is 26.

Tutorial – 7

1. Determine which of the following graphs are bipartite and which are not. If a graph is bipartite, state if it is completely bipartite.







Solution:

In the graph G_1, Since there is no edges between D, E and F, let us take it as one vertex set $V_1 = \{D, E, F\}$. Obviously the other vertex set will be $V_2 = \{A, B, C\}$.

Since there are edges between A and B, and B and C.

$\therefore G_1$ is not a bipartite graph.

In the graph G_2, Let $V_1 = \{A, C\}$ and $V_2 = \{B, D, E\}$. Since there is no edge between the vertices in the same vertex set, G_2 is a bipartite graph. Since there are edges between every vertices in the vertex set V_1 to every vertices in the vertex set V_2, G_2 is completely bipartite.
In the graph G_3, Let $V_1 = \{A, B, C\}$ and $V_2 = \{D, E, F\}$. Since there is no edge between the vertices in the same vertex set, G_3 is a bipartite graph. Since there is no edge between A and F, C and D, where $A, C \in V_1$ and $D, F \in V_2$. $\therefore$ G_3 is not completely bipartite.

2. Prove that the maximum number of edges in a simple disconnected graph with n vertices and k components is

$$\frac{(n-k)(n-k+1)}{2}.$$

Solution:
Let n_i be the number of vertices in i^{th} component.

$$\sum_{i=1}^{k} n_i = n \ldots (1)$$

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k \ldots (2) \; [from\ (1)]$$

Squaring on both sides of (2), we get

$$\left(\sum_{i=1}^{k} (n_i - 1)\right)^2 = (n-k)^2$$

$$\sum_{i=1}^{k} (n_i - 1)^2 + \sum_{i \neq j}^{k} (n_i - 1)(n_j - 1) = (n-k)^2$$

$$\sum_{i=1}^{k} (n_i - 1)^2 \leq (n-k)^2$$

$$\sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 - 2nk + k^2 \qquad [from\ (1)]$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k \ \ldots (3)$$

The maximum number of edges in i^{th} component is

$$\frac{n_i(n_i - 1)}{2}$$

$\therefore$The maximum number of edges in the graph is

$$\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}n \qquad [from\ (1)]$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2}n \qquad [from\ (3)]$$

$$\leq \frac{1}{2}(n^2 - 2nk + k^2 + n - k)$$

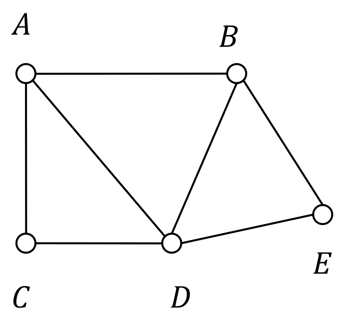
$$\leq \frac{1}{2}((n-k)^2 + n - k)$$

$$\leq \frac{1}{2}(n-k)(n-k+1)$$

∴The maximum number of edges in the graph is

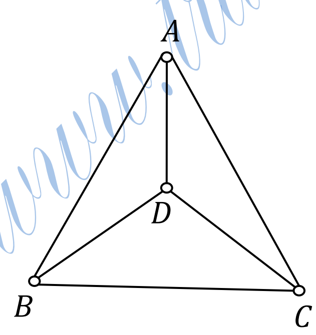
$$\leq \frac{(n-k)(n-k+1)}{2}$$

3. Draw the graph with 5 vertices, A, B, C, D, E such that $deg(A) = 3$, B is an odd vertex, $deg(C) = 2$ and $D$and E are adjacent.

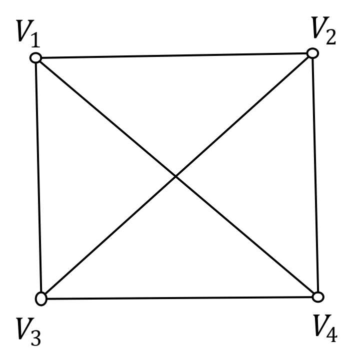


Tutorial - 8

1. Using circuits, examine whether the following pairs of graphs G_1, G_2 given below are isomorphic or not:



G_1



G_2

Solution:

In G_1, the number of vertices is 4, the number of edges is 6.

$$\deg(A) = 3 , \deg(B) = 3 , \deg(C) = 3 , \deg(D) = 3$$

In G_2, the number of vertices is 4, the number of edges is 6.

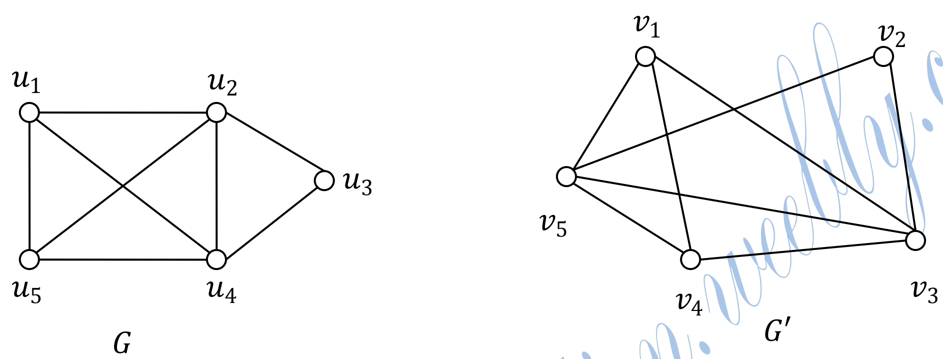
$$\deg(V_1) = 3 , \deg(V_2) = 3 , \deg(V_3) = 3 , \deg(V_4) = 3$$

There are same number of vertices and edges in both the graph G_1 and G_2.
Here in both graphs G_1 and G_2, all vertices are of degree 3.
The mapping between the vertices of two graphs is given below

$$A \to V_1, B \to V_2, C \to V_3, D \to V_4$$

There is one to one correspondences between the adjacency of the vertices in the graphs G_1 and G_2.
$\therefore$ The graphs $G_1$and G_2 are isomorphic.

2. Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reasons.



Solution:
In G, the number of vertices is 5, the number of edges is 8.

$$\deg(u_1) = 3, \deg(u_2) = 4, \deg(u_3) = 2, \deg(u_4) = 4, \deg(u_5) = 3$$

In G', the number of vertices is 5, the number of edges is 8.

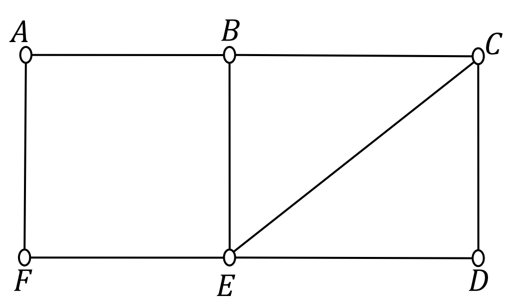
$$\deg(v_1) = 3, \deg(v_2) = 2, \deg(v_3) = 4, \deg(v_4) = 3, \deg(v_5) = 4$$

There are same number of vertices and edges in both the graph G and G'.
Here in both graphs G and G', two vertices are of degree 3, two vertices are of degree 4, and one vertex is of degree 2.

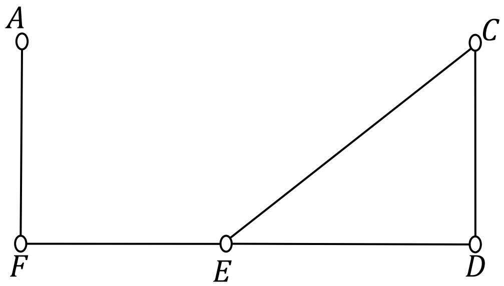
$$u_1 \to v_1, u_2 \to v_5, u_3 \to v_2, u_4 \to v_3, u_5 \to v_4$$

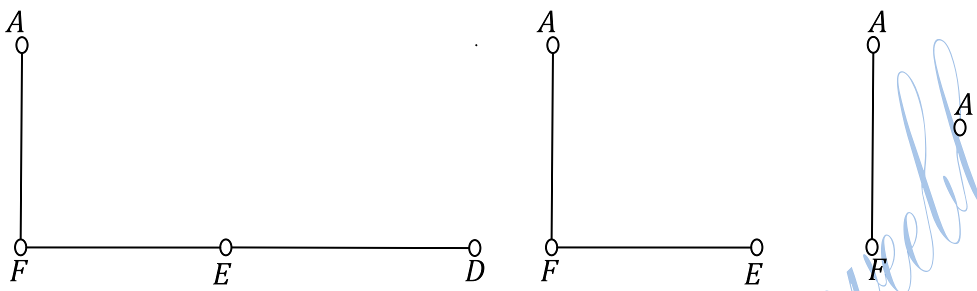
There is one to one correspondences between the graphs G and G'.
$\therefore$ The graphs $G$and G' are isomorphic.

3. Find the all the connected sub graph obtained form the graph given in the following Figure, by deleting each vertex. List out the simple paths from A to in each sub graph.



Solution:
The connected sub graph obtained form the graph given in the Figure, by deleting each vertex are

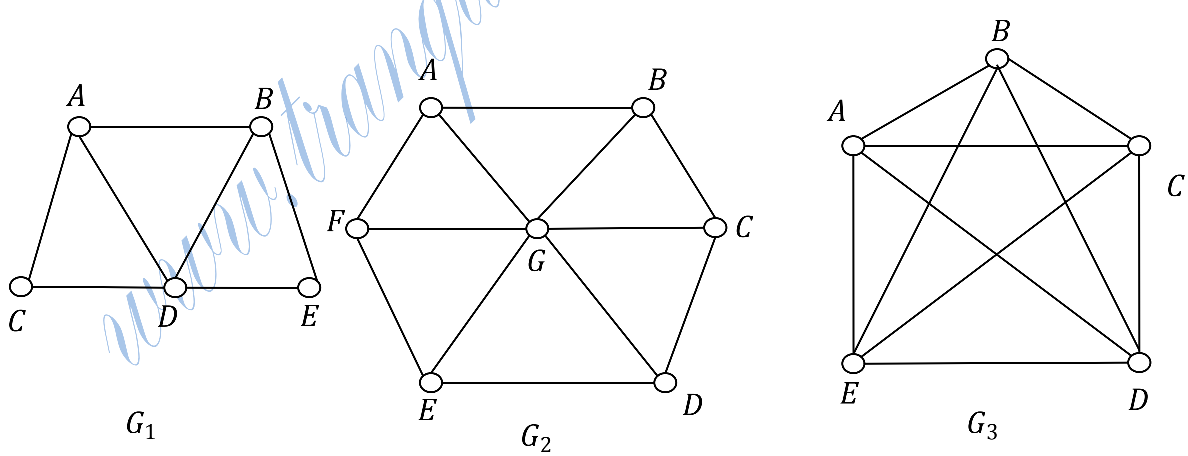




The simple paths from $\boldsymbol{A}$ to in each sub graph is
(1) $A \to F \to E \to D \to C,\ A \to F \to E \to C \to D$
(2) $A \to F \to E \to D$
(3) $A \to F \to E$
(4) $A \to F$

Tutorial - 9

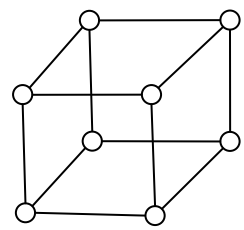
1. Find an Euler path or an Euler circuit, if it exists in each of the three graphs below. If it does not exist, explain why?



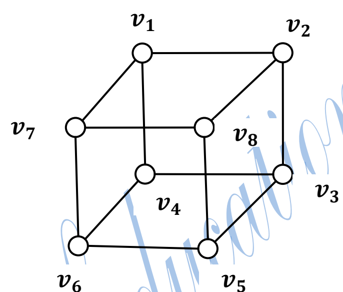
Solution:
In graph G_1, $\deg(A) = 3, \deg(B) = 3, \deg(C) = 2, \deg(D) = 4, \deg(E) = 2$
The graph G_1 contains only two vertices of odd degree and all the other vertices are of even degree.
$\therefore$ G_1 has an Eulerian path but not Eulerian circuit.
The Eulerian path for graph G_1 is $A \to B \to E \to D \to C \to A \to D \to B$
In graph G_2, $\deg(A) = 3, \deg(B) = 3, \deg(C) = 3, \deg(D) = 3, \deg(E) = 3\ , \deg(F) = 3, \deg(G) = 6$

The graph G_2 contains only one vertex of even degree and all the other vertices are of odd degree.
$\therefore G_2$ don't contain neither Eulerian path nor Eulerian circuit.
In graph G_3, $\deg(A) = 4, \deg(B) = 4, \deg(C) = 4, \deg(D) = 4, \deg(E) = 4$
The graph G_3 has all the vertices are of even degree.
$\therefore G_3$ has an Eulerian circuit.
The Eulerian circuit for graph G_3 is $A \to B \to C \to D \to E \to A \to D \to B \to E \to C \to A$

2. Check whether the graph given below is Hamiltonian or Eulerian or 2-colourable. Justify your answer.

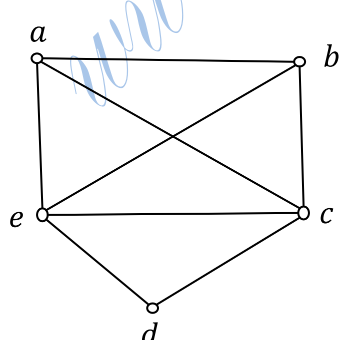


Solution:

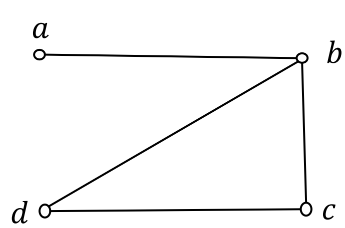


Since all the vertices of the above graph are of odd degree, $\therefore$ The graph is not Eulerian.
The graph is Hamiltonian, since there is a circuit which starts from the vertex v_1 and traversing through all the vertices of the graph only once and ends at the vertex v_1.
The circuit is $v_1 \to v_2 \to v_3 \to v_4 \to v_6 \to v_5 \to v_8 \to v_7 \to v_1$.
The graph is 2-colourable, since all the vertices are coloured in such a way that adjacent vertices doesn't have the same colour and the vertices are coloured with only two colours. Here v_1, v_3, v_6 and v_8 have one colour and v_2, v_4, v_5 and v_7 have another colour.

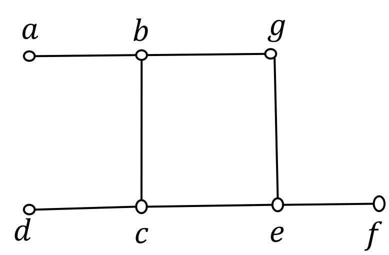
3. Which of the following simple graphs have a Hamiltonian circuit or, if not, a Hamiltonian path?



G_1



G_2



G_3

Solution:

The circuit $a \to b \to c \to d \to e \to a$ contains all the vertices of the graph G_1 by traversing through the vertices only once except end vertex. ∴The graph G_1 is a Hamiltonian circuit.
The path $a \to b \to c \to d$ contains all the vertices of the graph G_2 by traversing through the vertices only once. ∴The graph G_1 is a Hamiltonian path.
There is no path containing all the vertices of the graph G_3 by traversing through the vertices only once. ∴The graph G_3 is neither a Hamiltonian circuit nor a Hamiltonian path.

Tutorial – 10

1. If $(G,*)$ is an abelian group, show that $(a * b)^2 = a^2 * b^2$.
Proof:

$$(a * b)^2 = (a * b) * (a * b)$$
$$= a * (b * a) * b \quad [Associative\ law]$$
$$= a * (a * b) * b \quad [Commutative\ law]$$
$$= (a * a) * (b * b) \quad [Associative\ law]$$
$$(a * b)^2 = a^2 * b^2$$

2. Prove that the intersection of any two subgroups of a group $(G,*)$ is again a subgroup of $(G,*)$.
Proof:
Let H_1 and H_2 be two normal subgroups of a group $(G,*)$.
Let H_1 and H_2 are subgroups $(G,*)$.
Since $e \in H_1$ and $e \in H_2 \Rightarrow e \in H_1 \cap H_2$

$$\therefore H_1 \cap H_2 \text{ is non empty.}$$
$$\forall a, b \in H_1 \cap H_2$$
$$\Rightarrow a, b \in H_1 \text{ and } a, b \in H_2$$
$$\Rightarrow a * b^{-1} \in H_1 \text{ and } a * b^{-1} \in H_2 \quad [\because H_1 \text{ and } H_2 \text{ are subgroups}]$$
$$\Rightarrow a * b^{-1} \in H_1 \cap H_2 \quad [\because H_1 \text{ and } H_2 \text{ are subgroups}]$$

$\therefore H_1 \cap H_2$ is a subgroup.

3. Prove that the necessary and sufficient condition for a non empty subset H of a group $\{G,*\}$ to be a subgroup is $a, b \in H \Rightarrow a * b^{-1} \in H$.
Proof:
Necessary condition:
Let us assume that H is a subgroup of G.
H itself is a group.

$$a, b \in H \Rightarrow a * b \in H \ldots (1) \ [Closure\ ]$$
$$b \in H \Rightarrow b^{-1} \in H \ldots (2) \quad [Inverse\ property]$$
$$a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow a * b^{-1} \in H \quad [from\ (1)\ and\ (2)]$$
$$\therefore a, b \in H \Rightarrow a * b^{-1} \in H.$$

Sufficient condition:
Let $a, b \in H \Rightarrow a * b^{-1} \in H$ and H is a subset of G.
Closure property:
If $b \in H \Rightarrow b^{-1} \in H$

$$a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H \Rightarrow a * b \in H$$
$$a, b \in H \Rightarrow a * b \in H$$

Hence H is closed.

Associative property:

$\because$ H is a subset of G. All the elements in H are elements of G. Since G is associative under $*$.

$\therefore$ H is associative under $*$.

Identity property:

$$a, a \in H \Rightarrow a * a^{-1} \in H \Rightarrow e \in H$$

$\therefore$ $e \in H$ be the identity element.

Inverse property:

$$e, a \in H \Rightarrow e * a^{-1} \in H \Rightarrow a^{-1} \in H$$

$\therefore$ $a^{-1} \in H$ be the inverse of $a \in H$.

H itself is a group.

$\therefore$ H is a subgroup of G.

Tutorial - 11

1. State and Prove Lagrange's theorem.

Statement:

The order of a subgroup of a finite group is a divisor of the order of the group.

Proof:

Let aH and bH be two left cosets of the subgroup $\{H,*\}$ in the group $\{G,*\}$.

Let the two cosets aH and bH be not disjoint.

Then let c be an element common to aH and bH i.e., $c \in aH \cap bH$

$$\because c \in aH, c = a * h_1, for\ some\ h_1 \in H \ldots (1)$$

$$\because c \in bH, c = b * h_2, for\ some\ h_2 \in H \ldots (2)$$

From (1) and (2), we have

$$a * h_1 = b * h_2$$

$$a = b * h_2 * h_1^{-1} \ldots (3)$$

Let x be an element in aH

$x = a * h_3, for\ some\ h_3 \in H$

$$= b * h_2 * h_1^{-1} * h_3, using\ (3)$$

Since H is a subgroup, $h_2 * h_1^{-1} * h_3 \in H$

Hence, (3) means $x \in bH$

Thus, any element in aH is also an element in bH. $\therefore$ $aH \subseteq bH$

Similarly, we can prove that $bH \subseteq aH$

Hence $aH = bH$

Thus, if aH and bH are disjoint, they are identical.

The two cosets aH and bH are disjoint or identical. …(4)

Now every element $a \in G$ belongs to one and only one left coset of H in G,

For,

$a = ae \in aH, since\ e \in H \Rightarrow a \in aH$

$a \notin bH$, since aH and bH are disjoint i.e., a belongs to one and only left coset of H in G i.e., $aH \ldots (5)$

From (4) and (5), we see that the set of left cosets of H in G form the partition of G. Now let the order of H be m.

Let $H = \{h_1, h_2, \ldots, h_m\}, where\ h_i's$ are distinct

Then $aH = \{ah_1, ah_2, \ldots, ah_m\}$

The elements of aH are also distinct, for, $ah_i = ah_j \Rightarrow h_i = h_j$, which is not true.
Thus H and aH have the same number of elements, namely m.
In fact every coset of H in G has exactly m elements.
Now let the order of the group $\{G,*\}$ be n, i.e., there are n elements in G
Let the number of distinct left cosets of H in G be p.
$\therefore$ The total number of elements of all the left cosets $= pm$ $=$ the total number of elements of G. i.e., $n = pm$
i.e., m, the order of H is adivisor of n, the order of G.

2. If $(Z,+)$ and $(E,+)$ where Z is the set all integers and E is the set all even integers, show that the two semi groups $(Z,+)$ and $(E,+)$ are isomorphic.

Proof:

Let $f:(Z,+) \to (E,+)$ be the mapping between the two semi groups $(Z,+)$ and $(E,+)$ defined by

$$f(x) = 2x, \forall x \in Z$$

f is one to one:

$$f(x) = f(y)$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

$\therefore f$ is one to one.
f is onto:
Let $f(x) = y \Rightarrow y = 2x \Rightarrow x = \frac{y}{2} \in Z$ $\quad [\because y \text{ is an even number}]$
$\therefore \forall x \in E$ there is a preimage $\frac{x}{2} \in Z$.
$\therefore f$ is onto.
f is homomorphism:

$$\forall x, y \in Z, f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$$
$$f(x+y) = f(x) + f(y)$$

$\therefore f$ is homomorphism.
$\therefore f$ is isomorphism.
$\therefore$The two semi groups $(Z,+)$ and $(E,+)$ are isomorphic.

3. If $f: G \to G'$ is a group homomorphism from $(G,*)$ to (G',Δ) then prove that for any $a \in G$,

$$\boldsymbol{f(a^{-1}) = [f(a)]^{-1}}.$$

Solution:

Since f is a group homomorphism.

$$\forall a, b \in G, \qquad f(a*b) = f(a)\,\Delta\, f(b)$$

Since G is a group, $a \in G \Rightarrow a^{-1} \in G$.

$$f(a)\,\Delta\, f(a^{-1}) = f(a*a^{-1})$$
$$f(a)\,\Delta\, f(a^{-1}) = f(e)$$
$$f(a)\,\Delta\, f(a^{-1}) = e' \qquad [\text{where } e' \text{ is the identity element in } G']$$

The inverse of $f(a)$ is $f(a^{-1})$.

$$\therefore [f(a)]^{-1} = f(a^{-1})$$

Tutorial – 12

1. Let $(G,*)$ and (H,Δ) be groups and $g: (G,*) \rightarrow (H,\Delta)$ be group homomorphism. Then prove that kernel of g is a normal sub-group of $(G,*)$.

Proof:

Let $K = ker(g) = \{g(a) = e' \backslash a \in G, e' \in H\}$

To prove K is a subgroup of G:

We know that $g(e) = e' \Rightarrow e \in K$

$\therefore K$ is a non-empty subset of G.

By the definition of homomorphism $g(a * b) = g(a)\,\Delta\, g(b), \forall a, b \in G$

Let $a, b \in K \Rightarrow g(a) = e'$ and $g(b) = e'$

Now $g(a * b^{-1}) = g(a)\,\Delta\, g(b^{-1}) = g(a)\,\Delta\,\big(g(b)\big)^{-1} = e'\Delta\,(e')^{-1}$

$$= e'\Delta\, e' = e'$$

$$\therefore a * b^{-1} \in K$$

$\therefore K$ is a subgroup of G

To prove K is a normal subgroup of G:

For any $a \in G$ and $k \in K$,

$$g(a^{-1} * k * a) = g(a^{-1})\,\Delta\, g(k)\Delta g(a) = g(a^{-1})\Delta\, g(k)\,\Delta\, g(a)$$

$$= g(a^{-1})\Delta\, e'\Delta\, g(a) = g(a^{-1})\,\Delta\, g(a) = g(a^{-1} * a) = g(e) = e'$$

$$a^{-1} * k * a \in K$$

$\therefore K$ is a normal subgroup of G

2. Show that $(Z,+,\times)$ is an integral domain where Z is the set of all integers.

Proof:

Closure:

$$\forall a, b \in Z \Rightarrow a + b \in Z$$

$$\forall a, b \in Z \Rightarrow a \times b \in Z$$

$\therefore Z$ is closed under $+$ and $\times$.

Associative:

$$\forall a, b, c \in Z \Rightarrow (a + b) + c = a + (b + c)$$

$$\forall a, b \in Z \Rightarrow (a \times b) \times c = a \times (b \times c)$$

$\therefore Z$ is associative under $+$ and $\times$.

Identity:

Let $e \in Z$ be the identity element.

$$\forall a \in Z, a + e = e + a = a \Rightarrow a + e = a \Rightarrow e = 0$$

$\therefore 0 \in Z$ is the identity element with respect to the binary operation $+$.

$$\forall a \in Z, a \times e = e \times a = a \Rightarrow a \times e = a \Rightarrow e = 1$$

$\therefore 1 \in Z$ is the identity element with respect to the binary operation $+$.

Inverse:

Let $b \in Z$ be the inverse element of $a \in Z$.

$$a + b = b + a = 0 \Rightarrow a + b = 0 \Rightarrow b = -a \in Z$$

$-a \in Z$ is the inverse of $a \in Z$

$\therefore$ Every element has its inverse in Z under binary operation $+$.

Commutative:

$$\forall a, b \in Z, a + b = b + a$$

$$\forall a, b \in Z, a \times b = b \times a$$

$\therefore Z$ is Commutative under $+$ and $\times$.

Distributive:

$$\forall a, b, c \in Z, a \times (b + c) = a \times b + a \times c$$

$\therefore \times$ is distributive over $+$.

$$\forall a, b \in Z, a \times b = 0 \Rightarrow a = 0 \text{ or } b = 0$$

$\therefore Z$ has no zero divisors.

$\therefore (Z, +, \times)$ is an integral domain.

3. If $*$ is a binary operation on the set R of real numbers defined by $a * b = a + b + 2ab$,
(1) Show that $(R, *)$ is a semigroup ,
(2) Find the identity element if it exists
(3) Which elements has inverse and what are they?

Solution:

(1) i) Closure: $\forall a, b \in R, a + b + 2ab \in R \Rightarrow a * b \in R$

$\therefore R$ is closed under binary operation $*$.

ii) Associative: $\forall a, b, c \in R$,

$$a * (b * c) = a * (b + c + 2bc)$$
$$= a + (b + c + 2bc) + 2a(b + c + 2bc) = a + b + c + 2ab + 2bc + 2ac + 4abc$$
$$= (a + b + 2ab) + 2(a + b + 2ab)c + c$$
$$= (a + b + 2ab) * c = (a * b) * c$$
$$\therefore a * (b * c) = a * (b + c + 2bc)$$

$\therefore R$ is associative under binary operation $*$.

iii) Identity: Let $e \in R$ be the identity element in R

$$\forall a \in R, a * e = e * a = a$$
$$a + e + 2ae = a \Rightarrow e + 2ae = 0 \Rightarrow e = 0 \in R$$

$\therefore 0 \in R$ is the identity element.

$\therefore (R, *)$ is a semigroup.

(2) $0 \in R$ is the identity element.

(3) Let $a' \in R$ be the inverse element of $a \in R$

$$\forall a \in R, a * a' = a' * a = e$$
$$a + a' + 2aa' = 0 \Rightarrow a'(1 + 2a) = -a \Rightarrow a' = -\frac{a}{1 + 2a} \in R$$

$\therefore a' = -\frac{a}{1+2a} \in R - \left\{\frac{1}{2}\right\}$ is the inverse element for $\forall a \in R - \left\{\frac{1}{2}\right\}$.

Tutorial – 13

1. Show that $(N, \leq)$ is a partially ordered set where N is set of all positive integers and $\leq$ is defined by $m \leq n$ iff $n - m$ is a non-negative integer.

Proof:

Let R be the relation $m \leq n \; iff \; n - m$ is a non-negative integer.

$i) \; \forall x \in N, (x - x) = 0 \text{ is also a non negative integer} \Rightarrow (x, x) \in R$

$\therefore R$ is reflexive.

$ii) \forall x, y \in N,$

$(x, y) \in R \; \& \; (y, x) \in R$

$\Rightarrow \; (x - y) \text{ is a non negative integer} \; \& \; (y - x) \text{ is a non negative integer}$

It is possible only if $x - y = 0 \Rightarrow x = y$

$(x, y) \in R \ \& \ (y, x) \in R \Rightarrow x = y$
$\therefore R$ is Anti Symmetric.
$iii) \forall x, y, z \in N, (x, y) \in R \ and \ (y, z) \in R$
$x - z = (x - y) + (y - z)$
Since sum of two non-negative integer is also a non-negative integer.
$\Rightarrow (x - z) is \ also \ a \ non-negative \ integer \Rightarrow (x, z) \in R$
$(x, y) \in R \ and \ (y, z) \in R \Rightarrow (x, z) \in R$
$\therefore R$ is Transitive.
$\therefore (\boldsymbol{N}, \leq)$ is a partially ordered set.

2. Let L be lattice, where $a * b = glb(a, b)$ and $a \oplus b = lub(a, b)$ for all $a, b \in L$. Then both binary operations $*$ and $\oplus$ defined as in L satisfies commutative law, associative law, absorption law and idempotent law.

Solution:
Commutative law:
To prove: $\forall a, b \in L \Rightarrow a * b = b * a, a \oplus b = b \oplus a$
$a * b = glb(a, b) = glb(b, a) = b * a$
$a \oplus b = lub(a, b) = lub(b, a) = b \oplus a$
Associative law:
To prove: $\forall a, b, c \in L \Rightarrow (a * b) * c = a * (b * c), (a \oplus b) \oplus c = a \oplus (b \oplus c)$
We know that $a * b \leq a, a * b \leq b$

$$\Rightarrow (a * b) * c \leq a * b \leq a \ldots (1)$$
$$\Rightarrow (a * b) * c \leq a * b \leq b \ldots (2)$$
$$\Rightarrow (a * b) * c \leq c \ldots (3)$$

From (2) and (3), we get

$$(a * b) * c \leq b * c \ldots (4)$$

Now from (1) and (4), we get

$$(a * b) * c \leq a * (b * c) \ldots (5)$$

We know that $b * c \leq b, b * c \leq c$

$$\Rightarrow a * (b * c) \leq a \ldots (6)$$
$$\Rightarrow a * (b * c) \leq b * c \leq b \ldots (7)$$
$$\Rightarrow a * (b * c) \leq b * c \leq c \ldots (8)$$

From (6) and (7), we get

$$a * (b * c) \leq a * b \ldots (9)$$

Now from (9) and (8), we get

$$a * (b * c) \leq (a * b) * c \ldots (10)$$

From (5) and (10), we get

$$(a * b) * c = a * (b * c)$$

Similarly we can prove

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

Idempotent law:
To prove: $\forall a \in L \Rightarrow a * a = a, a \oplus a = a$
Since $a \leq a, a$ is a lower bound of $\{a\}$. If b is any lower bound of $\{a\}$, then we have $b \leq a$. Thus we have $a \leq a$ or $b \leq a$ equivalently, a is an lower bound for $\{a\}$and any other lower bound of $\{a\}$ is smaller than a. This shows that a is the greatest lower bound of $\{a\}$, i.e., $glb\{a, a\} = a$

$\therefore a * a = glb\{a, a\} = a$
Since $a \geq a, a$ is a upper bound of $\{a\}$. If b is any upper bound of $\{a\}$, then we have $b \geq a$. Thus we have $a \geq a$ or $b \geq a$ equivalently, a is an upper bound for $\{a\}$and any other upper bound of $\{a\}$ is greater than a. This shows that a is the least upper bound of $\{a\}$, i.e., $lub\{a, a\} = a$
$\therefore a * a = lub\{a, a\} = a$
Absorption law:
To prove: $\forall a, b \in L \Rightarrow (a * b) \oplus a = a, (a \oplus b) * a = a$
Form the definition of glb, $a * b \leq a$

$$\Rightarrow (a * b) \oplus a \leq a \oplus a$$

$$\Rightarrow (a * b) \oplus a \leq a \ldots (1) \; [\because a \oplus a = a]$$

Form the definition of lub, $(a * b) \oplus a \geq a \ldots (2)$
From (1) and (2), we get

$$(a * b) \oplus a = a$$

Form the definition of lub, $a \oplus b \geq a$

$$\Rightarrow (a \oplus b) * a \geq a * a$$

$$\Rightarrow (a \oplus b) * a \geq a \ldots (3) \; [\because a * a = a]$$

Form the definition of glb, $(a \oplus b) * a \leq a \ldots (4)$
From (3) and (4), we get

$$(a \oplus b) * a = a$$

3. Show that in a lattice if $a \leq b \leq c$, then
(1) $a \oplus b = b * c$ and
(2) $(a * b) \oplus (b * c) = b = (a \oplus b) * (a \oplus c)$
Solution:
(1) $a \oplus b = lub(a, b) = b$

$$b * c = glb(a, b) = b$$

$$a \oplus b = b * c$$

(2) $(a * b) \oplus (b * c) = lub(a * b, b * c) = lub\big(glb(a, b), glb(b, c)\big) = lub(a, b) = b$

$$(a \oplus b) * (a \oplus c) = glb\big(lub(a, b), lub(a, c)\big) = glb(b, c) = b$$

$$(a * b) \oplus (b * c) = b = (a \oplus b) * (a \oplus c)$$

Tutorial – 14

1. Prove that every chain is a distributive lattice.
Solution:
Let $(L, \leq)$ be a chain and $a, b, c \in L$. Consider the following cases:
(I) $a \leq b$ and $a \leq c$, and (II) $a \geq b$ and $a \geq c$
For (I)

$$a * (b \oplus c) = a \ldots (1)$$

$$(a * b) \oplus (a * c) = a \oplus a = a \ldots (2)$$

For (II)

$$a * (b \oplus c) = b \oplus c \ldots (3)$$

$$(a * b) \oplus (a * c) = b \oplus c \ldots (4)$$

$\therefore$From (1),(2) and (3),(4)

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

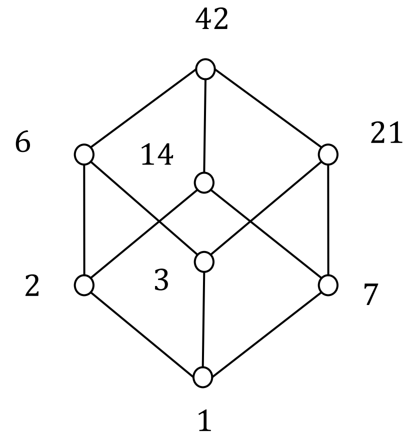
$\therefore$Every chain is a distributive lattice

2. If S_{42} is the set all divisors of 42 and D is the relation "divisor of" on S_{42} , prove that (S_{42}, D) is a complemented Lattice.

Solution:

$S_{42} = \{1,2,3,6,7,14,21,42\}$

The Hasse diagram for (S_{42}, D) is



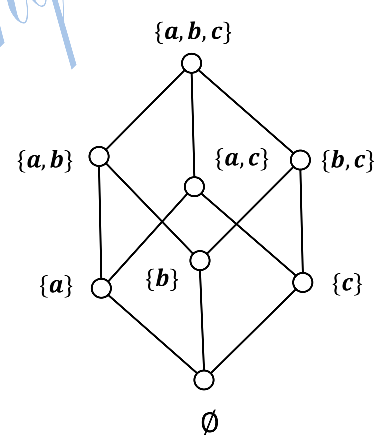
$$1 \vee 42 = 42, 1 \wedge 42 = 1$$
$$2 \vee 21 = 42, 2 \wedge 21 = 1$$
$$3 \vee 14 = 42, 3 \wedge 14 = 1$$
$$7 \vee 6 = 42, 7 \wedge 6 = 1$$

The complement of 1 is 42, The complement of 42 is 1, The complement of 2 is 21, The complement of 21 is 2, The complement of 3 is 14, The complement of 14 is 3, The complement of 7 is 6, The complement of 6 is 7. Since all the elements in (S_{24}, D) has a complement,

$\therefore$ (S_{24}, D) is a complemented lattice.

3. Draw the Hasse diagram representing the partial ordering $\{(A, B): A \subseteq B\}$ on the power set $P(S)$ Where $S = \{a, b, c\}$. Find the maximal, minimal, greatest and least elements of the Poset.

Solution:



The minimal element is ϕ

The maximal element is $\{a, b, c\}$

The least element is ϕ

The greatest element is $\{a, b, c\}$

Tutorial - 15

1. In a Boolean algebra, prove that $(\boldsymbol{a} \wedge \boldsymbol{b})' = \boldsymbol{a}' \vee \boldsymbol{b}'$.

Solution: Let $a, b \in (B, \wedge, \oplus, ', 0, 1)$

To prove $(a \wedge b)' = a' \vee b'$

$$(a \wedge b) \vee (a' \vee b') = \big(a \vee (a' \vee b')\big) \wedge \big(b \vee (a' \vee b')\big)$$
$$= \big(a \vee (a' \vee b')\big) \wedge \big((a' \vee b') \vee b\big)$$
$$= \big((a \vee a') \vee b'\big) \wedge \big(a' \vee (b' \vee b)\big)$$
$$= (1 \vee b') \wedge (a' \vee 1) = 1 \wedge 1$$
$$(a \wedge b) \vee (a' \vee b') = 1 \ldots (1)$$
$$(a \wedge b) \wedge (a' \vee b') = \big((a \wedge b) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big((b \wedge a) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big(b \wedge (a \wedge a')\big) \vee \big(a \wedge (b \wedge b')\big)$$
$$= (b \wedge 0) \vee (a \wedge 0) = 0 \vee 0$$
$$(a \wedge b) \wedge (a' \vee b') = 0 \ldots (2)$$

$From\ (1)\ and\ (2)\ we\ get,$

$$(a \wedge b)' = a' \vee b'$$

2. Simplify the Boolean expression $\boldsymbol{a}'.\boldsymbol{b}'.\boldsymbol{c} + \boldsymbol{a}.\boldsymbol{b}'.\boldsymbol{c} + \boldsymbol{a}'.\boldsymbol{b}'.\boldsymbol{c}'$ using Boolean algebra identities.

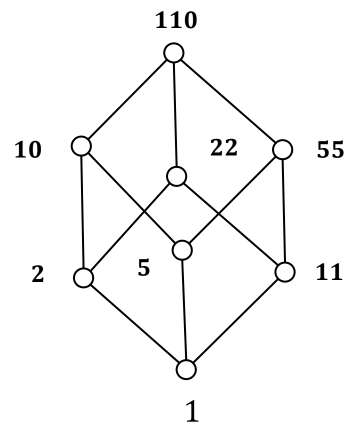
Solution:

$$a'.b'.c + a.b'.c + a'.b'.c' = (a' + a).b'.c + a'.b'.c' \qquad [Distributive\ law]$$
$$= 1.b'.c + a'.b'.c' \qquad [a' + a = 1]$$
$$= b'.c + a'.b'.c' \qquad [1.a = a]$$
$$= b'.c + b'.a'.c' \qquad [a.b = b.a]$$
$$= b'.(c + a'.c') \qquad [Distributive\ law]$$
$$= b'.\big((c + a').(c + c')\big) \qquad [Distributive\ law]$$
$$= b'.\big((c + a').1\big) \qquad [a' + a = 1]$$
$$= b'.(c + a') \qquad [1.a = a]$$
$$= b'.c + b'.a' \qquad [Distributive\ law]$$

3. Prove that $\boldsymbol{D_{110}}$, the set of all positive divisors of a positive integer 110, is a Boolean algebra 110 and find all its sub algebras.

Solution:

$$D_{110} = \{1,2,5,10,11,22,55,110\}$$



Since set all divisors D satisfies reflexive, anti-symmetric and transitive properties, D is a partial order relation.

$\therefore (D_{110}, D)$ is a Poset.

From the Hasse diagram, we observe that every element in the Poset (D_{110}, D) has a least upper bound and greatest lower bound. $\therefore (D_{110}, D)$ is a Lattice.

Here 1 is the least element and 110 is the greatest element.

From the Hasse diagram, we observe that $\forall\, a, b, c \in D_{110}, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\therefore D_{110}$ is a distributive Lattice.

The complement of 1 is 110. $[\because 1 \wedge 110 = 1\ \&\ 1 \vee 110 = 110\,]$

The complement of 2 is 55. $[\because 2 \wedge 55 = 1\ \&\ 2 \vee 55 = 110\,]$

The complement of 5 is 22. $[\because 5 \wedge 22 = 1\ \&\ 22 \vee 5 = 110\,]$

The complement of 10 is 11. $[\because 10 \wedge 11 = 1\ \&\ 10 \vee 11 = 110\,]$

The complement of 11 is 10.

The complement of 22 is 5.

The complement of 55 is 2.

The complement of 110 is 1.

$\because$ Every element in D_{110} has atleast one complement, D_{110} is a complemented Lattice.

The sub Boolean algebras are

$i)$ $\{1,110\}$

$ii)$ $\{1,2,5,10,11,22,55,110\}$

$iii)$ $\{1,2,5,110\}$

$iv)$ $\{1,2,11,110\}$

$v)$ $\{1,5,11,110\}$

$vi)$ $\{1,10,22,110\}$

$vii)$ $\{1,10,55,110\}$

$viii)$ $\{1,22,11,110\}$