Transportation Problem

1. Production costs at factories F_1, F_2, F_3 and F_4 are Rs. 2, 3, 1 and 5 respectively. The production capacities are 50, 70, 40 and 50 units respectively. Four stores S_1, S_2, S_3 and S_4 have requirements of 25, 35, 105 and 20 units respectively. Using transportation cost matrix find the transportation plan that is optimal considering the production costs also.

Stores

		S_1	S_2	S_3	S_4
	F_1	2	4	6	11
Factory	F_2	10	8	7	5
	F_3	13	3	9	12
	F_4	4	6	8	3

Solution:

Stores

		S_1	S_2	S_3	S_4	Supply
	F_1	2+2	4+2	6+2	11+2	50
Factory	F_2	10+3	8+3	7+3	5+3	70
	F_3	13+1	3+1	9+1	12+1	40
	F_4	4+5	6+5	8+5	3+5	50
Demand		25	35	105	20	

Total demand $= 25 + 35 + 105 + 20 = 185$

Total supply $= 50 + 70 + 40 + 50 = 210$

Total demand $\neq$ Total supply. Therefore the problem is unbalanced.

Since the total supply is more than the total demand add an extra column with zero entries and demand as the difference between their sums to make the problem balanced one.

4	6	8	13	0	50
13	11	10	8	0	70
14	4	10	13	0	40
9	11	13	8	0	50
25	35	105	20	25	

Vogel's approximation method:

[25] 4	6	[25] 8	13	0	50 /25/0 (4) /(2)/(4) /(5)
13	11	[45] 10	8	[25] 0	70 /45 (8)/(2)/(2)/(2)
14	[35] 4	[5] 10	13	0	40 /5 (4)/(6)/(3)/(3)
9	11	[30] 13	[20] 8	0	50/30 (8)/(1)/(1)/(5)
25/0	35/0	105/80	20/0	25/0	
(5)	(2)	(2)/(0)	(0)/(0)	(0)	

$$Minimum\ cost = 25 \times 4 + 25 \times 8 + 45 \times 10 + 25 \times 0 + 35 \times 4 + 5 \times 10 + 30 \times 13 + 20 \times 8 = 1490$$

The number of basic cells = 8 = number of rows +number of columns – 1. Therefore the solution is non degeneracy.

Modi method: For basic cells $u_i + v_j - c_{ij} = 0$

[25] 4	6	[25] 8	13	0	$u_1 = 0$
13	11	[45] 10	8	[25] 0	$u_2 = 2$
14	[35] 4	[5] 10	13	0	$u_3 = 2$
9	11	[30] 13	[20] 8	ϵ 0	$u_4 = 5$
$v_1 = 4$	$v_2 = 2$	$v_3 = 8$	$v_4 = 3$	$v_5 = -2$	

For non basic cells

$$u_1 + v_2 - c_{12} = 0 + 2 - 6 = -4 < 0$$
$$u_1 + v_4 - c_{14} = 0 + 3 - 13 = -10 < 0$$
$$u_1 + v_5 - c_{15} = 0 - 2 - 0 = -2 < 0$$
$$u_2 + v_1 - c_{21} = 2 + 4 - 13 = -7 < 0$$
$$u_2 + v_2 - c_{22} = 2 + 2 - 11 = -7 < 0$$
$$u_2 + v_4 - c_{24} = 2 + 3 - 8 = -3 < 0$$
$$u_3 + v_1 - c_{31} = 2 + 4 - 14 = -8 < 0$$
$$u_3 + v_4 - c_{34} = 2 + 3 - 13 = -8 < 0$$
$$u_3 + v_5 - c_{35} = 2 - 2 - 0 = 0$$
$$u_4 + v_1 - c_{41} = 5 + 4 - 9 = 0$$
$$u_4 + v_2 - c_{42} = 5 + 2 - 11 = -4 < 0$$
$$u_4 + v_5 - c_{45} = 5 - 2 - 0 = 3 > 0$$

[25] 4	6	[25] 8	13	0	$u_1 = 0$
13	11	[45] $+\epsilon$ 10	8	[25] $-\epsilon$ 0	$u_2 = 2$
14	[35] 4	[5] 10	13	0	$u_3 = 2$
9	11	[30] $-\epsilon$ 13	[20] 8	ϵ 0	$u_4 = 5$
$v_1 = 4$	$v_2 = 2$	$v_3 = 8$	$v_4 = 3$	$v_5 = -2$	

Put $\epsilon = 25$, we get

(25) 4	6	(25) 8	13	0	$u_1 = 0$
13	11	(70) 10	8	0	$u_2 = 2$
14	(35) 4	(5) 10	13	0	$u_3 = 2$
9	11	(5) 13	(20) 8	(25) 0	$u_4 = 5$
$v_1 = 4$	$v_2 = 2$	$v_3 = 8$	$v_4 = 3$	$v_5 = -5$	

For non basic cells

$$u_1 + v_2 - c_{12} = 0 + 2 - 6 = -4 < 0$$
$$u_1 + v_4 - c_{14} = 0 + 3 - 13 = -10 < 0$$
$$u_1 + v_5 - c_{15} = 0 - 5 - 0 = -5 < 0$$
$$u_2 + v_1 - c_{21} = 2 + 4 - 13 = -7 < 0$$
$$u_2 + v_2 - c_{22} = 2 + 2 - 11 = -7 < 0$$
$$u_2 + v_4 - c_{24} = 2 + 3 - 8 = -3 < 0$$
$$u_2 + v_5 - c_{25} = 2 - 5 - 0 = -3 < 0$$
$$u_3 + v_1 - c_{31} = 2 + 4 - 14 = -8 < 0$$
$$u_3 + v_4 - c_{34} = 2 + 3 - 13 = -8 < 0$$
$$u_3 + v_5 - c_{35} = 2 - 2 - 0 = 0$$
$$u_4 + v_1 - c_{41} = 5 + 4 - 9 = 0$$
$$u_4 + v_2 - c_{42} = 5 + 2 - 11 = -4 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

$$Minimum\ cost = 25 \times 4 + 25 \times 8 + 70 \times 10 + 35 \times 4 + 5 \times 10 + 5 \times 13 + 20 \times 8$$
$$+25 \times 0 = 1415$$

2. Find the minimum cost distribution plan to satisfy demand for cement at three construction sites from available capacities at the three cement plants given the following transportation costs (in Rs) per ton of cement moved from plants to sites.

From	Stores			Capacity (tons/month)
	1	2	3	
P_1	300	360	425	600
P_2	390	340	310	300
P_3	255	295	275	1000
Demand (tons/month)	400	500	800	

Solution:

Total demand $= 400 + 500 + 800 = 1700$

Total capacity $= 600 + 300 + 1000 = 1900$

Total demand ≠ Total capacity. Therefore the problem is unbalanced.

Since the total capacity is more than the total demand add an extra column with zero entries and demand as the difference between their sums to make the problem balanced one.

300	360	425	0	600
390	340	310	0	300
255	295	275	0	1000
400	500	800	200	

Vogel's approximation method:

[400] 300	[200] 360	425	0	600/200/0	(300) (60)(65)
390	340	[100] 310	[200] 0	300/100	(310)(30)(30)
255	[300] 295	[700] 275	0	1000/700	(255)(20) (20)
400 0	500 300	800 0	200 0		
(45)	(45) (45)	(35) (35)	(0)		

$Minimum\ cost$

$$= 400 \times 300 + 200 \times 360 + 100 \times 310 + 200 \times 0 + 300 \times 295 + 700 \times 275$$
$$= 504000$$

The number of basic cells = 6 = number of rows +number of columns – 1. Therefore the solution is non degeneracy.

Modi method:

For basic cells $u_i + v_j - c_{ij} = 0$

[400] 300	[200] 360	425	0	$u_1 = 0$
390	340	[100] 310	[200] 0	$u_2 = -30$
255	[300] 295	[700] 275	0	$u_3 = -65$
$v_1 = 300$	$v_2 = 360$	$v_3 = 340$	$v_4 = 30$	

For non basic cells

$$u_1 + v_3 - c_{13} = 0 + 340 - 425 = -85 < 0$$
$$u_1 + v_4 - c_{14} = 0 + 30 - 0 = 30 > 0$$
$$u_2 + v_1 - c_{21} = -30 + 300 - 390 = -120 < 0$$
$$u_2 + v_2 - c_{22} = -30 + 360 - 340 = -10 < 0$$
$$u_3 + v_1 - c_{31} = -65 + 300 - 255 = -20 < 0$$
$$u_3 + v_4 - c_{34} = -65 + 30 - 0 = -35 < 0$$

[400] 300	[$200 - \epsilon$] 360	425	ϵ 0
390	340	[$100 + \epsilon$] 310	[$200 - \epsilon$] 0
255	[$300 + \epsilon$] 295	[$700 - \epsilon$] 275	0

Put $\epsilon = 200$, we get

(400) 300	(0) 360	425	(200) 0	$u_1 = 0$
390	340	(300) 310	0	$u_2 = -30$
255	(500) 295	(500) 275	0	$u_3 = -65$
$v_1 = 300$	$v_2 = 360$	$v_3 = 340$	$v_4 = 0$	

For non basic cells

$$u_1 + v_3 - c_{13} = 0 + 340 - 425 = -85 < 0$$
$$u_2 + v_1 - c_{21} = -30 + 300 - 390 = -120 < 0$$
$$u_2 + v_2 - c_{22} = -30 + 360 - 340 = -10 < 0$$
$$u_2 + v_4 - c_{24} = -30 + 0 - 0 = -30 < 0$$
$$u_3 + v_1 - c_{31} = -65 + 300 - 255 = -20 < 0$$
$$u_3 + v_4 - c_{34} = -65 + 0 - 0 = -65 < 0$$

Since $u_i + v_j - c_{ij} \le 0, \forall i, j$. Optimum solution is reached.

$$Minimum\ cost = 400 \times 300 + 0 \times 360 + 200 \times 0 + 300 \times 310 + 500 \times 295 + 500 \times 275$$
$$= 498000$$

3. A dairy firm has three plants located in a state. The daily production at each plant is as follow: Plant 1: 6 million litres, Plant 2: 1 million litre and plant 3: 10 million litres. Each day, the firm must fulfill the needs of its four distribution centers. Minimum requirement at each center is as follows. Distribution center 1: 7 million litres, Distribution center 2: 5 million litres, Distribution center 3: 3 million litres, and Distribution center 4: 2 million litres. Cost in hundreds of rupees of shipping one million litre from each plant to each distribution centre is given in the following table:

Distribution Center

		D_1	D_2	D_3	D_4
Plant	P_1	2	3	11	7
	P_2	1	0	6	1
	P_3	5	8	15	9

Determine an optimal solution to minimize the total cost.

Solution:

		Distribution Center				Capacity
		D_1	D_2	D_3	D_4	
Plant	P_1	2	3	11	7	6
	P_2	1	0	6	1	1
	P_3	5	8	15	9	10
Requirement		7	5	3	2	

Since the sum of capacities is equal to the sum of requirements then the problem is balanced.

Vogel's approximation method:

[1] 2	[5] 3	11	7	6/1/0	(1) (5)
1	0	6	[1] 1	1/0	(1)(0)
[6] 5	8	[3] 15	[1] 9	10	(3)(4)
7	5	3	2		
6	0		1		
	(3)	(5)			
(1)	(5)	(4)	(2)		
(3)					

$$Minimum\ cost = 1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9 = 102$$

The number of basic cells = 6 = number of rows +number of columns – 1. Therefore the solution is non degeneracy.

Modi method:

For basic cells $u_i + v_j - c_{ij} = 0$

[1] 2	[5] 3	11	7	$u_1 = 0$
1	0	6	[1] 1	$u_2 = -5$
[6] 5	8	[3] 15	[1] 9	$u_3 = 3$
$v_1 = 2$	$v_2 = 3$	$v_3 = 12$	$v_4 = 6$	

For non basic cells

$$u_1 + v_3 - c_{13} = 0 + 12 - 11 = 1 > 0$$

$$u_1 + v_4 - c_{14} = 0 + 6 - 7 = -1 < 0$$

$$u_2 + v_1 - c_{21} = -5 + 2 - 1 = -4 < 0$$

$$u_2 + v_2 - c_{22} = -5 + 3 - 0 = -2 < 0$$

$$u_2 + v_3 - c_{23} = -5 + 12 - 6 = 1 > 0$$

$$u_3 + v_2 - c_{32} = 3 + 3 - 8 = -2 < 0$$

[$1-\epsilon$] 2	[5] 3	[ϵ] 11	7
1	0	6	[1] 1
[$6+\epsilon$] 5	8	[$3-\epsilon$] 15	[1] 9

Put $\epsilon = 1$, we get

2	[5] 3	[1] 11	7	$u_1 = 0$
1	0	6	[1] 1	$u_2 = -4$
[7] 5	8	[2] 15	[1] 9	$u_3 = 4$
$v_1 = 1$	$v_2 = 3$	$v_3 = 11$	$v_4 = 5$	

For non basic cells

$$u_1 + v_1 - c_{11} = 0 + 1 - 2 = -1 < 0$$

$$u_1 + v_4 - c_{14} = 0 + 5 - 7 = -2 < 0$$

$$u_2 + v_1 - c_{21} = -4 + 1 - 1 = -4 < 0$$

$$u_2 + v_2 - c_{22} = -4 + 3 - 0 = -1 < 0$$

$$u_2 + v_3 - c_{23} = -4 + 11 - 6 = 1 > 0$$

$$u_3 + v_2 - c_{32} = 3 + 4 - 8 = -1 < 0$$

2	[5] 3	[1] 11	7
1	0	ϵ 6	[$1-\epsilon$] 1
[7] 5	8	[$2-\epsilon$] 15	[$1+\epsilon$] 9

Put $\epsilon = 1$, we get

2	[5] 3	[1] 11	7	$u_1 = 0$
1	0	[1] 6	1	$u_2 = -5$
[7] 5	8	[1] 15	[2] 9	$u_3 = 4$
$v_1 = 1$	$v_2 = 3$	$v_3 = 11$	$v_4 = 5$	

For non basic cells

$$u_1 + v_1 - c_{11} = 0 + 1 - 2 = -1 < 0$$

$$u_1 + v_4 - c_{14} = 0 + 5 - 7 = -2 < 0$$

$$u_2 + v_1 - c_{21} = -5 + 1 - 1 = -5 < 0$$

$$u_2 + v_2 - c_{22} = -5 + 3 - 0 = -2 < 0$$

$$u_2 + v_4 - c_{24} = -5 + 5 - 1 = -1 < 0$$

$$u_3 + v_2 - c_{32} = 3 + 4 - 8 = -1 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

$$Minimum\ cost = 5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9 = 100$$

4. A company has factories at F_1, F_2 and F_3 which supply to warehouses at W_1, W_2 and W_3. Weekly factory capacities are 200, 160 and 90 units, respectively. Weekly warehouse requirement are 180, 120 and 150 units, respectively. Unit shipping costs (in rupees) are as follows:

		W_1	W_2	W_3
Factories	F_1	16	20	12
	F_2	14	8	18
	F_3	26	24	16

Determine optimal distribution for this company to minimize total shipping cost.

Solution:

	W_1	W_2	W_3	Capacities
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Requirement	180	120	150	

Since the sum of capacities is equal to the sum of requirements then the problem is balanced.

Vogel's approximation method:

[140] 16	20	[60] 12	200/140	(4) (4)
[40] 14	[120] 8	18	160 /40	(6) (4)
26	24	[90] 16	90/0	(8) (10)
180	120/ 0	150/ 60/0		
(2)/(2)	(12)	(4)/ (6)		

$$Minimum\ cost = 140 \times 16 + 60 \times 12 + 40 \times 14 + 120 \times 8 + 90 \times 16 = 5920$$

The number of basic cells = 5 = number of rows +number of columns – 1

Therefore the solution is non degeneracy.

Modi method:

For basic cells $u_i + v_j - c_{ij} = 0$

(140) 16	20	(60) 12	$u_1 = 0$
(40) 14	(120) 8	18	$u_2 = -2$
26	24	(90) 16	$u_3 = 4$
$v_1 = 16$	$v_2 = 10$	$v_3 = 12$	

For non basic cells

$$u_1 + v_2 - c_{12} = 0 + 10 - 20 = -10 < 0$$
$$u_2 + v_3 - c_{23} = -2 + 12 - 18 = -8 < 0$$
$$u_3 + v_1 - c_{31} = 4 + 16 - 26 = -6 < 0$$
$$u_3 + v_2 - c_{32} = 4 + 10 - 24 = -10 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

$Minimum\ cost = 140 \times 16 + 60 \times 12 + 40 \times 14 + 120 \times 8 + 90 \times 16 = 5920$

5. Solve the transportation problem using Vogel's approximation method and check its optimality using MODI method.

From	To			Availability
	A	B	C	
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirements	4	2	2	

Solution:

Since the sum of Availability is equal to the sum of requirements the problem is balanced.

Vogel's approximation method:

[1] 50	30	220	1	(20) (20)
[3] 90	45	170	3	(45) (45)
250	[2] 200	[2] 50	4/2/0	(150) (50)
4	2 0	2 0		
(40)	(15)	(120)		

$$Minimum\ cost = 1 \times 50 + 3 \times 90 + 2 \times 200 + 2 \times 50 = 820$$

The number of basic cells = 4 < number of rows +number of columns – 1

Therefore degeneracy occurs. To make the solution has non degeneracy select the minimum value in the non basic cell and make it as basic cell with 0 value entered in the box.

[1] 50	[0] 30	220	1
[3] 90	45	170	3
250	[2] 200	[2] 50	4
4	2	2	

The number of basic cells = 5 =number of rows +number of columns – 1

Therefore the solution is non degeneracy.

Modi method:

For basic cells $u_i + v_j - c_{ij} = 0$

[1] 50	[0] 30	220	$u_1 = 0$
[3] 90	45	170	$u_2 = 40$
250	[2] 200	[2] 50	$u_3 = 170$
$v_1 = 50$	$v_2 = 30$	$v_3 = -120$	

For non basic cells

$$u_1 + v_3 - c_{13} = 0 - 120 - 220 = -340 < 0$$
$$u_2 + v_2 - c_{22} = 40 + 30 - 45 = 25 > 0$$
$$u_2 + v_3 - c_{23} = 40 - 120 - 170 = -250 < 0$$
$$u_3 + v_1 - c_{31} = 170 + 50 - 250 = -30 < 0$$

[$1+\epsilon$] 50	[$0-\epsilon$] 30	220
[$3-\epsilon$] 90	[ϵ] 45	170
250	[2] 200	[2] 50

Put $\epsilon = 0$, we get

[1] 50	30	220	$u_1 = 0$
[3] 90	[0] 45	170	$u_2 = 40$
250	[2] 200	[2] 50	$u_3 = 195$
$v_1 = 50$	$v_2 = 5$	$v_3 = -145$	

For non basic cells

$$u_1 + v_2 - c_{12} = 0 + 5 - 30 = -25 > 0$$
$$u_1 + v_3 - c_{13} = 0 - 145 - 220 = -365 < 0$$
$$u_2 + v_3 - c_{23} = 40 - 145 - 170 = -275 < 0$$
$$u_3 + v_1 - c_{31} = 195 + 50 - 250 = -5 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

$$Minimum\ cost = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50 = 820$$

6. Consider the following transportation problem:

Factory	Godowns						
	1	2	3	4	5	6	Stock Available
A	7	5	7	7	5	3	60
B	9	11	6	11	-	5	20
C	11	10	6	2	2	8	90
D	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

It is not possible to transport any quantity from factory B to godown 5. Determine:

(i) Initial solution by Vogel's Approximation method.

(ii) Optimal basic feasible solution.

(iii) Is the optimal solution unique?

If not find the alternative optimum basic feasible solution.

Solution: Since the total stock availability is equal to the total demand, the problem is balanced.

Vogel's approximation method:

7	[20] 5	7	7	5	[40] 3	60/40 /0 (2) /(2)/(2) /(4)
[10] 9	11	[10] 6	11	∞	5	20 /10 (1)/(1)/(1)/(1)/(3)
11	10	[30] 6	[20] 2	[40] 2	8	90 /70 /30 /0 (0)/(0)/(4)/(2)/(5)
[50] 9	10	9	6	9	12	50 (3)/(3)/(0)/(0)/(0)
60 (2)/(0) (0)	20/0 (5)	40/10/0 (0)/(0)	20/0 (4)	40/0 (3)	40/0 (2)	

The number of basic cells = 8 < number of rows +number of columns – 1

Therefore degeneracy occurs. To make the solution has non degeneracy select the minimum value in the non basic cell and make it as basic cell with 0 value entered in the box.

7	[20] 5	7	7	[0] 5	[40] 3	60
[10] 9	11	[10] 6	11	∞	5	20
11	10	[30] 6	[20] 2	[40] 2	8	90
[50] 9	10	9	6	9	12	50
60	20	40	20	40	40	

$$Minimum\ cost = 20 \times 5 + 0 \times 5 + 40 \times 3 + 10 \times 9 + 10 \times 6 + 30 \times 6 + 20 \times 2$$
$$+40 \times 2 + 50 \times 9 = 1120$$

Modi method

7	[20] 5	7	7	[0] 5	[40] 3	$u_1 = 0$
[10] 9	11	[10] 6	11	∞	5	$u_2 = -3$
11	10	[30] 6	[20] 2	[40] 2	8	$u_3 = -3$
[50] 9	10	9	6	9	12	$u_4 = -3$
$v_1 = 12$	$v_2 = 5$	$v_3 = 9$	$v_4 = 5$	$v_5 = 5$	$v_6 = 3$	

For non basic cells

$$u_1 + v_1 - c_{11} = 0 + 12 - 7 = 5 > 0$$
$$u_1 + v_3 - c_{13} = 0 + 9 - 7 = 2 > 0$$
$$u_1 + v_4 - c_{14} = 0 + 5 - 7 = -2 < 0$$
$$u_2 + v_2 - c_{22} = -3 + 5 - 11 = -9 < 0$$
$$u_2 + v_4 - c_{24} = -3 + 5 - 11 = -9 < 0$$
$$u_2 + v_5 - c_{25} = -3 + 5 - \infty = -\infty < 0$$
$$u_2 + v_6 - c_{26} = -3 + 3 - 5 = -5 < 0$$
$$u_3 + v_1 - c_{31} = -3 + 12 - 11 = -2 < 0$$
$$u_3 + v_2 - c_{32} = -3 + 5 - 10 = -8 < 0$$
$$u_3 + v_6 - c_{36} = -3 + 3 - 8 = -8 < 0$$
$$u_4 + v_2 - c_{42} = -3 + 5 - 10 = -8 < 0$$
$$u_4 + v_3 - c_{43} = -3 + 9 - 9 = -3 < 0$$
$$u_4 + v_4 - c_{44} = -3 + 5 - 6 = -4 < 0$$
$$u_4 + v_5 - c_{45} = -3 + 5 - 9 = -7 < 0$$
$$u_4 + v_6 - c_{46} = -3 + 3 - 12 = -12 < 0$$

ϵ 7	[20] 5	7	7	[0] $-\epsilon$ 5	[40] 3
[10] $-\epsilon$ 9	11	[10] $+\epsilon$ 6	11	∞	5
11	10	[30] $-\epsilon$ 6	[20] 2	[40] $+\epsilon$ 2	8
[50] 9	10	9	6	9	12

Put $\epsilon = 0$

[0] 7	[20] 5	7	7	5	[40] 3	$u_1 = 0$
[10] 9	11	[10] 6	11	∞	5	$u_2 = 2$
11	10	[30] 6	[20] 2	[40] 2	8	$u_3 = 2$
[50] 9	10	9	6	9	12	$u_4 = 2$
$v_1 = 7$	$v_2 = 5$	$v_3 = 4$	$v_4 = 0$	$v_5 = 0$	$v_6 = 3$	

For non basic cells

$$u_1 + v_3 - c_{13} = 0 + 4 - 7 = -3 < 0$$
$$u_1 + v_4 - c_{14} = 0 + 0 - 7 = -7 < 0$$
$$u_1 + v_5 - c_{15} = 0 + 0 - 5 = -5 < 0$$
$$u_2 + v_2 - c_{22} = 2 + 5 - 11 = -4 < 0$$
$$u_2 + v_4 - c_{24} = 2 + 0 - 11 = -9 < 0$$
$$u_2 + v_5 - c_{25} = 2 + 0 - \infty = -\infty < 0$$
$$u_2 + v_6 - c_{26} = 2 + 3 - 5 = 0$$

$$u_3 + v_1 - c_{31} = 2 + 7 - 11 = -2 < 0$$
$$u_3 + v_2 - c_{32} = 2 + 5 - 10 = -3 < 0$$
$$u_3 + v_6 - c_{36} = 2 + 3 - 8 = -3 < 0$$
$$u_4 + v_2 - c_{42} = 2 + 5 - 10 = -3 < 0$$
$$u_4 + v_3 - c_{43} = 2 + 4 - 9 = -3 < 0$$
$$u_4 + v_4 - c_{44} = 2 + 0 - 6 = -4 < 0$$
$$u_4 + v_5 - c_{45} = 2 + 0 - 9 = -7 < 0$$
$$u_4 + v_6 - c_{46} = 2 + 3 - 12 = -7 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

$$Minimum\ cost = 0 \times 7 + 20 \times 5 + 40 \times 3 + 10 \times 9 + 10 \times 6 + 30 \times 6 + 20 \times 2$$
$$+40 \times 2 + 50 \times 9 = 1120$$

7. The Burgarf Co. distributes its products from 2 ware houses to 5 major metropolitan areas. Next months anticipated demand exceeds the warehouse supplies and the management of Burgraf would like to know how to distribute their product in order to maximize the revenue. Their distribution costs per unit are shown in the table. To be competitive Burgraf must charge a different price in different cities. Selling prices are Rs. 9.95, Rs. 10.5, Rs. 9.5, Rs. 11.15, Rs. 10.19 in cities 1, 2, 3, 4 and 5.

From Warehouse	1	2	3	4	5	Supply
1	1.5	1.65	2.05	1.4	1.35	800
2	1.6	2.1	1.8	1.65	2	1000
Demand	500	200	300	600	800	

Find the optimal allocation to maximize the revenue.

Solution:

From Warehouse	1	2	3	4	5	Supply
1	9.95 - 1.5	10.5 - 1.65	9.5 - 2.05	11.15 - 1.4	10.19 - 1.35	800
2	9.95 - 1.6	10.5 - 2.1	9.5 - 1.8	11.15 - 1.65	10.19 - 2	1000
Demand	500	200	300	600	800	

Total demand $= 500 + 200 + 300 + 600 + 800 = 2400$

Total supply $= 800 + 1000 = 1800$

Total demand ≠ Total supply. Therefore the problem is unbalanced.

Since the total supply is less than the total demand add an extra row with zero entries and demand as the difference between their sums to make the problem balanced one.

8.45	8.85	7.45	9.75	8.84	800
8.35	8.4	7.7	9.5	8.19	1000
0	0	0	0	0	600
500	200	300	600	800	

Convert the maximization problem into a minimization problem by subtracting all the values from the highest value.

1.3	0.9	2.3	[0] 0	[800] 0.91	800 (0.9)/(0.01)
[200] 1.4	[200] 1.35	2.05	[600] 0.25	1.56	1000 /400/200/0 (1.1) /(0.05) /(0.05)/(0.65)
[300] 9.75	9.75	[300] 9.75	9.75	9.75	600 (0) /(0) /(0)
500/300 (0.1) (8.35)	200 (0.45) (8.4)	300 (0.25) (7.7)	600/0 (0.25)	800 (0.65)	

1.3	0.9	2.3	[0] 0	[800] 0.91	$u_1 = 0$
[200] 1.4	[200] 1.35	2.05	[600] 0.25	1.56	$u_2 = 0.25$
[300] 9.75	9.75	[300] 9.75	9.75	9.75	$u_3 = 8.6$
$v_1 = 1.15$	$v_2 = 1.1$	$v_3 = 1.15$	$v_4 = 0$	$v_5 = 0.91$	

For non basic cells

$$u_1 + v_1 - c_{11} = 1.15 + 0 - 1.3 = -0.15 < 0$$
$$u_1 + v_2 - c_{12} = 0 + 1.1 - 0.9 = 0.2 > 0$$
$$u_1 + v_3 - c_{13} = 0 + 1.15 - 2.3 = -1.15 < 0$$
$$u_2 + v_3 - c_{23} = 0.25 + 1.15 - 2.05 = -0.65 < 0$$
$$u_2 + v_5 - c_{25} = 0.25 + 0.91 - 1.56 = -0.4 < 0$$
$$u_3 + v_2 - c_{32} = 8.6 + 1.1 - 9.75 = -0.05 < 0$$
$$u_3 + v_4 - c_{34} = 8.6 + 0 - 9.75 = -1.15 < 0$$
$$u_3 + v_6 - c_{36} = 8.6 + 0.91 - 9.75 = -0.24 < 0$$

1.3	ϵ 0.9	2.3	[0] $-\epsilon$ 0	[800] 0.91
[200] 1.4	[200] $-\epsilon$ 1.35	2.05	[600] $+\epsilon$ 0.25	1.56
[300] 9.75	9.75	[300] 9.75	9.75	9.75

Put $\epsilon = 0$

1.3	(0) 0.9	2.3	0	(800) 0.91	$u_1 = 0$
(200) 1.4	(200) 1.35	2.05	(600) 0.25	1.56	$u_2 = 0.45$
(300) 9.75	9.75	(300) 9.75	9.75	9.75	$u_3 = 8.8$
$v_1 = 0.95$	$v_2 = 0.9$	$v_3 = 0.95$	$v_4 = -0.2$	$v_5 = 0.91$	

For non basic cells

$$u_1 + v_1 - c_{11} = 0 + 0.95 - 1.3 = -0.35 < 0$$
$$u_1 + v_3 - c_{13} = 0 + 0.95 - 2.3 = -1.35 < 0$$
$$u_1 + v_4 - c_{14} = 0 - 0.2 - 0 = -0.2 < 0$$
$$u_2 + v_3 - c_{23} = 0.45 + 0.95 - 2.05 = -0.65 < 0$$
$$u_2 + v_5 - c_{25} = 0.45 + 0.91 - 1.56 = -0.2 < 0$$
$$u_3 + v_2 - c_{32} = 8.8 + 0.9 - 9.75 = -0.05 < 0$$
$$u_3 + v_4 - c_{34} = 8.8 - 0.2 - 9.75 = -1.15 < 0$$
$$u_3 + v_6 - c_{36} = 8.8 + 0.91 - 9.75 = -0.04 < 0$$

Since $u_i + v_j - c_{ij} \leq 0, \forall i, j$. Optimum solution is reached.

8.45	(0) 8.85	7.45	9.75	(800) 8.84	800
(200) 8.35	(200) 8.4	7.7	(600) 9.5	8.19	1000
(300) 0	0	(300) 0	0	0	600
500	200	300	600	800	

$$Maximum\ cost = 0 \times 8.85 + 800 \times 8.84 + 200 \times 8.35 + 200 \times 8.4 + 600 \times 9.5$$
$$+300 \times 0 + 300 \times 0 = 16122$$

Assignment Problem

1. At the end of a cycle schedule a trucking firm has a surplus of one variable in each of the cities 1, 2, 3, 4 and 5 and a deficit of one vehicle on each of the cities A, B, C, D, E and F. The costs in rupees of transportation and handling between the cities are given below. Find the assignment of surplus, vehicles to deficit cities which results in minimum total cost. Also find out which city will not receive a vehicle?

	A	B	C	D	E	F
1	134	116	167	233	194	197
2	114	195	260	166	175	130
3	129	117	48	94	66	101
4	71	156	92	143	114	136
5	97	134	125	83	1425	118

Solution:

Since the number of rows is less than the number of column, the problem is unbalanced. To make it balance (i.e., number of rows =number of columns) add a row with zero entries.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{pmatrix} 134 & 116 & 167 & 233 & 194 & 197 \\ 114 & 195 & 260 & 166 & 175 & 130 \\ 129 & 117 & 48 & 94 & 66 & 101 \\ 71 & 156 & 92 & 143 & 114 & 136 \\ 97 & 134 & 125 & 83 & 1425 & 118 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Row reduction:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{pmatrix} 18 & \boxed{0} & 51 & 117 & 78 & 81 \\ \boxed{0} & 81 & 146 & 52 & 61 & 16 \\ 81 & 117 & \boxed{0} & 46 & 18 & 53 \\ \not{0} & 85 & 21 & 72 & 43 & 65 \\ 14 & 51 & 42 & \boxed{0} & 1342 & 35 \\ \not{0} & \not{0} & \not{0} & \not{0} & \boxed{0} & \not{0} \end{pmatrix} \end{array}$$

Number of lines drawn ≠ Number of rows. Optimality condition is not satisfied.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{pmatrix}
18 & \boxed{0} & 51 & 117 & 62 & 65 \\
\not{0} & 81 & 146 & 52 & 45 & \boxed{0} \\
81 & 117 & \boxed{0} & 46 & 2 & 37 \\
\boxed{0} & 85 & 21 & 72 & 27 & 49 \\
14 & 51 & 42 & \boxed{0} & 1326 & 19 \\
16 & 16 & 16 & 16 & \boxed{0} & \not{0}
\end{pmatrix}
\end{array}
$$

Number of lines drawn = Number of rows. Therefore Optimality condition is satisfied.

$$1 \to B, 2 \to F, 3 \to C, 4 \to A, 5 \to D, 6 \to E$$

$$Minimum\ cost = 116 + 130 + 48 + 71 + 83 + 0 = 448$$

2. A company is faced with the problem of assigning 4 machines to different jobs (one machine to one job only). The profits are estimated as follows. Solve the problem to maximize the total profits.

Job	Machine			
	A	B	C	D
1	3	6	2	6
2	7	1	4	7
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solution:

Since the number of rows is more than the number of columns, the problem is unbalanced. To make it balance (i.e., number of rows =number of columns) add two columns with zero entries.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{pmatrix}
3 & 6 & 2 & 6 & 0 & 0 \\
7 & 1 & 4 & 7 & 0 & 0 \\
3 & 8 & 5 & 8 & 0 & 0 \\
6 & 4 & 3 & 7 & 0 & 0 \\
5 & 2 & 4 & 3 & 0 & 0 \\
5 & 7 & 6 & 4 & 0 & 0
\end{pmatrix}
\end{array}
$$

Converting maximization problem into a minimization problem by subtracting each value of the row from its highest element in the row.

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} 3 & 0 & 4 & 0 & 6 & 6 \\ 0 & 6 & 3 & 0 & 7 & 7 \\ 5 & 0 & 3 & 0 & 8 & 8 \\ 1 & 3 & 4 & 0 & 7 & 7 \\ 0 & 3 & 1 & 2 & 5 & 5 \\ 2 & 0 & 1 & 3 & 7 & 7 \end{pmatrix} \end{array}$$

Column reduction:

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} 3 & \boxed{0} & 3 & \cancel{0} & 1 & 1 \\ \boxed{0} & 6 & 2 & \cancel{0} & 2 & 2 \\ 5 & \cancel{0} & 2 & \cancel{0} & 3 & 3 \\ 1 & 3 & 3 & \boxed{0} & 2 & 2 \\ \cancel{0} & 3 & \cancel{0} & 2 & \boxed{0} & \cancel{0} \\ 2 & \cancel{0} & \boxed{0} & 3 & 2 & 2 \end{pmatrix} \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} 3 & \cancel{0} & 2 & \cancel{0} & \boxed{0} & \cancel{0} \\ \boxed{0} & 6 & 1 & \cancel{0} & 1 & 1 \\ 5 & \boxed{0} & 1 & \cancel{0} & 2 & 2 \\ 1 & 3 & 2 & \boxed{0} & 1 & 1 \\ 1 & 4 & \cancel{0} & 3 & \cancel{0} & \boxed{0} \\ 3 & 1 & \boxed{0} & 4 & 2 & 2 \end{pmatrix} \end{array}$$

Number of lines drawn = Number of rows. Therefore Optimality condition is satisfied.

$$1 \to E, 2 \to A, 3 \to B, 4 \to D, 5 \to F, 6 \to C$$

$$Maximum\ cost = 0 + 7 + 8 + 7 + 0 + 6 = 28$$

3. A computer centre has three expert programmers. The head of the centre wants to allocate projects to programmers. The development time in hours is given in the table below. Allocate the projects to the programmers, in-order to minimize the development time.

Projects	Programmers		
	1	2	3
A	120	100	80
B	80	90	110
C	110	140	120

Solution:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 120 & 100 & 80 \\ 80 & 90 & 110 \\ 110 & 140 & 120 \end{pmatrix} \end{array}$$

Since the number of rows is equal to the number of columns, the problem is balanced.

Row reduction:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 40 & 20 & 0 \\ 0 & 10 & 30 \\ 0 & 30 & 10 \end{pmatrix} \end{array}$$

Column reduction:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 40 & 10 & \boxed{0} \\ \not{0} & \boxed{0} & 30 \\ \boxed{0} & 20 & 10 \end{pmatrix} \end{array}$$

Number of lines drawn = Number of rows. Therefore Optimality condition is satisfied.

$$A \to 3, B \to 2, C \to A$$

$$Maximum\ cost = 80 + 90 + 110 = 280$$

4. A department has five employees with five jobs to be performed. The time (in hours) each mean will take to perform each job is given in the effectiveness matrix.

		Employees				
		I	II	III	IV	V
Jobs	A	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

Solution:

Since the number of rows is equal to the number of columns, the problem is balanced.

Row reduction:

$$\begin{array}{c|ccccc} & I & II & III & IV & V \\ A & 5 & \boxed{0} & 8 & 10 & 11 \\ B & \boxed{0} & 6 & 15 & 10 & 3 \\ C & 8 & 5 & \boxed{0} & \cancel{0} & \cancel{0} \\ D & \cancel{0} & 4 & 2 & \cancel{0} & 5 \\ E & 3 & 5 & 6 & \boxed{0} & 8 \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c|ccccc} & I & II & III & IV & V \\ A & 5 & \boxed{0} & 6 & 10 & 9 \\ B & \boxed{0} & 6 & 13 & 10 & 1 \\ C & 10 & 7 & \cancel{0} & 2 & \boxed{0} \\ D & \cancel{0} & 4 & \boxed{0} & \cancel{0} & 3 \\ E & 3 & 5 & 4 & \boxed{0} & 6 \end{array}$$

Number of lines drawn $=$ Number of rows. Therefore Optimality condition is satisfied.

$$A \to II, B \to I, C \to V, D \to III, E \to IV$$

$$Minimum\ cost = 5 + 3 + 2 + 9 + 4 = 23$$

5. Solve the assignment problem.

		P	Q	R	S
	A	18	26	17	11
Machine	B	13	28	14	26
	C	38	19	18	15
	D	19	26	24	10

Solution:

Since the number of rows is equal to the number of columns, the problem is balanced.

Row reduction:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{pmatrix} \end{array}$$

Column reduction:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \times\!\!\!\!0 & 13 \\ 23 & \boxed{0} & 2 & \times\!\!\!\!0 \\ 9 & 12 & 13 & \times\!\!\!\!0 \end{pmatrix} \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 5 & 11 & 3 & \boxed{0} \\ \boxed{0} & 13 & \times\!\!\!\!0 & 15 \\ 21 & \boxed{0} & \times\!\!\!\!0 & \times\!\!\!\!0 \\ 7 & 12 & 11 & \times\!\!\!\!0 \end{pmatrix} \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 2 & 8 & \boxed{0} & \times\!\!\!\!0 \\ \boxed{0} & 13 & \times\!\!\!\!0 & 18 \\ 21 & \boxed{0} & \times\!\!\!\!0 & 3 \\ 4 & 9 & 8 & \boxed{0} \end{pmatrix} \end{array}$$

Number of lines drawn $=$ Number of rows. Therefore Optimality condition is satisfied.

$$A \rightarrow R, B \rightarrow P, C \rightarrow Q, D \rightarrow S$$

$$Minimum\ cost = 17 + 13 + 19 + 10 = 59$$

6. A company has five jobs to be done. The following matrix shows the return in rupees of assigning i^{th} machine $(i = 1,2,3,4,5)$ to the j^{th} job $(j = A,B,C,D,E)$. Assign the five jobs to the five machines to maximize profit.

	A	B	C	D	E
1	5	11	10	12	4
2	2	4	6	3	5
3	3	12	5	14	6
4	6	14	4	11	7
5	7	9	8	12	5

Solution:

Since the number of rows is equal to the number of columns, the problem is balanced.

Row reduction:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 1 & 7 & 6 & 8 & 0 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 9 & 2 & 11 & 3 \\ 2 & 10 & 0 & 7 & 3 \\ 2 & 4 & 3 & 7 & 0 \end{pmatrix} \end{array}$$

Column reduction:

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 1 & 5 & 6 & 7 & \boxed{0} \\ \cancel{0} & \boxed{0} & 4 & \cancel{0} & 3 \\ \boxed{0} & 7 & 2 & 10 & 3 \\ 2 & 8 & \boxed{0} & 6 & 3 \\ 2 & 2 & 3 & 6 & \cancel{0} \end{pmatrix} \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 1 & 3 & 6 & 5 & \boxed{0} \\ 2 & \cancel{0} & 6 & \boxed{0} & 5 \\ \boxed{0} & 5 & 2 & 8 & 3 \\ 2 & 6 & \boxed{0} & 4 & 3 \\ 2 & \boxed{0} & 3 & 4 & \cancel{0} \end{pmatrix} \end{array}$$

Number of lines drawn $=$ Number of rows. Therefore Optimality condition is satisfied.

$$1 \to E, 2 \to D, 3 \to A, 4 \to C, 5 \to B$$

$$Minimum\ cost = 4 + 3 + 3 + 4 + 9 = 23$$

Travelling Salesman Problem

1. A delivery truck must leave a warehouse and visit four customer delivery points and return to the warehouse. Find out the best route where (1 represents the warehouse).

Distance (Miles)

	To	I	2	3	4	5
From	1	-	10	7	5	5
	2	11	-	4	6	8
	3	7	4	-	7	8
	4	5	5	7	-	3
	5	3	8	8	3	-

Solution:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \infty & 10 & 7 & 5 & 5 \\ 11 & \infty & 4 & 6 & 8 \\ 7 & 4 & \infty & 7 & 8 \\ 5 & 5 & 7 & \infty & 3 \\ 3 & 8 & 8 & 3 & \infty \end{pmatrix} \end{array}$$

Row reduction:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \infty & 5 & 2 & \boxed{0} & \cancel{0} \\ 7 & \infty & \boxed{0} & 2 & 4 \\ 3 & \boxed{0} & \infty & 3 & 4 \\ 2 & 2 & 4 & \infty & \boxed{0} \\ \boxed{0} & 5 & 5 & \cancel{0} & \infty \end{pmatrix} \end{array}$$

Number of lines drawn = Number of rows. Therefore Optimality condition is satisfied.

$$1 \to 4 \to 5 \to 1, 2 \to 3 \to 2$$

$$Minimum\ cost = 5 + 3 + 3 + 4 + 4 = 19$$

$$12\ and\ 34 = 5 + 3 = 8$$

$$13\ and\ 24 = 2 + 2 = 4$$

$$42\ and\ 35 = 2 + 4 = 6$$

$$43\ and\ 25 = 4 + 4 = 8$$

$$52 \; and \; 31 = 5 + 3 = 8$$
$$53 \; and \; 21 = 5 + 7 = 12$$

Since 13 and 24 give the minimum extra amount we select this route.

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 1$$
$$Minimum \; cost = 7 + 4 + 6 + 3 + 3 = 23$$

2. A salesman has to travel five cities in a month without revisiting any city. Determine an optimal travel plan for the time matrix (in hours) given.

To City

From City		A	B	C	D	E
	A	-	4	7	3	4
	B	4	-	6	3	4
	C	7	6	-	7	5
	D	3	3	7	-	7
	E	4	4	5	7	-

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} \infty & 4 & 7 & 3 & 4 \\ 4 & \infty & 6 & 3 & 4 \\ 7 & 6 & \infty & 7 & 5 \\ 3 & 3 & 7 & \infty & 7 \\ 4 & 4 & 5 & 7 & \infty \end{pmatrix} \end{array}$$

Row reduction:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix} \end{array}$$

Column reduction:

$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} \infty & 1 & 3 & \boxed{0} & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty \end{pmatrix} \end{array}$$

Number of lines drawn $\neq$ Number of rows. Optimality condition is not satisfied.

$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} \infty & \boxed{0} & 2 & \cancel{0} & 1 \\ \cancel{0} & \infty & 1 & \boxed{0} & 1 \\ 1 & \cancel{0} & \infty & 2 & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 5 \\ \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{pmatrix} \end{array}$$

Number of lines drawn $=$ Number of rows. Therefore Optimality condition is satisfied.

$$A \rightarrow B \rightarrow D \rightarrow A, C \rightarrow E \rightarrow C$$

$$Minimum\ cost = 4 + 3 + 3 + 5 + 5 = 20$$

$$AC\ and\ EB = 2 + 0 = 2$$

$$AE\ and\ CB = 1 + 0 = 1$$

$$BC\ and\ ED = 1 + 4 = 5$$

$$BE\ and\ CD = 1 + 2 = 3$$

$$DC\ and\ EA = 3 + 0 = 3$$

$$DE\ and\ CA = 5 + 1 = 6$$

Since AE and CB give the minimum extra amount we select this route.

$$A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$$

$$Minimum\ cost = 4 + 5 + 6 + 3 + 3 = 21$$