UNIT-I

FOURIER SERIES

Periodic function :

A function $f(x)$ is said to be periodic, if and only if $f(x+T)=f(x)$ where T is called period for the function $f(x)$.

Eg: $\sin x \ and \cos x$ are periodic functions with period $2\pi. i.e \sin(2\pi + x) = \sin x$

$\cos(2\pi + x) = \cos x$

Fourier series:

Fourier series is an infinite trigonometric series defined by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right), c\leq x\leq c+2l$$

Where $a_0, a_n \ and \ b_n$ are called Euler's constants or Fourier coefficients.

Dirichlet's conditions:

A function defined in $c \leq x \leq c+2l$ can be expanded as an infinite trigonometric series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right), \text{provided}$$

i) $f(x)$ is single valued and periodic in $(c, c+2l)$

ii) $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuous in $(c, c+2l)$

iii) $f(x)$ has no or finite number of maxima or minima in $(c, c+2l)$.

Fourier coefficients *for the function f(x) in the interval $c \leq x \leq c+2l$ is given by*

$$a_0=\frac{1}{l}\int_{c}^{c+2l} f(x)dx, a_n=\frac{1}{l}\int_{c}^{c+2l} f(x)cos\ \frac{n\pi x}{l}dx,$$

$$b_n=\frac{1}{l}\int_{c}^{c+2l} f(x)sin\ \frac{n\pi x}{l}dx$$

Fourier coefficients *for the function f(x) in the interval $0 \leq x \leq 2l$ is given by*

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx\,, a_n = \frac{1}{l}\int_0^{2l} f(x)cos\,\frac{n\pi x}{l}dx\,, b_n = \frac{1}{l}\int_0^{2l} f(x)sin\,\frac{n\pi x}{l}dx$$

Fourier coefficients *for the function f(x) in the interval $-l \le x \le l$ is given by*

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx\,, a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\,\frac{n\pi x}{l}dx\,, b_n = \frac{1}{l}\int_{-l}^{l} f(x)sin\,\frac{n\pi x}{l}dx$$

Fourier series for the function f(x) in the interval $0 \le x \le 2\pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cos\,nxdx, b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sin\,nxdx$$

Fourier series for the function f(x) in the interval $-\pi \le x \le \pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\,, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos\,nxdx\,, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin\,nxdx$$

Fourier series for the even function f(x) in the interval $-\pi \le x \le \pi$. (or)

Half range cosine series in the interval $0 \le x \le \pi$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx\,, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx\,, b_n = 0$$

then the Fourier series is reduced to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Fourier series for the odd function f(x) in the interval$-\pi \le x \le \pi$. (or)

Half range sine series in the interval $0 \le x \le \pi$.

$$a_0 = 0, \;\; a_n = 0, b_n = \frac{2}{\pi}\int_0^{\pi} f(x) sin\, nx dx,$$

then the Fourier series is reduced to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier series for the even function f(x) in the interval $-l \le x \le l$. (or)

Half range Fourier cosine series in the interval $0 \le x \le l$

$$a_0 = \frac{2}{l}\int_0^{l} f(x) dx \,, a_n = \frac{2}{l}\int_0^{l} f(x) cos\, \frac{n\pi x}{l} dx \,, b_n = 0$$

then the Fourier series is reduced to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

Fourier series for the odd function f(x) in the interval $-l \le x \le l$. (or)

Half range Fourier sine series in the interval $0 \le x \le l$

$$a_0 = 0, \;\; a_n = 0, b_n = \frac{2}{l}\int_0^{l} f(x) sin\, \frac{n\pi x}{l} dx$$

then the Fourier series is reduced to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Fourier series for the complex function f(x) in the interval $-\pi \le x \le \pi$

$$f(x) \;\; = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

Parseval's identity for the Fourier series in the interval $(\mathbf{0}, \mathbf{2\pi})$

$$\frac{1}{\pi}\int_{0}^{2\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Parseval's identity for the Fourier series in the interval $(\mathbf{0}, \mathbf{2l})$

$$\frac{1}{l}\int_{0}^{2l} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Parseval's identity for the Fourier series in the interval $(-\boldsymbol{\pi}, \boldsymbol{\pi})$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Parseval's identity for the Fourier series in the interval $(-\boldsymbol{l}, \boldsymbol{l})$

$$\frac{1}{\pi}\int_{-l}^{l} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Parseval's identity for the half range Fourier cosine series in the interval $(\mathbf{0}, \boldsymbol{\pi})$

$$\frac{2}{\pi}\int_{0}^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2]$$

Parseval's identity for the half range Fourier sine series in the interval $(\mathbf{0}, \boldsymbol{\pi})$

$$\frac{2}{\pi}\int_{0}^{\pi} [f(x)]^2\, dx = \sum_{n=1}^{\infty} [b_n^2]$$

Parseval's identity for the half range Fourier cosine series in the interval $(\mathbf{0}, \boldsymbol{l})$

$$\frac{2}{l}\int_{0}^{l} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} [a_n^2]$$

Parseval's identity for the half range Fourier sine series in the interval $(\mathbf{0}, \mathbf{2l})$

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty}[b_n^2]$$

Root-Mean Square value of a function $f(x)$

Solution:

RMS value of a function $f(x)$

$$= \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

Problems:

1.Find the Fourier series for $f(x) = x\,sinx, 0 < x < 2\pi.$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\ \ dx$$

$$= \frac{1}{\pi}[x(-\cos x) - 1(-\sin x)]_0^{2\pi}$$

$$= \frac{1}{\pi}[2\pi(-\cos 2\pi) - 1(-\sin 2\pi) - 0] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) cos\ nxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x\ cos\ nxdx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(1+n)x + \sin(1-n)x]dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(1+n)x}{1+n}\right) - 1.\left(\frac{-\sin(1+n)x}{(1+n)^2}\right)\right]_0^{2\pi}$$

$$+\frac{1}{2\pi}\left[x\left(\frac{-\cos(1-n)x}{1-n}\right)-1.\left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi}$$

$$=-\left(\frac{1}{1+n}+\frac{1}{1-n}\right)$$

$$=-\frac{2}{1-n^2}, n \neq 1$$

When $n = 1$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) cos\, xdx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x \sin x\, cos\, xdx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\, dx$$

$$=\frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-1.\left(\frac{-\sin 2x}{2^2}\right)\right]_0^{2\pi}=-\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) sin\, nxdx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x \sin x\, sin\, nxdx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x[\cos(1-n)x-\cos(1+n)x]dx$$

$$=\frac{1}{2\pi}\left[x\left(\frac{\sin(1-n)x}{1-n}\right)-1.\left(\frac{-\cos(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[x\left(\frac{\sin(1+n)x}{1+n}\right)-1.\left(\frac{-\cos(1+n)x}{(1+n)^2}\right)\right]_0^{2\pi}=0, n \neq 1$$

When $n = 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) sin\, xdx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \, sin\, xdx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[1 - \cos 2x]dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} - \frac{\cos 2x}{2^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(4\pi^2 - \frac{4\pi^2}{2}\right) = \pi$$

$$f(x) = -1 - \frac{1}{2}cosx + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi \sin x$$

2. Show that for

$$0 < x < l, x = \frac{2l}{\pi}\left(sin\frac{\pi x}{l} - \frac{1}{2}sin\frac{2\pi x}{l} + \frac{1}{3}sin\frac{3\pi x}{l} - \cdots\right)$$

Using root mean square value of x, deduce the value of

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Solution: We know that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

Where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x \; dx$$

$$= \frac{2}{l}\int_0^l x \sin\frac{n\pi}{l}x \; dx$$

$$= \frac{2}{l}\left[x\left(-\frac{\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2l}{n\pi}(-1)^{n+1}$$

$$x = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\frac{n\pi}{l}x$$

$$x = \frac{2l}{\pi}\left(sin\frac{\pi x}{l} - \frac{1}{2}sin\frac{2\pi x}{l} + \frac{1}{3}sin\frac{3\pi x}{l} - \cdots\right)$$

Using RMS value

$$\frac{2}{l}\int_0^l [f(x)]^2\, dx = \sum_{n=1}^{\infty}[b_n^2]$$

$$\frac{2}{l}\int_0^l x^2\, dx = \sum_{n=1}^{\infty}\left(\frac{2l}{n\pi}(-1)^{n+1}\right)^2$$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}(-1)^{n+1}\right)^2$$

$$\frac{2}{l}\frac{l^3}{3} = \frac{4l^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

3. Obtain Fourier series for $f(x) = x^2$ in $0 \le x \le 2\pi$.

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x^2\, dx$$

$$=\frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi}=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)cos\,nxdx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x^2cos\,nxdx$$

$$=\frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)sin\,nxdx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x^2sin\,nxdx$$

$$=\frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right)-2x\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}=\frac{-4}{n}\pi$$

$$f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx-\frac{\pi}{n}\sin nx\right)$$

4. Prove that in the interval

$$0\ <\ x\ <l, x=\frac{l}{2}-\frac{4l}{\pi^2}\left(cos\frac{\pi x}{l}+\frac{1}{3^2}cos\frac{3\pi x}{l}+\cdots\right)$$

and deduce that

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}$$

Solution: We know that

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l x\,dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{l}{n\pi}\right)^2(-1)^{n+1} - \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2l}{n^2\pi^2}((-1)^n - 1)$$

When n is odd

$$a_n = -\frac{4l}{n^2\pi^2}$$

When n is even

$$a_n = 0$$

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(cos\frac{\pi x}{l} + \frac{1}{3^2}cos\frac{3\pi x}{l} + \cdots\right)$$

Using Parseval's identity

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{l}\int_0^l x^2\,dx = \frac{l^2}{2} + \sum_{n=1,3,5,\dots}^{\infty}\left(-\frac{4l}{n^2\pi^2}\right)^2$$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{2} + \frac{16l^2}{\pi^4}\sum_{n=1,3,5,\dots}^{\infty}\left(\frac{1}{n^2}\right)^2$$

$$\frac{2l^2}{3} - \frac{l^2}{2} = \frac{16l^2}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$$\frac{16l^2}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) = \frac{l^2}{6}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

5. Find the complex form of Fourier series of the function $f(x) = e^{ax}, -\pi \leq x \leq \pi$

In the form

$$e^{ax} = \frac{\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n\,\frac{a+in}{a^2+n^2}e^{inx}$$

And hence prove that

$$\frac{\pi}{\text{a}\sinh a\pi} = \sum_{-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}$$

Solution: We know that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\left(e^{(a-in)\pi} - e^{(a-in)(-\pi)}\right)$$

$$= \frac{(a+in)}{2\pi(a-in)(a+in)}\left(e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right)$$

$$= \frac{(a+in)}{2\pi(a^2+n^2)}\left(e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i\sin n\pi)\right)$$

$$= \frac{(a+in)(-1)^n}{\pi(a^2+n^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$c_n = \frac{(a+in)(-1)^n}{\pi(a^2+n^2)}\sinh a\pi$$

$$e^{ax} = \frac{\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}e^{inx}$$

Put $x = 0$in the above series, We get

$$1 = \frac{\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}$$

Equating real parts on both sides, we get

$$1 = \frac{\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a}{a^2+n^2}$$

$$\frac{\pi}{\text{a}\sinh a\pi} = \sum_{-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}$$

6. Find the Fourier series for $f(x) = \begin{cases} l-x, 0 < x < l \\ 0,\ l < x < 2l \end{cases}$

Hence deduce the sum to infinity of the series

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

Where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx = \frac{1}{l}\int_0^{l}(l-x)dx$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{1}{l}\left[l^2 - \frac{l^2}{2}\right]$$

$$a_0 = \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)cos\ \frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\int_0^{l}(l-x)cos\ \frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{sin\ \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{cos\ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\frac{-cos\ n\pi}{\left(\frac{n\pi}{l}\right)^2} + \frac{1}{\left(\frac{n\pi}{l}\right)^2}\right] = \frac{l}{n^2\pi^2}[(-1)^{n+1} + 1]$$

$$a_n = 0, when\ n\ is\ even$$

$$a_n = \frac{2l}{n^2\pi^2}, when\ n\ is\ odd$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)sin\ \frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\int_0^{l}(l-x)sin\ \frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(l-x)\left(-\frac{cos\ \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{sin\ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$b_n = \frac{1}{l}\left(\frac{l^2(-1)^n}{n\pi}\right) = \frac{l(-1)^n}{n\pi}$$

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2} cos\ \frac{\pi x}{l} + \frac{1}{3^2} cos\ \frac{3\pi x}{l} + \frac{1}{5^2} cos\ \frac{5\pi x}{l} + \cdots\right]$$

$$+\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\ \frac{n\pi x}{l}$$

$$l - x = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1^2} cos\ \frac{\pi x}{l} + \frac{1}{3^2} cos\ \frac{3\pi x}{l} + \frac{1}{5^2} cos\ \frac{5\pi x}{l} + \cdots\right]$$

$$+\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\ \frac{n\pi x}{l}$$

Here ‘0’ is a point of discontinuity which is an end point of the given interval. Therefore the value of the Fourier series at $x = 0$ is the average value of $f(x)$ at $x = 0$ and $x = 2l$

Putting $x = 0$, we get

$$\frac{f(0) + f(2l)}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{l-0}{2} - \frac{l}{4} = \frac{l}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

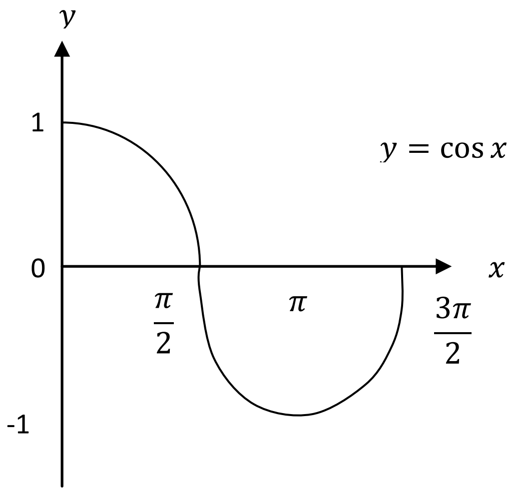
7. Find the Fourier series for $f(x) = |\cos x|$ in $(-\pi, \pi)$

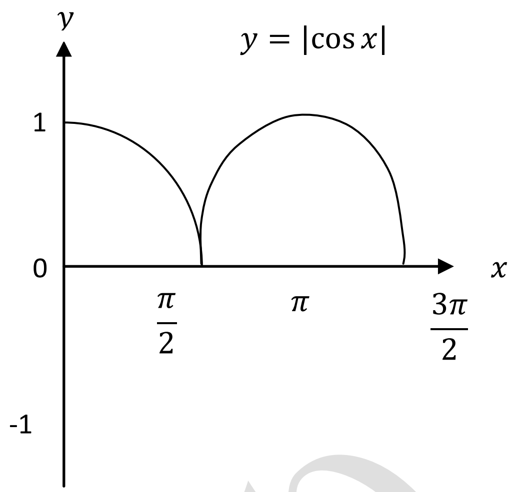
Solution:

$$f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

$\therefore f(x)$ is an even function.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx$$





$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) dx\right] \qquad \left[\because \ |\cos x| = \begin{cases} \cos x \ \ if \ 0 < x < \frac{\pi}{2} \\ -\cos x \ \ if \ \frac{\pi}{2} < x < \ \pi \end{cases}\right]$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\pi}\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) cos \ nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \ cos \ nx dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \ cos \ nx \ dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \cos nx \ dx\right]$$

$$\left[\because \ |\cos x| = \begin{cases} \cos x \ \ if \ 0 < x < \frac{\pi}{2} \\ -\cos x \ \ if \ \frac{\pi}{2} < x < \ \pi \end{cases}\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos(\mathrm{n}+1)\, x + \ cos\ (n-1)x]dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}[\cos(\mathrm{n}+1)x \cos(\mathrm{n}-1)x] \ dx]\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(\mathrm{n}+1)\, x}{n+1} + \frac{\sin(\mathrm{n}-1)\, x}{n-1}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{\sin(\mathrm{n}+1)\, x}{n+1} + \frac{\sin(\mathrm{n}-1)\, x}{n-1}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\sin(\mathrm{n}+1)\frac{\pi}{2}}{n+1}+\frac{\sin(\mathrm{n}-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin \mathrm{n}\frac{\pi}{2}\cos\frac{\pi}{2}+\cos n\frac{\pi}{2}\sin\frac{\pi}{2}}{n+1}+\frac{\sin \mathrm{n}\frac{\pi}{2}\cos\frac{\pi}{2}-\cos n\frac{\pi}{2}\sin\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\frac{\pi}{2}(n-1-n-1)}{n^2-1}\right]$$

$$a_n = -\frac{4}{\pi(n^2-1)}\cos n\frac{\pi}{2}, n \neq 1.$$

when $n = 1$, we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)cos\, xdx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\, cos\, xdx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\, cos\, x\, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}[\cos 2x + 1]dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}[\cos 2x + 1]dx]\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin 2x}{2}+x\right]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2x}{2}+x\right]_{\frac{\pi}{2}}^{\pi}\right]$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\frac{\pi}{2}\right] = 0$$

$$b_n = 0,$$

Then the Fourier series is reduced to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2}^{\infty} \cos nx \ \frac{\cos n\frac{\pi}{2}}{(n^2-1)}$$

8. Find Half Range Cosine Series $f(x) = x$ in $0 < x <$ ▲

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx = \frac{2}{\pi} \int_0^{\pi} x \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[x\left(\frac{\sin nx}{\mathrm{n}}\right) - 1\left(-\frac{\cos\ nx}{\mathrm{n}^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{1}{n}\right)^2 (-1)^{\mathrm{n}+1} - \left(\frac{1}{n}\right)^2\right]$$

$$= \frac{2}{\pi n^2}((-1)^{\mathrm{n}+1} - 1)$$

When n is odd

$$a_n = -\frac{4}{\pi n^2}$$

When n is even

$$a_n = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x\ + \frac{1}{3^2}\cos 3x + \cdots\right)$$

9. Find the Fourier series of periodicity 2 π for $(x) = x^2, -\pi < x < \pi$ and deduce

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} \dots = \frac{\pi^4}{90}$$

Solution:

$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is an even function.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cos\ nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 cos\ nxdx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{2\,\pi\cos n\pi}{n^2}\right) = -\frac{4(-1)^n}{n^2}$$

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

By Parseval's identity

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}[a_n^2]$$

$$\frac{2}{\pi}\int_0^{\pi}[x^2]^2\,dx = \frac{\left(\frac{2\pi^2}{3}\right)^2}{2}+\sum_{n=1}^{\infty}\left[-\frac{4(-1)^n}{n^2}\right]^2$$

$$\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right]^2$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5}-\frac{2\pi^4}{9}$$

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{8\pi^4}{45}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

10. Find the Fourier series for $f(x) = \begin{cases} l+x\,, -l<x<0 \\ l-x, 0<x<l \end{cases}$ and deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\ \ldots = \frac{\pi^2}{8}$$

Solution:

$$f(-x) = \begin{cases} l-x\,, -l<-x<0 \\ l+x, 0<-x<l \end{cases} = \begin{cases} l-x\,, l>x>0 \\ l+x, 0>x>-l \end{cases} = f(x)$$

$\therefore f(x)$ is an even function.

$$a_0 = \frac{2}{l}\int_0^{l} f(x)dx = \frac{2}{l}\int_0^{l}(l-x)dx$$

$$= \frac{2}{l}\left[lx-\frac{x^2}{2}\right]_0^{l} = \frac{2}{l}\left[l^2-\frac{l^2}{2}\right]$$

$$a_0 = l$$

$$a_n = \frac{2}{\pi}\int_0^l f(x)cos\ \frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l (l-x)cos\ \frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left(-\frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^2} + \frac{1}{\left(\frac{n\pi}{l}\right)^2}\right)$$

$$= \frac{2l}{n^2\pi^2}(1-(-1)^n)$$

$$a_n = 0, when\ n\ is\ even$$

$$a_n = \frac{4l}{n^2\pi^2}, when\ n\ is\ odd$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$f(x) = \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\cos\frac{n\pi x}{l}$$

Putting $x = 0$, we get

$$f(0) = \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}$$

Here ‘0’ is a point of discontinuity, then

$$f(0) = \frac{f(0+0) + f(0-0)}{2}$$

$$f(0) = \frac{l+l}{2} = l$$

$$l = \frac{l}{2} + \frac{4l}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}$$

$$\frac{4l}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} = l - \frac{l}{2} = \frac{l}{2}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$$

11. Find Half Range Fourier Sine series $f(x) = x\, sinx$ in $0 < x < \pi$.

Solution: We know that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) sin\, nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x\, sin\, nx dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x[\cos(n-1)x - \cos(n+1)x] dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{\sin(n-1)x}{n-1} \right) - 1. \left(\frac{-\cos(n-1)x}{(n-1)^2} \right) \right]_0^{\pi}$$

$$- \frac{1}{\pi} \left[x \left(\frac{\sin(n+1)x}{n+1} \right) - 1. \left(\frac{-\cos(n+1)x}{(n+1)^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{\cos(n+1)\pi}{(n+1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^{n-1}}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{(-1)^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2} \right]$$

$$= \frac{1}{\pi}\left[\frac{-(n+1)^2(-1)^n + (n-1)^2 + (n-1)^2(-1)^n - (n+1)^2}{(n^2-1)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(n^2+2n+1)(-1)^n + (n^2-2n+1) + (n^2-2n+1)(-1)^n - (n^2+2n+1)}{(n^2-1)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4n(-1)^n - 4n}{(n^2-1)^2}\right],$$

$n \neq 1$

$$b_n = \begin{cases} 0 & when\ n \neq 1\ and\ n\ is\ odd \\ \dfrac{-8n}{\pi(n^2-1)^2} & when\ n \neq 1\ and\ n\ is\ even \end{cases}$$

When $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x) sin\ x dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x\ sin\ x dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[1 - \cos 2x] dx$$

$$= \frac{1}{\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} - \frac{\cos 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$f(x) = \frac{\pi}{2}\sin x - \frac{8}{\pi}\sum_{n=2,4,6,\dots}^{\infty} \frac{n}{(n^2-1)^2}\sin nx$$

12. Find Half Range Fourier Sine series $f(x) = x(\pi - x)$ in $0 < x < \pi$ and Prove that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$$

Solution: We know that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) sin\, nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,(\pi - x)\, sin\, nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{4}{\pi}\left[\frac{\cos n\pi - 1}{n^3}\right] = -\frac{4}{\pi}\left[\frac{(-1)^n - 1}{n^3}\right]$$

$$b_n = \begin{cases} \dfrac{8}{\pi n^3} & when \;\; n\; is\; odd \\ 0 & when\;\; n\; is\; even \end{cases}$$

$$f(x) = \frac{8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}\sin nx$$

Put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}\sin n\frac{\pi}{2}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}\sin n\frac{\pi}{2} = \frac{\pi^3}{32}$$

$$\frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \ldots = \frac{\pi^3}{32}$$

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots = \frac{\pi^3}{32}$$

13. Find the Fourier series for $f(x) = \begin{cases} 1, 0 < x < \pi \\ 2,\; \pi < x < 2\pi \end{cases}$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1dx + \int_{\pi}^{2\pi} 2\, dx\right] = \frac{1}{\pi}[\pi + 4\pi - 2\pi] = 3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) cos\, nxdx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\, dx + \int_{\pi}^{2\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\, dx + 2\int_{\pi}^{2\pi} \cos nx\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi} + 2\left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) sin\, nxdx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\, dx + \int_{\pi}^{2\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\, dx + 2\int_{\pi}^{2\pi} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_0^{\pi} + 2\left(-\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos n\pi - 1}{n}\right) + 2\left(-\frac{\cos 2n\pi - \cos n\pi}{n}\right)\right] = -\frac{1}{n\pi}[(-1)^n - 1 + 2 - 2(-1)^n]$$

$$b_n = -\frac{1}{n\pi}[1-(-1)^n]$$

$$= \begin{cases} -\frac{2}{n\pi} & when\ n\ is\ odd \\ 0 & when\ n\ is\ even \end{cases}$$

$$f(x) = \frac{3}{2} - \frac{2}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin nx$$

14. Find the Fourier sine series for $f(x) = \begin{cases} x\ for\ 0 < x < \frac{l}{2} \\ l - x \quad for\ \frac{l}{2} < x < l \end{cases}$

Solution: We know that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{\frac{l}{2}}$$

$$+\frac{2}{l}\left[(l-x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{\frac{l}{2}}^{l}$$

$$= \frac{2}{l}\left[\frac{l}{2}\left(-\frac{\cos\frac{n\pi\frac{l}{2}}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi\frac{l}{2}}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) - 0\right]$$

$$+\frac{2}{l}\left[0 - \left(l - \frac{l}{2}\right)\left(-\frac{\cos\frac{n\pi\frac{l}{2}}{l}}{\frac{n\pi}{l}}\right) + (-1)\left(-\frac{\sin\frac{n\pi\frac{l}{2}}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{\cos\frac{n\pi}{2}}{2n\pi} + \frac{l^2 sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{\cos\frac{n\pi}{2}}{2n\pi} + \frac{l^2 sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$b_n = \frac{4l\, sin\frac{n\pi}{2}}{n^2\pi^2}$$

$$f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{l}$$

15. Find the Fourier series for $f(x) = \begin{cases} 0 & for\ -1 < x < 0 \\ 1 & for\ 0 < x < 1 \end{cases}$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

Here $l = 1$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x)$$

Where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx = (x)_0^1 = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\, dx$$

$$= \int_{-1}^{0} f(x)\cos n\pi x\ \ dx + \int_{0}^{1} f(x)\cos n\pi x\ \ dx$$

$$= \int_{-1}^{0} 0.\cos n\pi x\ \ dx + \int_{0}^{1} 1.\cos n\pi x\ \ dx = \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 = 0.$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\, dx$$

$$= \int_{-1}^{0} f(x)\sin n\pi x\ \ dx + \int_{0}^{1} f(x)\sin n\pi x\ \ dx$$

$$= \int_{-1}^{0} 0.\sin n\pi x\ \ dx + \int_{0}^{1} 1.\sin n\pi x\ \ dx = \left[-\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{\cos n\pi}{n\pi} + \frac{\cos 0}{n\pi} = \frac{1}{n\pi}(1-(-1)^n)$$

$$b_n = \begin{cases} 0 & when\ n\ is\ even \\ \dfrac{2}{n\pi} & when\ n\ is\ odd \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin n\pi x$$

16. Find Half Range Fourier Sine series $f(x) = x\cos x$ in $0 < x < \pi$.

Solution: We know that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x cos\, x \sin nx dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n-1)x + \sin(n+1)x]dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n-1)x}{n-1}\right) - 1.\left(\frac{-\sin\,(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) - 1.\left(\frac{-\sin\,(n+1)x}{(n+1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos(n-1)\pi}{n-1}\right) + \pi\left(-\frac{\cos(n+1)\pi}{n+1}\right)\right]$$

$$= -\left[\frac{(-1)^{n-1}}{n-1} + \frac{(-1)^{n+1}}{n+1}\right]$$

$$= -\left[\frac{-(n+1)(-1)^n - (n-1)(-1)^n}{n^2-1}\right], n \neq 1$$

$$b_n = \frac{2n(-1)^n}{n^2-1}$$

When $n = 1$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x) sin\, x dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos x\, sin\, x dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin 2x]dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{2\pi}(-\pi\cos 2\pi) = -\frac{1}{2}$$

$$f(x) = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx$$

17. Find the Fourier series for $f(x) = 1 + \ x \ + x^2$ in $(-\pi, \pi)$ and deduce that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{6}$$

Solution: We know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + \ x \ + x^2) dx$$

$$= \frac{1}{\pi} \left(x + \frac{x^2}{2} + \frac{x^3}{3} \right)_{-\pi}^{\pi} = \frac{1}{\pi} \left[\left(\pi + \frac{\pi^2}{2} + \frac{\pi^3}{3} \right) - \left(-\pi + \frac{(-\pi)^2}{2} + \frac{(-\pi)^3}{3} \right) \right]$$

$$= \frac{2}{\pi} \left[\pi + \frac{\pi^3}{3} \right] = 2 + \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + \ x \ + x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} cos\ nxdx + \int_0^{\pi} x^2 cos\ nxdx \right]$$

$$= \frac{2}{\pi} \left[\frac{\sin nx}{n} + x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) sin\ nxdx = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + \ x \ + x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x sin\ nxdx$$

$$= \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - 1 \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} = \frac{-2(-1)^n}{n}$$

$$f(x) = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\cos nx - \frac{(-1)^n}{2n}\sin nx\right)$$

Here 'π' is a point of discontinuity which is an end point of the given interval. Therefore the value of the Fourier series at $x = \pi$ is the average value of $f(x)$ at $x = -\pi$ and $x = \pi$

Putting $x = \pi$, we get

$$\frac{f(-\pi) + f(\pi)}{2} = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\cos n\pi - \frac{(-1)^n}{2n}\sin n\pi\right)$$

$$\frac{1 - \pi + \pi^2 + 1 + \pi + \pi^2}{2} = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$

$$1 + \pi^2 = 1 + \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Harmonic analysis:

18. Find the Fourier series expansion of period 2π for the function $y = f(x)$ which is defined in $(0, 2\pi)$ by means of the table of values given below. Find the series upto the third harmonic.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Solution: Let $y = f(x)$

Here $n = 6$ Since the last value of y is repetition of the first, only the first six values will be used.

We know that

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

Where

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}\sum y = \frac{1}{3}(8.7) = 2.9$$

$$a_1 = \frac{2}{n}\sum y \cos x = \frac{2}{6}\sum y \cos x = \frac{1}{3}(-1.1) = -2.53$$

$$a_2 = \frac{2}{n}\sum y \cos 2x = \frac{2}{6}\sum y \cos 2x = \frac{1}{3}(-0.3) = -0.1$$

$$a_3 = \frac{2}{n}\sum y \cos 3x = \frac{2}{6}\sum y \cos 3x = \frac{1}{3}(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\sum y \sin x = \frac{2}{6}\sum y \sin x = \frac{1}{3}(0.524) = 0.175$$

$$b_2 = \frac{2}{n}\sum y \sin 2x = \frac{2}{6}\sum y \sin 2x = \frac{1}{3}(-0.178) = -0.059$$

$$b_3 = \frac{2}{n}\sum y \sin 3x = \frac{2}{6}\sum y \sin 3x = \frac{1}{3}(0) = 0$$

$$f(x) = 1.45 - 2.53 \cos x - 0.1 \cos 2x + 0.033 \cos 3x + 0.175 \sin x - 0.059 \sin 2x$$

x	y	$\cos x$	$\cos 2x$	$\cos 3x$	$\sin x$	$\sin 2x$	$\sin 3x$	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.0	1	1	1	0	0	0	1	1	1	0	0	0
$\frac{\pi}{3}$	1.4	0.5	-0.5	-1	0.866	0.866	0	0.7	-0.7	-1.4	1.212	1.212	0
$\frac{2\pi}{3}$	1.9	-0.5	-0.5	1	0.866	-0.866	0	-0.95	-0.95	1.9	1.65	-1.65	0
π	1.7	-1	1	-1	0	0	0	-1.7	1.7	-1.7	0	0	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.5	1	-0.866	0.866	0	-0.75	-0.75	1.5	-1.299	1.299	0
$\frac{5\pi}{3}$	1.2	0.5	-0.5	-1	-0.866	-0.866	0	0.6	-0.6	-1.2	-1.039	-1.039	0
SUM	8.7							-1.1	-0.3	0.1	0.524	-0.178	0

19. Find the Fourier series as far as the second harmonic to represent the function given in the following data

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution: Let $y = f(x)$, Here $n = 6$ and $2l = 6 \Rightarrow l = 3$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos 2\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l} + b_2 \sin 2\frac{\pi x}{l}$$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos 2\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} + b_2 \sin 2\frac{\pi x}{3}$$

Where

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}\sum y = \frac{1}{3}(125) = 41.67$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}\sum y\cos\frac{\pi x}{3} = \frac{1}{3}(-25) = -8.33$$

$$a_2 = \frac{2}{n}\sum y\cos 2\frac{\pi x}{3} = \frac{2}{6}\sum y\cos 2\frac{\pi x}{3} = \frac{1}{3}(-7) = -2.33$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}\sum y\sin\frac{\pi x}{3} = \frac{1}{3}(-3.464) = -1.155$$

$$b_2 = \frac{2}{n}\sum y\sin 2\frac{\pi x}{3} = \frac{2}{6}\sum y\sin 2\frac{\pi x}{3} = \frac{1}{3}(0) = 0$$

$$f(x) = 20.83 - 8.33\cos\frac{\pi x}{3} - 2.33\cos 2\frac{\pi x}{3} - 1.155\sin\frac{\pi x}{3}$$

x	y	$\cos\frac{\pi x}{3}$	$\cos 2\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\sin 2\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\cos 2\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\sin 2\frac{\pi x}{3}$
0	9	1	1	0	0	9	9	0	0
1	18	0.5	-0.5	0.866	0.866	9	-9	15.588	15.588
2	24	-0.5	-0.5	0.866	-0.866	-12	-12	20.784	-20.784
3	28	-1	1	0	0	-28	28	0	0
4	26	-0.5	-0.5	-0.866	0.866	-13	-13	-22.516	22.516
5	20	0.5	-0.5	-0.866	-0.866	10	-10	-17.32	-17.32
SUM	125					-25	-7	-3.464	0

UNIT-II

FOURIER TRANSFORMS

Fourier integral theorem.

If is a given function $f(x)$ defined in $(-l, l)$ and satisfies Dirichlet's conditions, then

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dx\,d\lambda$$

Infinite Fourier transform and its inverse:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$F[f(x)]$ and $F^{-1}[F(s)]$ are Fourier transform pairs.

Infinite Fourier cosine transform and its inverse

The infinite Fourier cosine transform is given by

$$F_c[f(x)] = F_c[s] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\,dx$$

Then the inverse Fourier cosine transform is given by

$$F_c^{-1}\big[F_c[s]\big] = f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c[s]\cos sx\,ds$$

Parseval's identity for Fourier transform.

If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

Fourier sine transform and its inverse of $\boldsymbol{f(x)}$ in $(\boldsymbol{0}, \boldsymbol{l})$.

The finite Fourier sine transform of a function $f(x)$ in $(0, l)$is given by

$$f_s(n) = \int_0^l f(x) sin\frac{n\pi x}{l}\, dx,$$

Then the inversion formula is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\, sin\frac{n\pi x}{l}.$$

Fourier cosine transform and its inverse of $f(x)$ in $(0, l)$.

The finite Fourier cosine transform of a function $f(x)$ in $(0, l)$is given by

$$f_c(n) = \int_0^l f(x) cos\frac{n\pi x}{l}\, dx,$$

Then the inversion formula is given by

$$f(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\, cos\frac{n\pi x}{l}.$$

Infinite Fourier sine transform and its inverse

The infinite Fourier sine transform is given by

$$F_s[f(x)] = F_s[s] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$$

Then the inverse Fourier cosine transform is given by

$$F_s^{-1}\big[F_s[s]\big] = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s[s]\sin sx\, ds$$

Parseval's identity for Fourier sine transform.

If $F_S(s)$ is the Fourier transform of $f(x)$, then

$$\int_0^{\infty} |f(x)|^2\, dx = \int_0^{\infty} |F_S(s)|^2\, ds$$

Parseval's identity for Fourier cosine transform.

If $F_C(s)$ is the Fourier transform of $f(x)$, then

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_C(s)|^2\,ds$$

properties of Fourier transform:

Linearity property:

If $f(x)$ and $g(x)$are any two functions then

$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$, where a and b are constants.

Shifting property:

If $F[f(x)] = F(s)$, then $F[f(x-a)] = e^{ias}F(s)$

Fourier transforms of the derivatives of a function.

$$F\left(\frac{d^n f(x)}{dx^n}\right) = (-is)^n F(s).$$

$\boldsymbol{F_c[f(ax)] = \frac{1}{a}F_c\left(\frac{s}{a}\right), a > 0.}$

Solution:

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx\,dx$$

Put $ax = y$ when $x = 0, y = 0.$

$dx = \frac{dy}{a}$ When $x = \infty, y = \infty$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos\left(\frac{sy}{a}\right)\frac{dy}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos\left(\frac{s}{a}\right) y\,dy$$

$$= \frac{1}{a}F_c\left(\frac{s}{a}\right)$$

If $\boldsymbol{F(s) = F(f(x))}$, then prove that $\boldsymbol{F(xf(x)) = (-i)\frac{d[F(s)]}{ds}}$

Solution: $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$$\frac{d[F(s)]}{ds} = \frac{d}{ds}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial s}(e^{isx})dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)ix(e^{isx})dx$$

$$= iF\big(xf(x)\big)$$

$$F\big(xf(x)\big) = (-i)\frac{d[F(s)]}{ds}$$

If $F_s[s]$ is the Fourier sine transform of $f(x)$, show that

$$\boldsymbol{F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]}$$

Solution:

$$F_s[f(x)cosax\,] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos ax\ \sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}.\frac{1}{2}}\int_{0}^{\infty} f(x)\,\{\sin(a+s)x - \sin(a-s)\,x\,\}\,dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\ \sin(a+s)x\ dx - \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\ \sin(a-s)x\ dx$$

$$= \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

If $F(s)$ is the Fourier transform of $f(x)$, the Fourier transform of $f(x)cos\ ax$

$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{e^{i(s+a)x}}{2}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{e^{i(s-a)x}}{2}dx$$

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

If $f(x)$ is an even function of x, its Fourier transform $F(s)$ will also be an even function of s.

Proof: By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx \;\; \ldots \;\; (1)$$

Changing s into –s in both sides of (1),

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \;\; \ldots \;\; (2)$$

In the right hand side integral in (2), put $x = -u$.

then $dx = -du$; when $x = \infty, u = -\infty$

and when $x = -\infty, u = \infty$.

So (2) becomes

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u)e^{isu}.(-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-u)e^{isu}\,du$$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)e^{isx}\,dx$ [Changing the dummy variable u into x]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = F(S)\; by\; (1).$$

Here $F(S)$ is an even function of s.

If $f(x)$ is an odd function of x, its Fourier transform $F(s)$ will also be an odd function of s.

Proof: By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx \;\; \ldots \;\; (1)$$

Changing s into –s in both sides of (1),

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \;\; \dots \;\; (2)$$

In the right hand side integral in (2), put $x = -u$.

then $dx = -du$; when $x = \infty, u = -\infty$

and when $x = -\infty, u = \infty$.

So (2) becomes

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u) e^{isu} . (-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-u) e^{isu}\, du$$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{isx}\, dx$ [Changing the dummy variable u into x]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -f(x) e^{isx}\, dx = -F(S)\; by\; (1).$$

Here $F(S)$ is an odd function of s.

If $F_c(f(x)) = F_c(s)$, prove that $F_c(F_c(x)) = f(s)$

Proof:

By inverse cosine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c[s] \cos sx\, ds$$

Interchanging $s\; and\; x$, we get

$$f(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c[x] \cos sx\, dx$$

$$= F_c(F_c(x)) \;\; [By\; definition\; of\; Fourier\; cosine\; transform]$$

Fourier transforms of derivates of a function:

$$F\left[\frac{df(x)}{dx}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) ise^{isx} dx \right\}$$

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

(assuming $f(x) \to 0 \; as \; x \to \pm\infty$)

$$= -isF(s).$$

$$F\left[\frac{d^2 f(x)}{dx^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d[f'(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f'(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) ise^{isx} dx \right\}$$

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{isx} dx$$

(assuming $f'(x) \to 0 \; as \; x \to \pm\infty$)

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d[f(x)]$$

$$= -\frac{is}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) ise^{isx} dx \right\}$$

$$= \frac{(-is)^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

(assuming $f'(x) \to 0 \; as \; x \to \pm\infty$)

$$= (-is)^2 F(s).$$

In general,

$$F\left[\frac{d^n f(x)}{dx^n}\right] = (-is)^n F(s).$$

Convolution theorem of the Fourier transform.

If $F(s)$ and $G(s)$ are the functions of $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform

$$F[(f * g)x] = F(S).G(s)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f * g)(x)e^{isx}\,dx = \left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx\right\}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{isx}\,dx\right\}$$

1. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|\, for\, |x| \le 1 \\ 0\, for\, |x| > 1 \end{cases}$

And deduce that

$$\int_0^{\infty}\frac{sin^4 t}{t^4}dt = \frac{\pi}{3}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{0}\big(1-(-x)\big)e^{isx}\,dx + \int_{0}^{1}(1-x)e^{isx}\,dx\right] \quad \left[\because |x| = \begin{cases} -x, & x<0 \\ x, & x \ge 0\end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{0}(1+x)e^{isx}\,dx + \int_{0}^{1}(1-x)e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[(1+x)\left(\frac{e^{isx}}{is}\right) - 1\left(\frac{e^{isx}}{(is)^2}\right)\right]_{-1}^{0} + \left[(1-x)\left(\frac{e^{isx}}{is}\right) - (-1)\left(\frac{e^{isx}}{(is)^2}\right)\right]_{0}^{1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[(1+0)\left(\frac{e^{is0}}{is}\right) - \left(\frac{e^{is0}}{(is)^2}\right)\right] - \left[(1-1)\left(\frac{e^{-is}}{is}\right) - \left(\frac{e^{-is}}{(is)^2}\right)\right]\right]$$

$$+ \frac{1}{\sqrt{2\pi}}\left[\left[(1-1)\left(\frac{e^{is}}{is}\right) + \left(\frac{e^{is}}{(is)^2}\right)\right] - \left[(1-0)\left(\frac{e^{is0}}{is}\right) + \left(\frac{e^{is0}}{(is)^2}\right)\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is} - \frac{1}{(is)^2} - \left[0 - \frac{e^{-is}}{(is)^2}\right] + 0 + \frac{e^{is}}{(is)^2} - \left[\frac{1}{is} + \frac{1}{(is)^2}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{(is)^2} + \frac{e^{-is}}{(is)^2} + \frac{e^{is}}{(is)^2} - \frac{1}{(is)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2} - \frac{e^{-is}}{s^2} - \frac{e^{is}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{s^2} - \left(\frac{e^{is}+e^{-is}}{s^2}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{s^2} - \left(\frac{2\cos s}{s^2}\right)\right] = \frac{1}{\sqrt{2\pi}}\frac{2}{s^2}[1-\cos s]$$

$$= \sqrt{\frac{2}{\pi}}\frac{2sin^2\frac{s}{2}}{s^2}$$

By Parseval's identity

$$\int_{-\infty}^{\infty}[f(x)]^2dx = \int_{-\infty}^{\infty}\left[F[f(x)]\right]^2ds$$

$$\int_{-1}^{1}[1-|x|]^2dx = \int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{2sin^2\frac{s}{2}}{s^2}\right]^2 ds$$

$$2\int_{0}^{1}[1-x]^2dx = 2\int_{0}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{2sin^2\frac{s}{2}}{s^2}\right]^2 ds$$

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{sin^4\frac{s}{2}}{s^4}ds = \left[\frac{(1-x)^3}{-3}\right]_0^1 = \left[\frac{(1-1)^3}{-3} - \left(\frac{(1-0)^3}{-3}\right)\right] = \frac{1}{3}$$

$$\int_{0}^{\infty}\frac{sin^4\frac{s}{2}}{s^4}ds = \frac{\pi}{24}$$

$$Put\ t = \frac{s}{2} \Rightarrow s = 2t, ds = 2dt, when\ s = 0 \Rightarrow t = 0\ and\ s = \infty \Rightarrow t = \infty$$

$$\int_{0}^{\infty}\frac{sin^4 t}{(2t)^4}2dt = \frac{\pi}{24}$$

$$\int_{0}^{\infty}\frac{sin^4 t}{t^4}dt = \frac{\pi}{3}$$

2. Using Parseval's identity calculate

$$i)\int_{0}^{\infty}\frac{dx}{(x^2+a^2)^2} \qquad ii)\int_{0}^{\infty}\frac{x^2dx}{(x^2+a^2)^2} \qquad if\ a > 0$$

Solution: i) We know that

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

Here $f(x) = e^{-ax}$

Using Parseval's identity

$$\int_{0}^{\infty}[f(x)]^2dx = \int_{0}^{\infty}\left[F_c[f(x)]\right]^2ds$$

$$\int_{0}^{\infty}[e^{-ax}]^2dx = \int_{0}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2 ds$$

$$\int_0^\infty e^{-2ax}\,dx = \frac{2a^2}{\pi}\int_0^\infty \left[\frac{1}{a^2+s^2}\right]^2 ds$$

$$\frac{2a^2}{\pi}\int_0^\infty \left[\frac{1}{a^2+s^2}\right]^2 ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a}$$

$$\frac{2a^2}{\pi}\int_0^\infty \left[\frac{1}{a^2+s^2}\right]^2 ds = \frac{1}{2a}$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \qquad (\because s \text{ is dummy variable it is replaced by } x)$$

ii) We know that

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}$$

Here $f(x) = e^{-ax}$

Using Parseval's identity

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty \left[F_s[f(x)]\right]^2 ds$$

$$\int_0^\infty [e^{-ax}]^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}\right]^2 ds$$

$$\int_0^\infty e^{-2ax}\,dx = \frac{2}{\pi}\int_0^\infty \left[\frac{s}{a^2+s^2}\right]^2 ds$$

$$\frac{2}{\pi}\int_0^\infty \left[\frac{s}{a^2+s^2}\right]^2 ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a}$$

$$\frac{2}{\pi}\int_0^\infty \left[\frac{s}{a^2+s^2}\right]^2 ds = \frac{1}{2a}$$

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi}{4a} \qquad (\because s \text{ is dummy variable it is replaced by } x)$$

3. Find the Fourier transform $f(x) = \begin{cases} 1 & for\ |x| \le a \\ 0 & for\ |x| > a \end{cases}$ $if\ a > 0$

Deduce that

$$(i)\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2} \qquad ii)\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is} \right] = \frac{1}{\sqrt{2\pi}} \frac{1}{s} \left[\frac{e^{isa} - e^{-isa}}{i} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2 \sin as}{s} = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$

i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] e^{-isx}\, ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} e^{-isx}\, ds \ldots (1)$$

Put $x = 0$ and $a = 1$ in $(1), we\ get$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} ds$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin s}{s} ds$$

$$\int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

ii) By Parseval's identity, we get

$$\int_{-\infty}^{\infty} [f(x)]^2 dx = \int_{-\infty}^{\infty} \left[F[f(x)]\right]^2 ds$$

$$\int_{-a}^{a} [1]^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \frac{\sin as}{s} \right]^2 ds$$

$$2 \int_{0}^{a} dx = 2 \frac{2}{\pi} \int_{0}^{\infty} \left[\frac{\sin as}{s} \right]^2 ds$$

$$\frac{2}{\pi} \int_{0}^{\infty} \left[\frac{\sin as}{s} \right]^2 ds = [x]_0^a$$

$$\int_{0}^{\infty} \left[\frac{\sin as}{s} \right]^2 ds = a \frac{\pi}{2} \ldots (2)$$

Put $a = 1$ in $(2), we\ get$

$$\int_{0}^{\infty} \left[\frac{\sin s}{s} \right]^2 ds = \frac{\pi}{2}$$

4. Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 \ for \ |x| \le 1 \\ 0 \ for \ |x| > 1 \end{cases}$

Hence prove that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\left(\frac{e^{isx}}{is}\right) - (-2x)\left(\frac{e^{isx}}{(is)^2}\right) + (-2)\left(\frac{e^{isx}}{(is)^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-1)\left(\frac{e^{is}}{is}\right) + 2\left(\frac{e^{is}}{(is)^2}\right) - 2\left(\frac{e^{is}}{(is)^3}\right)\right]$$

$$- \frac{1}{\sqrt{2\pi}}\left[(1-1)\left(\frac{e^{-is}}{is}\right) + 2(-1)\left(\frac{e^{-is}}{(is)^2}\right) - 2\left(\frac{e^{-is}}{(is)^3}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\left(\frac{e^{is}}{(is)^2}\right) + 2\left(\frac{e^{-is}}{(is)^2}\right) - 2\left(\frac{e^{is}}{(is)^3}\right) + 2\left(\frac{e^{-is}}{(is)^3}\right)\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left(-\frac{e^{is}}{s^2}\right) + \left(-\frac{e^{-is}}{s^2}\right) - \left(-\frac{e^{is}}{is^3}\right) + \left(-\frac{e^{-is}}{is^3}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{s^2}\left(e^{is} + e^{-is}\right) + \frac{1}{s^3}\left(\frac{e^{is} - e^{-is}}{i}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\frac{1}{s^3}(\sin s - s\cos s)$$

By Inverse Fourier transform, we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\,ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\frac{1}{s^3}(\sin s - s\cos s)e^{-isx}\,ds$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(\sin s - s\cos s)}{s^3}(\cos sx - i\sin sx)ds$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(\sin s - s\cos s)}{s^3}\cos sx\,ds - \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(\sin s - s\cos s)}{s^3}\sin sx\,ds$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)}{s^3}\cos sx\,ds - 0 \ldots (1)$$

$[\because \textit{The function in first intergral is even and the function in second integral is odd}]$

$$\textit{Put } x = \frac{1}{2} \textit{ in } (1), \textit{we get}$$

$$f\left(\frac{1}{2}\right) = \frac{4}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)}{s^3}\cos\frac{s}{2}\,ds$$

$$1-\left(\frac{1}{2}\right)^2 = \frac{4}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)}{s^3}\cos\frac{s}{2}\,ds$$

$$\frac{4}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)}{s^3}\cos\frac{s}{2}\,ds = 1-\frac{1}{4} = \frac{3}{4}$$

$$\int_0^\infty \frac{(\sin s - s\cos s)}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$$

5. Find the Fourier cosine transform of $e^{-a^2x^2}, a > 0$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2} R.P\ of\ e^{isx}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}e^{isx}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2\left(x^2-\frac{isx}{a^2}\right)}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2\left(x^2-\frac{isx}{a^2}+\left(\frac{is}{2a^2}\right)^2-\left(\frac{is}{2a^2}\right)^2\right)}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2\left(\left(x-\frac{is}{2a^2}\right)^2-\left(\frac{is}{2a^2}\right)^2\right)}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(\left(ax-\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right)}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(ax-\frac{is}{2a}\right)^2}e^{\left(\frac{is}{2a}\right)^2}\,dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$$Put\ y = ax - \frac{is}{2a}, dy = adx \Rightarrow dx = \frac{dy}{a}, when\ x = \pm\infty \Rightarrow y = \pm\infty$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{a}$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \frac{\sqrt{\pi}}{a}$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

6. Show that $f(x) = e^{-x^2/2}$ is self reciprocal under the Fourier cosine transform, deduce that $g(x) = xe^{-x^2/2}$ is self reciprocal under the Fourier sine transform.

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

$$F_c\left[e^{-\frac{x^2}{2}}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}} \cos sx\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} R.P\ of\ e^{isx}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+isx}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2isx)}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2isx+(is)^2-(is)^2)}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}((x-is)^2-(is)^2)}\, dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} e^{\left(\frac{is}{\sqrt{2}}\right)^2} dx$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx$$

$$Put\ y = \frac{x - is}{\sqrt{2}}, dy = \frac{dx}{\sqrt{2}} \Rightarrow dx = \sqrt{2}dy, when\ x = \pm\infty \Rightarrow y = \pm\infty$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}dy$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \sqrt{2}\sqrt{\pi}$$

$$F_c\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$$

$$F_s[xf(x)] = -\frac{d}{ds} F_c[f(x)]$$

$$F_s\left[xe^{-\frac{x^2}{2}}\right] = -\frac{d}{ds} F_c\left[e^{-\frac{x^2}{2}}\right]$$

$$= -\frac{d}{ds} e^{-\frac{s^2}{2}} = -\left(-\frac{2s}{2} e^{-\frac{s^2}{2}}\right)$$

$$F_s\left[xe^{-\frac{x^2}{2}}\right] = se^{-\frac{s^2}{2}}$$

7. Find the Fourier sine transform of

$$\frac{e^{-ax}}{x}, a > 0$$

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx$$

Differentiating with respect to s, we get

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx$$

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \frac{d}{ds} \sin sx\, dx$$

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} x \cos sx\, dx$$

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-\infty}}{a^2 + s^2} (-a \cos s\infty + s \sin s\infty) - \frac{e^0}{a^2 + s^2} (-a \cos s0 + s \sin s0)\right]$$

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\int\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}ds=\sqrt{\frac{2}{\pi}}a\int\frac{ds}{a^2+s^2}$$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}a\frac{1}{a}tan^{-1}\frac{s}{a}$$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}tan^{-1}\frac{s}{a}$$

8. Find the Fourier cosine transform of e^{-x^2}

Solution:

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}R.P\ of\ e^{isx}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}e^{isx}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2+isx}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(x^2-isx)}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(x^2-isx+\left(\frac{is}{2}\right)^2-\left(\frac{is}{2}\right)^2\right)}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(\left(x-\frac{is}{2}\right)^2-\left(\frac{is}{2}\right)^2\right)}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(x-\frac{is}{2}\right)^2}e^{\left(\frac{is}{2}\right)^2}\,dx$$

$$=R.P\ of\ \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4}}\int_{-\infty}^\infty e^{-\left(x-\frac{is}{2}\right)^2}\,dx$$

$$Put\ y = x - \frac{is}{2}, dy = dx \Rightarrow dx = dy, when\ x = \pm\infty \Rightarrow y = \pm\infty$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

$$= R.P\ of\ \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4}} \sqrt{\pi}$$

$$F_c\left[e^{-x^2}\right] = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

9. Find the Fourier sine transform of $f(x) = \begin{cases} x\ for\ 0 \le x \le 1 \\ 2 - x\ for\ 1 \le x \le 2 \\ 0\ for\ x > 2 \end{cases}$

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} f(x) \sin sx\, dx + \int_1^{2} f(x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} x \sin sx\, dx + \int_1^{2} (2 - x) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x\left(-\frac{\cos sx}{s}\right) - 1\left(-\frac{\sin sx}{s^2}\right)\right]_0^1 + \left[(2 - x)\left(-\frac{\cos sx}{s}\right) - (-1)\left(-\frac{\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[1\left(-\frac{\cos s}{s}\right) + 1\left(\frac{\sin s}{s^2}\right) - 0 + (2 - 2)\left(-\frac{\cos 2s}{s}\right) - \left(\frac{\sin 2s}{s^2}\right)\right.$$

$$\left. - \left[(2 - 1)\left(-\frac{\cos s}{s}\right) - \left(\frac{\sin s}{s^2}\right)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{2 \sin s \cos s}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2\sin s}{s^2} [1 - \cos s] = \sqrt{\frac{2}{\pi}} \frac{2\sin s}{s^2} \left[2 sin^2 \frac{s}{2}\right]$$

$$F_s[f(x)] = 4\sqrt{\frac{2}{\pi}} \frac{\sin s}{s^2} sin^2 \frac{s}{2}$$

10. Find the complex Fourier transform of $f(x) = \begin{cases} a^2 - x^2 & for\ |x| \le a \\ 0 & for\ |x| > a \end{cases}$

Hence deduce

$$i) \int_0^{\infty} \frac{(\sin x - x\cos x)}{x^3} dx = \frac{\pi}{4} \qquad ii) \int_0^{\infty} \left(\frac{(\sin x - x\cos x)}{x^3}\right)^2 dx = \frac{\pi}{15}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[(a^2 - x^2)\left(\frac{e^{isx}}{is}\right) - (-2x)\left(\frac{e^{isx}}{(is)^2}\right) + (-2)\left(\frac{e^{isx}}{(is)^3}\right)\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(a^2 - a^2)\left(\frac{e^{isa}}{is}\right) + 2a\left(\frac{e^{isa}}{(is)^2}\right) - 2\left(\frac{e^{isa}}{(is)^3}\right)\right]$$

$$- \frac{1}{\sqrt{2\pi}} \left[(a^2 - a^2)\left(\frac{e^{-isa}}{is}\right) + 2(-a)\left(\frac{e^{-isa}}{(is)^2}\right) - 2\left(\frac{e^{-isa}}{(is)^3}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\left(\frac{e^{isa}}{(is)^2}\right) + 2a\left(\frac{e^{-isa}}{(is)^2}\right) - 2\left(\frac{e^{isa}}{(is)^3}\right) + 2a\left(\frac{e^{-isa}}{(is)^3}\right)\right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(-\frac{e^{isa}}{s^2}\right) + \left(-\frac{e^{-isa}}{s^2}\right) - \left(-\frac{e^{isa}}{is^3}\right) + \left(-\frac{e^{-isa}}{is^3}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{a}{s^2}\left(e^{isa} + e^{-isa}\right) + \frac{1}{s^3}\left(\frac{e^{isa} - e^{-isa}}{i}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2\, a\cos as}{s^2} + \frac{2\sin as}{s^3}\right] = 2\sqrt{\frac{2}{\pi}} \frac{1}{s^3} (\sin as - as\cos as)$$

i) By Inverse Fourier transform, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] e^{-isx} ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \frac{1}{s^3} (\sin as - as\cos as) e^{-isx} ds \ldots (1)$$

Put $x = 0\ in\ (1),\ we\ get$

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3} ds$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3} ds = a^2 - 0 \ldots (2)$$

Put $a = 1\ in\ (2),\ we\ get$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = 1$$

$$\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = 1$$

$$\int_{0}^{\infty}\frac{(\sin x - x\cos x)}{x^3}dx = \frac{\pi}{4} \qquad (\because s\ is\ dummy\ variable\ it\ is\ replaced\ by\ x)$$

ii) By Parseval's identity, we get

$$\int_{-\infty}^{\infty}[f(x)]^2dx = \int_{-\infty}^{\infty}\left[F[f(x)]\right]^2ds$$

$$\int_{-a}^{a}[a^2-x^2]^2dx = \int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\frac{(\sin as - as\cos as)}{s^3}\right]^2 ds$$

$$2\int_{0}^{a}[a^2-x^2]^2dx = 2\int_{0}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\frac{(\sin as - as\cos as)}{s^3}\right]^2 ds$$

$$\int_{0}^{a}[a^4-2a^2x^2+x^4]dx = \frac{8}{\pi}\int_{0}^{\infty}\left[\frac{(\sin as - as\cos as)}{s^3}\right]^2 ds$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{(\sin as - as\cos as)}{s^3}\right]^2 ds = \left[a^4x - 2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]_0^a$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{(\sin as - as\cos as)}{s^3}\right]^2 ds = a^5 - \frac{2a^5}{3}+\frac{a^5}{5} = \frac{8a^5}{15} \ldots (3)$$

$$put\ a = 1\ in\ (3), we\ get$$

$$\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{(\sin s - s\cos s)}{s^3}\right]^2 ds = \frac{1}{15}$$

$$\int_{0}^{\infty}\left(\frac{(\sin x - x\cos x)}{x^3}\right)^2 dx = \frac{\pi}{15} \qquad (\because s\ is\ dummy\ variable\ it\ is\ replaced\ by\ x)$$

11. If $F(s)$ is the Fourier transform of $f(x)$, find the Fourier transform of

$f(x-a) and\ f(ax)$

Solution:

$$F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)e^{isx}\,dx$$

$put\ y = x - a \Rightarrow x = y + a, dx = dy, when\ x = \pm\infty \Rightarrow y = \pm\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{is(y+a)}dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{isy}\,e^{isa}\,dy$$

$$= e^{isa}\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{isy}\,dy = e^{isa}\,F[s]$$

$$F[f(x-a)] = e^{isa}\,F[s]$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)e^{isx}\,dx$$

$put\ y = ax \Rightarrow x = \frac{y}{a}, dx = \frac{dy}{a}, when\ x = \pm\infty \Rightarrow y = \pm\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{is\frac{y}{a}}\frac{dy}{a} = \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{i\frac{s}{a}y}\,dy$$

$$F[f(ax)] = \frac{1}{a}F\left[\frac{s}{a}\right]$$

12. Evaluate the following integral using Fourier transforms

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Solution: We know that

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{(s^2+a^2)}, F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}}\frac{b}{(s^2+b^2)}$$

Here $f(x) = e^{-ax}$ $\quad and\ g(x) = e^{-bx}$

By convolution theorem, we get

$$\int_{0}^{\infty} f(x).\,g(x)dx = \int_{0}^{\infty} F_c[f(x)]F_c[g(x)]\,ds$$

$$\int_0^\infty e^{-ax}.e^{-bx}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{(s^2+a^2)}\sqrt{\frac{2}{\pi}}\frac{b}{(s^2+b^2)}ds$$

$$\int_0^\infty e^{-(a+b)x}dx = \frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{e^{-\infty}}{-(a+b)} - \frac{e^0}{-(a+b)} = \frac{1}{a+b}$$

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)} \qquad (\because s \text{ is dummy variable it is replaced by } x)$$

13. Find the Fourier transform of $e^{-a|x|}$, $a > 0$ and hence deduce that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-a(-x)}e^{isx}\,dx + \int_0^\infty e^{-ax}e^{isx}\,dx\right] \quad \left[\because |x| = \begin{cases} -x, & x<0 \\ x, & x \ge 0\end{cases}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(a+is)x}dx + \int_0^\infty e^{-(a-is)x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{(a+is)x}}{a+is}\right]_{-\infty}^{0} - \left[\frac{e^{-(a-is)x}}{a-is}\right]_0^\infty\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^0}{a+is} - \frac{e^{-\infty}}{a+is} - \left[\frac{e^{-\infty}}{a-is} - \frac{e^0}{a-is}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{a-is+a+is}{a^2-(is)^2}\right]$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+s^2}\right]$$

By Parseval's identity, we get

$$\int_{-\infty}^{\infty}[f(x)]^2 dx = \int_{-\infty}^{\infty}\left[F[f(x)]\right]^2 ds$$

$$\int_{-\infty}^{\infty}\left[e^{-a|x|}\right]^2 dx = \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+s^2}\right]\right]^2 ds$$

$$2\int_{0}^{\infty}[e^{-ax}]^2 dx = 2\int_{0}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\left[\frac{2a}{a^2+s^2}\right]\right]^2 ds$$

$$\int_{0}^{\infty} e^{-2ax}\, dx = \frac{2}{\pi}\int_{0}^{\infty}\left[\frac{a}{a^2+s^2}\right]^2 ds$$

$$\frac{2a^2}{\pi}\int_{0}^{\infty}\left[\frac{1}{a^2+s^2}\right]^2 ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty} = \frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a} = \frac{1}{2a}$$

$$\frac{2a^2}{\pi}\int_{0}^{\infty}\left[\frac{1}{a^2+s^2}\right]^2 ds = \frac{1}{2a}$$

$$\int_{0}^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3} \quad (\because s \text{ is dummy variable it is replaced by } x)$$

14. Find the Fourier cosine transform of $f(x) = e^{-4x}$. Deduce that

$$\int_{0}^{\infty}\frac{\cos 2x\, dx}{x^2+16} = \frac{\pi}{8}e^{-8} \text{ and } \int_{0}^{\infty}\frac{x \sin 2x\, dx}{x^2+16} = \frac{\pi}{2}e^{-8}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\, dx$$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-4x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4x}}{(-4)^2+s^2}(-4\cos sx + s\sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4(\infty)}}{16+s^2}(-4\cos s\infty + s\sin s\infty) - \frac{e^{-4(0)}}{16+s^2}(-4\cos s0 + s\sin s0)\right]$$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\frac{4}{16+s^2}$$

By inverse cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\, ds$$

$$e^{-4x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{4}{16+s^2}\cos sx\, ds$$

$$\int_0^\infty \frac{1}{16+s^2}\cos sx\, ds = \frac{\pi}{8}e^{-4x}$$

Interchanging x *and* s, we get

$$\int_0^\infty \frac{1}{16+x^2}\cos sx\, dx = \frac{\pi}{8}e^{-4s} \ldots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}\int_0^\infty \frac{1}{16+x^2}\cos sx\, ds = \frac{d}{ds}\frac{\pi}{8}e^{-4s}$$

Differentiating under integral sign we get,

$$\int_0^\infty \frac{1}{16+x^2}\frac{d}{ds}\cos sx\, ds = \frac{d}{ds}\frac{\pi}{8}e^{-4s}$$

$$\int_0^\infty \frac{1}{16+x^2}(-x\sin sx)\, ds = \frac{\pi}{8}(-4e^{-4s})$$

$$\int_0^\infty \frac{x}{16+x^2}\sin sx\, ds = \frac{\pi}{2}e^{-4s} \ldots (2)$$

$$Put\ s = 2\ in\ (1)\ and\ (2), we\ get$$

$$\int_0^\infty \frac{\cos 2x\, dx}{x^2+16} = \frac{\pi}{8}e^{-8}$$

$$\int_0^\infty \frac{x \sin 2x\, dx}{x^2+16} = \frac{\pi}{2}e^{-8}$$

15. Find the Fourier sine and cosine transform $f(x) = e^{-x}$ and hence find the Fourier cosine transform of $\frac{1}{1+x^2}$ and Fourier sine transform of $\frac{x}{1+x^2}$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-\infty}}{1+s^2}(-\cos s\infty + s\sin s\infty) - \frac{e^{-0}}{1+s^2}(-\cos s0 + s\sin s0)\right]$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}}\frac{1}{1+s^2}$$

By inverse cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{1+s^2}\cos sx\, ds$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+s^2}\cos sx\, ds = \sqrt{\frac{\pi}{2}}e^{-x}$$

Interchanging $x \; and \; s$, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos sx \, dx = \sqrt{\frac{\pi}{2}}e^{-s}$$

$$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{\pi}{2}}e^{-s}$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx \, dx$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx\,)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-\infty}}{1+s^2}(-\sin s\infty - s\cos s\infty\,) - \frac{e^{-0}}{1+s^2}(-\sin s0 - s\cos s0\,)\right]$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}$$

By inverse sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx \, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{2}{1+s^2}\sin sx \, ds$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{1+s^2}\sin sx \, ds = \sqrt{\frac{\pi}{2}}e^{-x}$$

Interchanging $x \; and \; s$, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{1+x^2}\cos sx \, dx = \sqrt{\frac{\pi}{2}}e^{-s}$$

$$F_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{\pi}{2}}e^{-s}$$

Aliter: For finding $F_s\left[\frac{x}{1+x^2}\right]$

$$F_s[xf(x)] = -\frac{d}{ds}F_c[f(x)]$$

$$F_s\left[x\frac{1}{1+x^2}\right] = -\frac{d}{ds}F_c\left[\frac{1}{1+x^2}\right]$$

$$= -\frac{d}{ds}\sqrt{\frac{\pi}{2}}e^{-s} = -\sqrt{\frac{\pi}{2}}(-e^{-s})$$

$$F_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{\pi}{2}}e^{-s}$$

Partial Differential equation

Partial differential equation:

The equations which involving coefficients as partial derivatives are called partial differential equations.

Solution of a partial differential equation:

The relation between the dependent and the independent variables without involving any partial derivatives is called solution of a partial differential equation.

Complete integral:

A solution of a partial differential equation which contains the maximum possible number of arbitrary constants is called complete integral.

Particular integral:

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

General integral:

A solution of a partial differential equation which contains the maximum possible number of arbitrary functions is called general integral.

Let $F(x, y, z, p, q) = 0 \dots (1)$be a partial differential equation and its complete integral be

$$\Phi(x, y, z, a, b) = 0 \dots (2)$$

Substituting $b = f(a)\ in\ (2), we\ get$

$$\Phi(x, y, z, a, f(a)) = 0 \dots (3)$$

Differentiating (3) partially w.r.t. a, we get

$$\frac{\partial \Phi}{\partial a} = 0 \dots (4)$$

The elimination of a from (3) and (4), if it exits, is called general integral.

Singular integral:

Let $F(x, y, z, p, q) = 0 \dots (1)$be a partial differential equation and its complete integral be

$$\Phi(x, y, z, a, b) = 0 \dots (2)$$

Differentiating (2) partially w.r.t. a and b in turn, we get

$$\frac{\partial \Phi}{\partial a} = 0 \ldots (3)$$

$$\frac{\partial \Phi}{\partial b} = 0 \ldots (4)$$

The elimination of a and b from (2), (3) and (4), if it exits, is called singular integral.

1. Form partial differential equation by eliminating the arbitrary functions f and g in

$$z = f(x^3 + 2y) + g(x^3 - 2y).$$

Solution:

$$z = f(x^3 + 2y) + g(x^3 - 2y) \ldots (1)$$

Differentiating (1) partially with respect to x,

$$p = \frac{\partial z}{\partial x} = f'(x^3 + 2y)3x^2 + g'(x^3 - 2y)3x^2 \ldots (2)$$

Differentiating (1) partially with respect to y,

$$q = \frac{\partial z}{\partial y} = f'(x^3 + 2y)2 + g'(x^3 - 2y)(-2) \ldots (3)$$

Differentiating (2) partially with respect to x,

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x^3 + 2y)9x^4 + g''(x^3 - 2y)9x^4 + f'(x^3 + 2y)6x + g'(x^3 - 2y)6x \ldots (4)$$

Differentiating (3) partially with respect to y,

$$t = \frac{\partial^2 z}{\partial y^2} = f''(x^3 + 2y)4 + g''(x^3 - 2y)4 \ldots (5)$$

Differentiating (3) partially with respect to x,

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''(x^3 + 2y)6x^2 - g''(x^3 - 2y)6x^2 \ldots (6)$$

$$r = 9x^4\big(f''(x^3 + 2y) + g''(x^3 - 2y)\big) + 6x\big(f'(x^3 + 2y) + g'(x^3 - 2y)\big)$$

$$r = 9x^4\frac{t}{4} + 6x\frac{p}{3x^2} = 9x^4\frac{t}{4} + \frac{2p}{x} \qquad (from\ (2)\ \&\ (5))$$

$$4xr = 9x^5 t + 8p$$

2. Form partial differential equation by eliminating the arbitrary functions f and g in

$$z = xf(2x + y) + g(2x + y).$$

Solution:

$$z = xf(2x+y) + g(2x+y) \dots (1)$$

Differentiating (1) partially with respect to x,

$$p = \frac{\partial z}{\partial x} = xf'(2x+y)2 + f(2x+y) + g'(2x+y)2 \dots (2)$$

Differentiating (1) partially with respect to y,

$$q = \frac{\partial z}{\partial y} = xf'(2x+y) + g'(2x+y) \dots (3)$$

Differentiating (2) partially with respect to x,

$$r = \frac{\partial^2 z}{\partial x^2} = xf''(2x+y)4 + f'(2x+y)2 + f'(2x+y)2 + g''(2x+y)4$$

$$r = \frac{\partial^2 z}{\partial x^2} = xf''(2x+y)4 + f'(2x+y)4 + g''(2x+y)4 \dots (4)$$

Differentiating (3) partially with respect to y,

$$t = \frac{\partial^2 z}{\partial y^2} = xf''(2x+y) + g''(2x+y) \dots (5)$$

Differentiating (2) partially with respect to y,

$$s = \frac{\partial^2 z}{\partial y \partial x} = xf''(2x+y)2 + f'(2x+y) + g''(2x+y)2 \dots (6)$$

From (4), (5) and (6), we get

$$4s - r = 4xf''(2x+y) + 4\ g''(2x+y) = 4t$$

$$4s - r = 4t$$

3. Find the singular solution of $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution:

$$z = px + qy + \sqrt{1 + p^2 + q^2} \dots (1)$$

(1) is of Type-II

Let $z = ax + by + c \dots (2)$ *be the complete solution of* (1)

Differentiating partially with respect to x *and* y, *we get*

$$p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

Substituting the values of p and q in (1),

$$z = ax + by + \sqrt{1 + a^2 + b^2} \ldots (3) is\ the\ complete\ Solution\ of\ (1).$$

To find singular solution:

$$\frac{\partial z}{\partial a} = 0\ \&\ \frac{\partial z}{\partial b} = 0$$

$$\frac{\partial z}{\partial a} = x + \frac{2a}{2\sqrt{1 + a^2 + b^2}} = 0 \Rightarrow x = \frac{-a}{\sqrt{1 + a^2 + b^2}} \ldots (4)$$

$$\frac{\partial z}{\partial b} = y + \frac{2b}{2\sqrt{1 + a^2 + b^2}} = 0 \Rightarrow y = \frac{-b}{\sqrt{1 + a^2 + b^2}} \ldots (5)$$

$$1 - x^2 - y^2 = 1 - \frac{a^2}{1 + a^2 + b^2} - \frac{b^2}{1 + a^2 + b^2} = \frac{1 + a^2 + b^2 - a^2 - b^2}{1 + a^2 + b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2} \Rightarrow \sqrt{1 - x^2 - y^2} = \frac{1}{\sqrt{1 + a^2 + b^2}} \ldots (6)$$

Substituting (4 and (5) in (6), we get

$$x = -a\sqrt{1 - x^2 - y^2} \Rightarrow a = \frac{-x}{\sqrt{1 - x^2 - y^2}} \ldots (7)$$

$$y = -b\sqrt{1 - x^2 - y^2} \Rightarrow b = \frac{-y}{\sqrt{1 - x^2 - y^2}} \ldots (8)$$

Substituting (6),(7) and (8) in (3), we get

$$z = \frac{-x}{\sqrt{1 - x^2 - y^2}}x + \frac{-y}{\sqrt{1 - x^2 - y^2}}y + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}$$

$$z = \sqrt{1 - x^2 - y^2}\ is\ the\ singular\ solution.$$

4. Solve $p(1 - q^2) = q(1 - z)$

Solution:

$$p(1 - q^2) = q(1 - z) \ldots (1)$$

This is of type $F(z, p, q) = 0$

Let $u = x + ay \ldots (2)$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}.1$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}.a$$

Substituting the values of p and q in (1),

$$\frac{\partial z}{\partial u}\left(1 - \left(\frac{\partial z}{\partial u}a\right)^2\right) = \frac{\partial z}{\partial u}a(1 - z)$$

$$= a(1 - z)$$

$$1 - a(1 - z) = \left(\frac{\partial z}{\partial u}a\right)^2 \Rightarrow \frac{1 - a - az}{a^2} = \left(\frac{\partial z}{\partial u}\right)^2$$

$$\frac{\partial z}{\partial u} = \frac{1}{a}\sqrt{(1 - a) - az}$$

$$\frac{a\partial z}{\sqrt{(1 - a) - az}} = \partial u$$

Integrating on both sides,

$$\int \frac{a\partial z}{\sqrt{(1 - a) - az}} = \int \partial u$$

$$-2\sqrt{(1 - a) - az} = u + c \ldots (3)$$

Substituting (2) in (3)

$-2\sqrt{(1 - a) - az} = x + ay + c$ is the complete solution.

5. Solve $z = p^2 + q^2$

Solution:

$$z = p^2 + q^2 \ldots (1)$$

This is of type $F(z, p, q) = 0$

Let $u = x + ay \ldots (2)$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}.1$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}.a$$

Substituting the values of p and q in (1),

$$z = \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}.a\right)^2$$

$$z = \left(\frac{\partial z}{\partial u}\right)^2 (1 + a^2)$$

$$\frac{z}{(1 + a^2)} = \left(\frac{\partial z}{\partial u}\right)^2$$

$$\sqrt{\frac{z}{(1 + a^2)}} = \frac{\partial z}{\partial u}$$

$$\int \frac{\sqrt{(1 + a^2)}\partial z}{\sqrt{z}} = \int \partial u$$

$$2\sqrt{z}\sqrt{(1 + a^2)} = u + c \ldots (3)$$

Substituting (2) in (3)

$2\sqrt{z}\sqrt{(1 + a^2)} = x + ay + c$ is the complete solution.

6. Solve $(y - xz)p + (yz - x)q = (x + y)(x - y)$

Solution:

The equation is of the form $Pp + Qq = R$

Here $P = (y - xz), Q = (yz - x), R = (x + y)(x - y)$

$$\frac{dx}{y - xz} = \frac{dy}{yz - x} = \frac{dz}{(x + y)(x - y)}$$

$$\frac{ydx + xdy}{y^2 - xyz + xyz - x^2} = \frac{dz}{x^2 - y^2}$$

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{dz}{x^2 - y^2}$$

$$ydx + xdy = -dz$$

$$d(xy) = -dz$$

Integrating on both sides, we get

$$xy = -z + c_1 \Rightarrow c_1 = xy + z$$

$$\frac{xdx + ydy}{xy - x^2z + y^2z - xy} = \frac{dz}{x^2 - y^2}$$

$$\frac{xdx + ydy}{-z(x^2 - y^2)} = \frac{dz}{x^2 - y^2}$$

$$xdx + ydy = -zdz$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = -\frac{z^2}{2} + c_2 \Rightarrow c_3 = x^2 + y^2 + z^2$$

The general solution is

$$\Phi(c_1, c_3) = 0$$

$$\Phi(xy + z, x^2 + y^2 + z^2) = 0$$

7. Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

Solution:

The equation is of the form $Pp + Qq = R$

Here $P = x(z^2 - y^2)$, $Q = y(x^2 - z^2)$, $R = z(y^2 - x^2)$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy}{z^2 - y^2 + x^2 - z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy}{-(y^2 - x^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{1}{x}dx + \frac{1}{y}dy = -\frac{dz}{z}$$

Integrating on both sides, we get

$$\log x + \log y = -\log z + \log c_1$$

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1 \Rightarrow c_1 = xyz$$

$$\frac{xdx + ydy}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{xdx + ydy}{x^2z^2 - y^2z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{xdx + ydy}{-z^2(y^2 - x^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$xdx + ydy = -zdz$$

Integrating on both sides, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = -\frac{z^2}{2} + c_2 \Rightarrow c_3 = x^2 + y^2 + z^2$$

The general solution is

$$\Phi(c_1, c_3) = 0$$

$$\Phi(xyz, x^2 + y^2 + z^2) = 0$$

8. Find the singular solution of $z = px + qy + p^2 - q^2$

Solution:

$$z = px + qy + p^2 - q^2 \dots (1)$$

(1) is of Type-II

Let $z = ax + by + c \dots (2)$ *be the complete solution of* (1)

Differentiating partially with respect to x *and* y, *we get*

$$p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$$

Substituting the values of p and q in (1),

$$z = ax + by + a^2 - b^2 \dots (3) \text{is the complete Solution of } (1).$$

To find singular solution:

$$\frac{\partial z}{\partial a} = 0 \ \& \ \frac{\partial z}{\partial b} = 0$$

$$\frac{\partial z}{\partial a} = x + 2a = 0 \Rightarrow a = \frac{-x}{2} \dots (4)$$

$$\frac{\partial z}{\partial b} = y - 2b = 0 \Rightarrow b = \frac{y}{2} \dots (5)$$

Substituting (4) and (5) in (3), we get

$$z = \frac{-x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = -\frac{x^2}{4} + \frac{y^2}{4} \Rightarrow 4z = y^2 - x^2 \textit{ is the singular solution.}$$

9. Solve $p^2y(1+x^2) = qx^2$

Solution:

$$p^2y(1+x^2) = qx^2 \Rightarrow \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} \dots (1)$$

It is of variable separable type.

$$Let\ \frac{p^2(1+x^2)}{x^2} = a^2 \Rightarrow p^2 = \frac{a^2x^2}{(1+x^2)} \Rightarrow p = \frac{ax}{\sqrt{1+x^2}} \dots (2)$$

$$Let\ \frac{q}{y} = a \Rightarrow q = ay \dots (3)$$

$$dz = pdx + qdy \dots (4)$$

Substituting (2) and (3) in (4), we get

$$dz = \frac{ax}{\sqrt{1+x^2}}dx + aydy$$

Integrating on both sides, we get

$$\int dz = \frac{1}{2}\int \frac{2ax}{\sqrt{1+x^2}}dx + \int aydy$$

$$z = \frac{a}{2}2\sqrt{1+x^2} + a\frac{y^2}{2} + c$$

$$z = a\sqrt{1+x^2} + a\frac{y^2}{2} + c \quad \textit{is the complete solution of } (1).$$

10. Solve $z^2(p^2 + q^2) = x^2 + y^2$.

Solution:

$$(zp)^2 + (zq)^2 = x^2 + y^2 \dots (1)$$

It is of the form $F(z^kp, z^kq) = 0$

Here $k = 1, Put\ Z = z^{k+1} = z^2 \ldots (2)$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = 2zp \Rightarrow zp = \frac{P}{2} \ldots (3)$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = 2zq \Rightarrow zq = \frac{Q}{2} \ldots (4)$$

Substituting (3) and (4) in (1), we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2 + y^2$$

$$P^2 + Q^2 = 4x^2 + 4y^2$$

It is of variable separable type

$$P^2 - 4x^2 = 4y^2 - Q^2 = a^2$$

$$P^2 - 4x^2 = a^2 \Rightarrow P^2 = 4x^2 + a^2 \Rightarrow P = 2\sqrt{x^2 + \left(\frac{a}{2}\right)^2} \ldots (5)$$

$$4y^2 - Q^2 = a^2 \Rightarrow 4y^2 - a^2 = Q^2 \Rightarrow Q = 2\sqrt{y^2 - \left(\frac{a}{2}\right)^2} \ldots (6)$$

$$dZ = Pdx + Qdy \ldots (7)$$

$$dZ = 2\sqrt{x^2 + \left(\frac{a}{2}\right)^2}\,dx + 2\sqrt{y^2 - \left(\frac{a}{2}\right)^2}\,dy \qquad from\ (5), (6) and\ (7)$$

Integrating on both sides, we get

$$\int dZ = 2\int \sqrt{x^2 + \left(\frac{a}{2}\right)^2}\,dx + 2\int \sqrt{y^2 - \left(\frac{a}{2}\right)^2}\,dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2 + \left(\frac{a}{2}\right)^2} + \frac{a^2}{8} sinh^{-1}\left(\frac{2x}{a}\right) + \frac{y}{2}\sqrt{y^2 - \left(\frac{a}{2}\right)^2} - cosh^{-1}\left(\frac{2y}{a}\right)\right] + c$$

$$z^2 = 2\left[\frac{x}{2}\sqrt{x^2 + \left(\frac{a}{2}\right)^2} + \frac{a^2}{8} sinh^{-1}\left(\frac{2x}{a}\right) + \frac{y}{2}\sqrt{y^2 - \left(\frac{a}{2}\right)^2} - cosh^{-1}\left(\frac{2y}{a}\right)\right] + c$$

$[from\ (2)]$

is the general solution of (1).

11. Solve $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$

Solution:

The equation is of the form $Pp + Qq = R$

Here $P = x^2 + y^2 + yz\,, Q = x^2 + y^2 - xz, R = z(x + y)$

$$\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x + y)}$$

$$\frac{dx - dy}{x^2 + y^2 + yz - x^2 - y^2 + xz} = \frac{dz}{z(x + y)}$$

$$\frac{dx - dy}{yz + xz} = \frac{dz}{z(x + y)} \Rightarrow dx - dy = dz$$

Integrating on both sides, we get

$$x - y = z + c_1 \Rightarrow c_1 = x - y - z$$

$$\frac{xdx + ydy}{x^3 + xy^2 + xyz + yx^2 + y^3 - xyz} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{x^3 + xy^2 + yx^2 + y^3} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{x^3 + yx^2 + xy^2 + y^3} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{x^2(x + y) + y^2(x + y)} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{(x + y)(x^2 + y^2)} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{x^2 + y^2} = \frac{dz}{z}$$

Integrating on both sides, we get

$$\frac{1}{2}\log(x^2 + y^2 + z^2) = \log z + log\ c_2$$

$$\log(x^2 + y^2) = 2\log z\, c_2$$

$$\log(x^2 + y^2) = \log z^2\, {c_2}^2$$

$$x^2 + y^2 = z^2\, c_3 \Rightarrow c_3 = \frac{x^2 + y^2}{z^2}$$

The general solution is

$$\Phi(c_1, c_3) = 0$$

$$\Phi\left(x - y - z, \frac{x^2 + y^2}{z^2}\right) = 0$$

12. Solve $(y^2 + z^2)p - xyq + xz = 0$

Solution:

$$(y^2 + z^2)p - xyq = -xz$$

The equation is of the form $Pp + Qq = R$

Here $P = (y^2 + z^2)$, $Q = -xy, R = -xz$

$$\frac{dx}{(y^2 + z^2)} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

$$\frac{dy}{-xy} = \frac{dz}{-xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides, we get

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1 \Rightarrow c_1 = \frac{y}{z}$$

$$\frac{xdx + ydy + zdz}{(xy^2 + xz^2) - xy^2 - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 \Rightarrow c_3 = x^2 + y^2 + z^2$$

The general solution is

$$\Phi(c_1, c_3) = 0$$

$$\Phi\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0$$

13. Solve $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$

Solution:

$$(x^2 - y^2 - z^2)p + 2xyq = 2xz$$

The equation is of the form $Pp + Qq = R$

Here $P = (x^2 - y^2 - z^2)$, $Q = 2xy, R = 2xz$

$$\frac{dx}{(x^2 - y^2 - z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides, we get

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1 \Rightarrow c_1 = \frac{y}{z}$$

$$\frac{dx}{(x^2 - y^2 - z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{xdx + ydy + zdz}{x^3 - \ xy^2 - xz^2 + 2\, xy^2 + 2xz^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{xdx + ydy}{x^3 + \ xy^2 + xz^2} = \frac{dy}{2xy}$$

$$\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2\ )} = \frac{dy}{2xy}$$

$$\frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2\ )} = \frac{dy}{y}$$

Integrating on both sides, we get

$$\int \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2\ )} = \int \frac{dy}{y}$$

$$\log(x^2 + y^2 + z^2\ ) = \log y + log\, c_2$$

$$\log(x^2 + y^2 + z^2\ ) = \log y$$

$$x^2+y^2+z^2 = yc_2 \Rightarrow c_2 = \frac{x^2+y^2+z^2}{y}$$

The general solution is

$$\Phi(c_1, c_2) = 0$$

$$\Phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0$$

14. Solve $\left(D^2 - DD' - 2D'^2\right)z = 2x + 3y + e^{3x+4y}$

Solution:

The auxiliary equation is $m^2 - m - 2 = 0$

$$m = -1,2$$

The complementary function CF is $z = \phi_1(y - x) + \phi_2(y + 2x)$

To find particular integral:

$$P.I_1 = \frac{2x + 3y}{D^2 - DD' - 2D'^2}$$

$$= \frac{2x + 3y}{D^2\left(1 - \frac{D'}{D} - 2\frac{D'^2}{D^2}\right)} = \frac{2x + 3y}{D^2\left(1 - \left(\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 - \left(\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)\right)^{-1}(2x + 3y)}{D^2}$$

$$= \frac{\left(1 + \left(\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)\right)(2x + 3y)}{D^2} \qquad [(1 + x)^{-1} = 1 - x + x^2 - \cdots]$$

$$= \frac{\left((2x + 3y) + \left(\frac{D'}{D}(2x + 3y) + 2\frac{D'^2}{D^2}(2x + 3y)\right)\right)}{D^2}$$

$$= \frac{\left((2x+3y)+\left(\frac{3}{D}\right)\right)}{D^2} = \frac{1}{D^2}(2x+3y) + \frac{3}{D^3}$$

$$P.I_1 = \frac{2x^3}{3!} + \frac{3x^2y}{2!} + \frac{3x^3}{3!} = \frac{5x^3}{3!} + \frac{3x^2y}{2!} \qquad \left[\frac{1}{D^n}x^m = \frac{m!\,x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^{3x+4y}}{D^2 - DD' - 2D'^2}$$

$$= \frac{e^{3x+4y}}{3^2 - (3)(4) - 2(4)^2} = \frac{e^{3x+4y}}{9 - 12 - 32}$$

$$P.I_2 = -\frac{e^{3x+4y}}{35}$$

The general solution is

$$CF + P.I_1 + P.I_2$$

$$z = \phi_1(y-x) + \phi_2(y+2x) + \frac{5x^3}{3!} + \frac{3x^2y}{2!} - \frac{e^{3x+4y}}{35}$$

15. Solve $(D^2 + 3DD' - 4D'^2)z = \sin y$

Solution:

The auxiliary equation is $m^2 + 3m - 4 = 0$

$$m = -4, 1$$

The complementary function CF is $z = \phi_1(y-4x) + \phi_2(y+x)$

To find particular integral:

$$P.I = \frac{\sin y}{D^2 + 3DD' - 4D'^2}$$

$$= \frac{Imaginary\ part\ of\ e^{iy}}{D^2 + 3DD' - 4D'^2}$$

$$= \frac{Imaginary\ part\ of\ e^{iy}}{(0)^2 + 3(0)(i) - 4(i)^2} = \frac{Imaginary\ part\ of\ e^{iy}}{4}$$

$$P.I = \frac{\sin y}{4}$$

The general solution is $C.F \; + \; P.I$

$$z = \phi_1(y-4x) + \phi_2(y+x) + \frac{\sin y}{4}$$

16. Solve $\left(D^2 + DD' - 6D'^2\right)z = x^2y + e^{3x+y}$

Solution:

The auxiliary equation is $m^2 + m - 6 = 0$

$$m = -3,2$$

The complementary function CF is $z = \phi_1(y-3x) + \phi_2(y+2x)$

To find particular integral:

$$P.I_1 = \frac{x^2y}{D^2 + DD' - 6D'^2}$$

$$= \frac{x^2y}{D^2\left(1 + \frac{D'}{D} - 6\frac{D'^2}{D^2}\right)} = \frac{x^2y}{D^2\left(1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right)^{-1} x^2y}{D^2}$$

$$= \frac{\left(1 - \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right)x^2y}{D^2} \qquad [(1+x)^{-1} = 1 - x + x^2 - \cdots]$$

$$= \frac{\left(x^2y - \left(\frac{D'}{D}x^2y - 6\frac{D'^2}{D^2}x^2y\right)\right)}{D^2}$$

$$= \frac{\left(x^2y - \left(\frac{x^2}{D}\right)\right)}{D^2} = \frac{1}{D^2}x^2y - \frac{1}{D^3}x^2$$

$$P.I_1 = \frac{2x^4y}{4!} - \frac{2x^5}{5!} = \frac{x^4y}{12} - \frac{x^5}{60} \qquad \left[\frac{1}{D^n}x^m = \frac{m!\,x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^{3x+y}}{D^2 + DD' - 6D'^2} = \frac{e^{3x+y}}{3^2 + (3)(1) - 6(1)^2}$$

$$P.I_2 = \frac{e^{3x+y}}{6}$$

The general solution is

$$CF \ + P.I_1 + P.I_2$$

$$z = \phi_1(y - 3x) + \phi_2(y + 2x) + \frac{x^4y}{12} - \frac{x^5}{60} + \frac{e^{3x+y}}{6}$$

17. Solve $(D^3 - 2D^2D')z = 4\sin(x + y) + e^{x+2y}$

The auxiliary equation is $m^3 - 2m^2 = 0 \Rightarrow m^2(m - 2) = 0$

$$m = 0,0,2$$

The complementary function CF is $z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$

To find particular integral:

$$P.I_1 = \frac{4\sin(x + y)}{D^3 - 2D^2D'}$$

$$= \frac{4\ Imaginary\ part\ of\ e^{i(x+y)}}{D^3 - 2D^2D'}$$

$$= \frac{4\ Imaginary\ part\ of\ e^{i(x+y)}}{(i)^3 - 2(i)^2(i)} = \frac{4\ Imaginary\ part\ of\ e^{i(x+y)}}{-i + 2(i)}$$

$$= \frac{4\ Imaginary\ part\ of\ e^{i(x+y)}}{i}$$

$$= -4\ Imaginary\ part\ of\ ie^{i(x+y)} \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= -4\ Imaginary\ part\ of\ i(\cos(x + y) + isin\ (x + y))$$

$$= -4\ Imaginary\ part\ of\ (i\cos(x + y) + i^2sin\ (x + y))$$

$$P.I_1 = -4\ \cos(x+y)$$

$$P.I_2 = \frac{e^{x+2y}}{D^3 - 2D^2D'}$$

$$P.I_2 = \frac{e^{x+2y}}{1^3 - 2(1)^2(2)} = -\frac{e^{x+2y}}{3}$$

The general solution is

$$CF\ + P.I_1 + P.I_2$$

$$z = \phi_1(y) + \phi_2\left(y + \sqrt{2}x\right) + \phi_3\left(y - \sqrt{2}x\right) - 4\ \cos(x+y) - \frac{e^{x+2y}}{3}$$

18. Solve $\left(D^2 - DD' - 20D'^2\right)z = \sin(4x - y) + e^{5x+y}$

Solution:

The auxiliary equation is $m^2 - m - 20 = 0$

$$m = -4{,}5$$

The complementary function CF is $z = \phi_1(y - 4x) + \phi_2(y + 5x)$

To find particular integral:

$$P.I_1 = \frac{\sin(4x - y)}{D^2 - DD' - 20D'^2}$$

$$= \frac{Imaginary\ part\ of e^{i(4x-y)}}{D^2 - DD' - 20D'^2}$$

$$= \frac{Imaginary\ part\ of e^{i(4x-y)}}{(4i)^2 - (4i)(-i) - 20(-i)^2}$$

$$= \frac{Imaginary\ part\ of e^{i(4x-y)}}{-16 - 4 + 20}$$

$$\begin{bmatrix} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{bmatrix}$$

$$= \frac{Imaginary\ part\ of xe^{i(4x-y)}}{2D - D'}$$

$$= \frac{Imaginary\ part\ of xe^{i(4x-y)}}{2(4i)-(-i)}$$

$$= \frac{Imaginary\ part\ of xe^{i(4x-y)}}{9i}$$

$$= -\frac{Imaginary\ part\ of ixe^{i(4x-y)}}{9} \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= -\frac{Imaginary\ part\ of ix(\cos(4x-y) + isin\ (4x-y))}{9}$$

$$= -\frac{Imaginary\ part\ of (ix\cos(4x-y) + xi^2 sin\ (4x-y))}{9}$$

$$P.I_1 = -\frac{x\cos(4x-y)}{9}$$

$$P.I_2 = \frac{e^{5x+y}}{D^2 - DD' - 20D'^2}$$

$$= \frac{e^{5x+y}}{5^2 - (5)(1) - 20(1)^2}$$

$$\left[\begin{matrix} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{matrix}\right]$$

$$= \frac{e^{5x+y}}{2D - D'} = \frac{e^{5x+y}}{2(5)-1}$$

$$P.I_2 = \frac{e^{5x+y}}{9}$$

The general solution is

$$CF\ + P.I_1 + P.I_2$$

$$z = \phi_1(y-4x) + \phi_2(y+5x) - \frac{x\cos(4x-y)}{9} + \frac{e^{5x+y}}{9}$$

19. Solve $\left(D^2 - 4DD' + 4D'^2\right)z = xy + e^{2x+y}$

Solution:

The auxiliary equation is $m^2 - 4m + 4 = 0$

$$m = 2,2$$

The complementary function CF is $z = \phi_1(y + 2x) + x\phi_2(y + 2x)$

To find particular integral:

$$P.I_1 = \frac{xy}{D^2 - 4DD' + 4D'^2}$$

$$= \frac{xy}{D^2\left(1 - 4\frac{D'}{D} + 4\frac{D'^2}{D^2}\right)} = \frac{xy}{D^2\left(1 - \left(4\frac{D'}{D} - 4\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 - \left(4\frac{D'}{D} - 4\frac{D'^2}{D^2}\right)\right)^{-1} xy}{D^2}$$

$$= \frac{\left(1 + \left(4\frac{D'}{D} - 4\frac{D'^2}{D^2}\right)\right)xy}{D^2} \qquad [(1 - x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \frac{\left(xy + \left(4\frac{D'}{D}xy - 4\frac{D'^2}{D^2}xy\right)\right)}{D^2}$$

$$= \frac{\left(xy + \left(4\frac{x}{D}\right)\right)}{D^2} = \frac{1}{D^2}xy + \frac{4}{D^3}x$$

$$P.I_1 = \frac{x^3y}{6} + \frac{4x^4}{24} = \frac{x^3y}{6} + \frac{x^4}{6} \qquad \left[\frac{1}{D^n}x^m = \frac{m!\,x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2} = \frac{e^{2x+y}}{2^2 - 4(2)(1) + 4(1)^2}$$

$$= \frac{xe^{2x+y}}{2D - 4D'} = \frac{xe^{2x+y}}{2(2) - 4(1)}$$

$$\begin{bmatrix} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{bmatrix}$$

$$P.I_2 = \frac{x^2 e^{2x+y}}{2}$$

The general solution is

$$CF\ + P.I_1 + P.I_2$$

$$z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^3 y}{6} + \frac{x^4}{6} + \frac{x^2 e^{2x+y}}{2}$$

20. Solve $(D^2 - 6DD' + 5D'^2)z = xy + e^x \sin hy$

Solution:

The auxiliary equation is $m^2 - 6m + 5 = 0$

$$m = 1,5$$

The complementary function CF is $z = \phi_1(y+x) + \phi_2(y+5x)$

To find particular integral:

$$P.I_1 = \frac{xy}{D^2 - 6DD' + 5D'^2}$$

$$= \frac{xy}{D^2\left(1 - 6\frac{D'}{D} + 5\frac{D'^2}{D^2}\right)} = \frac{xy}{D^2\left(1 - \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 - \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right)^{-1} xy}{D^2}$$

$$= \frac{\left(1 + \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right) xy}{D^2} \qquad [(1-x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \frac{\left(xy + \left(6\frac{D'}{D}xy - 5\frac{D'^2}{D^2}xy\right)\right)}{D^2}$$

$$= \frac{\left(xy + \left(6\frac{x}{D}\right)\right)}{D^2} = \frac{1}{D^2}xy + \frac{6}{D^3}x$$

$$P.I_1 = \frac{x^3y}{6} + \frac{6x^4}{24} = \frac{x^3y}{6} + \frac{x^4}{4} \qquad \left[\frac{1}{D^n}x^m = \frac{x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^x \sin hy}{D^2 - 6DD' + 5D'^2} = \frac{e^x\left(\frac{e^y - e^{-y}}{2}\right)}{D^2 - 6DD' + 5D'^2}$$

$$= \frac{e^{x+y}}{2(D^2 - 6DD' + 5D'^2)} - \frac{e^{x-y}}{2(D^2 - 6DD' + 5D'^2)}$$

$$= \frac{e^{x+y}}{2(1^2 - 6(1)(1) + 5(1)^2)} - \frac{e^{x-y}}{2(1^2 - 6(1)(-1) + 5(-1)^2)}$$

$$= \frac{xe^{x+y}}{2(2D - 6D')} - \frac{e^{x-y}}{2(11)}$$

$$\left[\begin{array}{l} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{array}\right]$$

$$P.I_2 = \frac{xe^{x+y}}{2(2(1) - 6(1))} - \frac{e^{x-y}}{22} = \frac{xe^{x+y}}{-8} - \frac{e^{x-y}}{22}$$

The general solution is

$$CF + P.I_1 + P.I_2$$

$$z = \phi_1(y + x) + \phi_2(y + 5x) + \frac{x^3y}{6} + \frac{x^4}{4} - \frac{xe^{x+y}}{8} - \frac{e^{x-y}}{22}$$

21. Solve $(D^2 - DD' - 30D'^2)z = xy + e^{6x+y}$

Solution:

The auxiliary equation is $m^2 - m - 30 = 0$

$$m = -5,6$$

The complementary function CF is $z = \phi_1(y - 5x) + \phi_2(y + 6x)$

To find particular integral:

$$P.I_1 = \frac{xy}{D^2 - DD' - 30D'^2}$$

$$= \frac{xy}{D^2\left(1 - \frac{D'}{D} - 30\frac{D'^2}{D^2}\right)} = \frac{xy}{D^2\left(1 - \left(\frac{D'}{D} + 30\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 - \left(\frac{D'}{D} + 30\frac{D'^2}{D^2}\right)\right)^{-1} xy}{D^2}$$

$$= \frac{\left(1 + \left(\frac{D'}{D} + 30\frac{D'^2}{D^2}\right)\right) xy}{D^2} \qquad [(1 - x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \frac{\left(xy + \left(\frac{D'}{D} xy + 30\frac{D'^2}{D^2} xy\right)\right)}{D^2}$$

$$= \frac{\left(xy + \left(\frac{1}{D} x\right)\right)}{D^2} = \frac{1}{D^2} xy + \frac{x}{D^3}$$

$$P.I_1 = \frac{x^3 y}{3!} + \frac{x^4}{4!} \qquad \left[\frac{1}{D^n} x^m = \frac{m!\, x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^{6x+y}}{D^2 - DD' - 30D'^2}$$

$$= \frac{e^{6x+y}}{6^2 - (6)(1) - 30(1)^2}$$

$$\left[\begin{array}{l} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{array}\right]$$

$$= \frac{xe^{6x+y}}{2D - D'} = \frac{xe^{6x+y}}{2(6) - 1}$$

$$P.I_2 = \frac{xe^{6x+y}}{11}$$

The general solution is

$$CF \ + P.I_1 + P.I_2$$

$$z = \phi_1(y - 5x) + \phi_2(y + 6x) + \frac{x^3 y}{3!} + \frac{x^4}{4!} + \frac{xe^{6x+y}}{11}$$

UNIT-IV

Applications to partial differential equations

Classification of second order partial differential equation:

$$Let\ A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0 \ldots (1)$$

$be\ the\ second\ order\ partial\ differential\ equation.$

The classifications of (1) are as follows

$i)\ Parabola\ if\ B^2 - 4AC = 0$

$ii)\ Ellipse\ if\ B^2 - 4AC < 0$

$iii)\ Hyperbola\ if\ B^2 - 4AC > 0$

1. Classify the partial differential equation $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$.

Solution: Given $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$.

$$A = 3, B = 4, C = 0$$

$B^2 - 4AC = 16 > 0$, Hyperbolic

2. Classify the partial differential equation $u_{xx} + xu_{xy} = 0$.

Solution:

Here $A = 1, B = x, C = 0$

$$B^2 - 4AC = x^2$$

$$(i)\ Parabolic\ \ if\ x = 0.$$

$$(ii)\ Hyperbolic\ \ if\ x > 0\ and\ x < 0.$$

Initial and boundary value problems:

The values of a required solution, on the boundary of some domain will be given. These are called boundary conditions. In other cases, when time t is one of the variables, the values of the solution at t=0 may be presented. These are called initial conditions.

The partial differential equation together with these conditions constitutes a boundary value problem or an initial value problem, according to the nature of the condition.

One dimensional wave equation:

One dimensional wave equation with initial and boundary conditions in which the initial position of the string is $f(x)$ and the initial velocity imparted at each point x is $g(x)$ is

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

$$(i)\ \ y(0,t) = 0 \qquad (ii)\ y(l,t) = 0, \qquad (iii)\ y(x,0) = f(x)\ \ (iv)\ \ \frac{\partial y\,(x,0)}{\partial t} = g(x)$$

Possible solutions of one dimensional wave equation:

The possible solutions of one dimensional wave equation are

$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat}) \ldots (1)$

$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat) \ldots (2)$

$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}) \ldots (3)$

Assumptions made in the derivation of one dimensional wave equation:

(i) The mass of the string per unit length is constant

(ii) The string is perfectly elastic and does not offer any resistance to bending

(iii) The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.

(iv) The string performs a small transverse motion in a vertical plane that is every particle of the string moves strictly vertically so that the deflection and the slope at every point of the string remain small in absolute value.

Steady state conditions of one dimensional heat flow equation:

when steady state conditions exist the heat flow equation is independent of time t.

$$\frac{\partial u}{\partial t} = 0$$

The heat flow equation becomes

$$\frac{\partial^2 u}{\partial x^2} = 0$$

The solution of heat flow equation is $u(x) = c_1 x + c_2$

Steady state heat flow equation in two dimensions in Cartesian form:

The two dimensional heat flow equation for the steady state is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. [Laplace\ equation]$$

Possible solutions of the Laplace equation:

The possible solutions of the Laplace equation are

$u(x,y)=(c_1e^{px}+c_2e^{-px})(c_3\cos py+c_4\sin py)\dots(1)$

$u(x,y)=(c_5\cos px+c_6\sin px)(c_7e^{py}+c_8e^{-py})\dots(2)$

$u(x,y)=(c_9x+c_{10})(c_{11}y+c_{12})\dots(3)$

The polar form of two dimensional heat flow equation in steady state:

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0. [Laplace\ equation]$$

Differential equation for two dimensional heat flow equation for the unsteady state:

The two dimensional heat flow equation for the unsteady state is given by

$$\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}$$

1. A string is stretched and fastened to two points $x=0$ and $x=l$ apart. Motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}\dots(1)$$

The boundary conditions are

$i)\ y(0,t)=0, t>0$

$ii)\ y(l,t)=0, t>0$

$iii)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)}=0, 0<x<l$

$iv)\ y(x,0)=k(lx-x^2), 0<x<l$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t) = (At + B)(Cx + D)$

$II)\ y(x,t) = (Ae^{-\alpha at} + Be^{\alpha at})(Ce^{-\alpha x} + De^{\alpha x})$

$III)\ y(x,t) = (A\ cos\ \alpha at + B\ sin\ \alpha at)(C\ cos\ \alpha x + D\ sin\ \alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)C = 0$

$\therefore C = 0 \qquad\qquad because\ A \cos \alpha at + B \sin \alpha at \neq 0$

$\left[If\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ \Rightarrow y(x,t) = 0\right]$

Now applying the boundary condition (i) and put $C = 0$ for solution III, we get

$y(l,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)D\ sin\ \alpha l = 0$

$sin\ \alpha l = 0 \Rightarrow \alpha l = n\pi \Rightarrow \alpha = \frac{n\pi}{l}$

$$\left[\begin{matrix} A \cos \alpha at + B \sin \alpha at \neq 0\ and\ D \neq 0 \\ Since\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ and\ \ D = 0\ \Rightarrow\ y(x,t) = 0 \end{matrix}\right]$$

Substituting $\alpha = \frac{n\pi}{l}$ $and\ C = 0$ in solution III, we get

$$y(x,t) = \left(A\ cos\ \frac{n\pi at}{l} + B\ sin\ \frac{n\pi at}{l}\right) D\ sin\ \frac{n\pi x}{l} \ldots (2)$$

By putting $AD = A_n\ and\ BD = B_n\ in\ (2),\ we\ get$

$$y(x,t) = \left(A_n\ cos\ \frac{n\pi at}{l} + B_n\ sin\ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (3)$$

The most general form of (3) is

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n\ cos\ \frac{n\pi at}{l} + B_n\ sin\ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (4)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n\ \frac{n\pi a}{l} sin\ \frac{n\pi at}{l} + B_n\ \frac{n\pi a}{l} cos\ \ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (5)$$

Now applying the boundary condition (iii) for (5), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0 \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0$$

$\therefore B_n = 0$

Substituting $B_n = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi at}{l} \sin \frac{n\pi x}{l} \ldots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$y(x,0) = k(lx - x^2) \Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = k(lx - x^2)$$

The above equation is Fourier Sine series.

$$A_n = \frac{2}{l}\int_0^l k(lx - x^2) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[(l^2 - l^2)\left(\frac{-\cos \frac{n\pi l}{l}}{\frac{n\pi}{l}}\right) - (l - 2l)\left(\frac{-\sin \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$- \frac{2k}{l}\left[(0) - (l - 0)\left(\frac{-\sin 0}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$= \frac{2k}{l}\left[(-2)\left(\frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^3}\right) - (-2)\left(\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right)\right] = \frac{4k}{l}\left(\frac{l}{n\pi}\right)^3 [1 - (-1)^n]$$

$$A_n = \frac{4kl^2}{n^3\pi^3}[1 - (-1)^n]$$

$$A_n = \begin{cases} \frac{8kl^2}{n^3\pi^3} \text{ for } n \text{ is odd} \\ 0 \text{ for } n \text{ is even} \end{cases}$$

Substituting the value of A_n in (6), we get

$$y(x,t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8kl^2}{n^3\pi^3} \cos \frac{n\pi at}{l} \sin \frac{n\pi x}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \cos \frac{n\pi at}{l} \sin \frac{n\pi x}{l}$$

2. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l - x)$, then show that

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1,3,5}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x}{l} \sin \frac{na\pi t}{l}$$

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \dots (1)$$

The boundary conditions are

$i)\ y(0,t) = 0, t > 0$

$ii)\ y(l,t) = 0, t > 0$

$iii)\ y(x,0) = 0\ , 0 < x < l$

$iv)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l - x), 0 < x < l$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t) = (At + B)(Cx + D)$

$II)\ y(x,t) = (Ae^{-\alpha at} + Be^{\alpha at})(Ce^{-\alpha x} + De^{\alpha x})$

$III)\ y(x,t) = (A \cos \alpha at + B \sin \alpha at)(C \cos \alpha x + D \sin \alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t) = 0 \Rightarrow (A \cos \alpha at + B \sin \alpha at)C = 0$

$\therefore C = 0 \qquad\qquad because\ A \cos \alpha at + B \sin \alpha at \neq 0$

$\left[If\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ \Rightarrow y(x,t) = 0\right]$

Now applying the boundary condition (i) and put $C = 0$ for solution III, we get

$y(l,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)D\ sin\ \alpha l = 0$

$sin\ \alpha l = 0 \Rightarrow \alpha l = n\pi \Rightarrow \alpha = \dfrac{n\pi}{l}$

$\left[\begin{array}{c} A\cos\alpha at + B\sin\alpha at \neq 0\ and\ D \neq 0 \\ Since\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ and\ \ D = 0\ \Rightarrow\ y(x,t) = 0 \end{array}\right]$

Substituting $\alpha = \frac{n\pi}{l}$ $and\ C = 0$ in solution III, we get

$y(x,t) = \left(A\ cos\ \frac{n\pi at}{l} + B\ sin\ \frac{n\pi at}{l}\right) D\ sin\ \frac{n\pi x}{l} \ldots (2)$

By putting $AD = A_n\ and\ BD = B_n\ in\ (2),\ we\ get$

$y(x,t) = \left(A_n\ cos\ \dfrac{n\pi at}{l} + B_n\ sin\ \dfrac{n\pi at}{l}\right)\ sin\ \dfrac{n\pi x}{l} \ldots (3)$

The most general form of (3) is

$y(x,t) = \displaystyle\sum_{n=1}^{\infty} \left(A_n\ cos\ \dfrac{n\pi at}{l} + B_n\ sin\ \dfrac{n\pi at}{l}\right)\ sin\ \dfrac{n\pi x}{l} \ldots (4)$

Now applying the boundary condition (iii) for (4), we get

$$y(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n\ \ sin\ \frac{n\pi x}{l} = 0$$

$$\therefore A_n\ = 0$$

Substituting $A_n\ = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n\ sin\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l} \ldots (5)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n\ \frac{n\pi a}{l} cos\ \ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l} \ldots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l-x) \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = \lambda(lx - x^2)$$

The above equation is Fourier Sine series.

$$B_n \frac{n\pi a}{l} = \frac{2}{l}\int_0^l \lambda(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[(l^2-l^2)\left(\frac{-\cos\frac{n\pi l}{l}}{\frac{n\pi}{l}}\right) - (l-2l)\left(\frac{-\sin\frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$- \frac{2\lambda}{l}\left[(0) - (l-0)\left(\frac{-\sin 0}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$= \frac{2\lambda}{l}\left[(-2)\left(\frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^3}\right) - (-2)\left(\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right)\right] = \frac{4\lambda}{l}\left(\frac{l}{n\pi}\right)^3[1-(-1)^n]$$

$$B_n \frac{n\pi a}{l} = \begin{cases} \frac{8\lambda l^2}{n^3\pi^3} & for\ n\ is\ odd \\ 0 & for\ n\ is\ even \end{cases}$$

$$B_n = \begin{cases} \frac{8\lambda l^3}{n^4\pi^4 a} & for\ n\ is\ odd \\ 0 & for\ n\ is\ even \end{cases}$$

Substituting the value of B_n in (5), we get

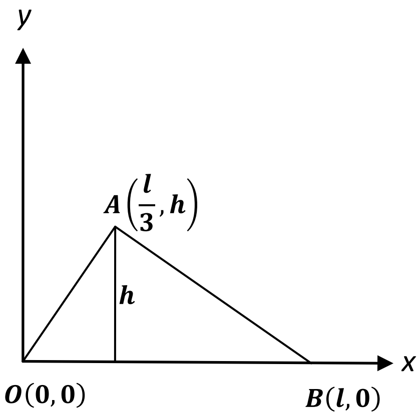
$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{na\pi t}{l}$$

3. A taut string of length l has its ends $x=0, x=l$ fixed. The point where $x=\frac{l}{3}$ is drawn aside a small distance h, the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t.

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \dots (1)$$



The equation of line OA

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

$$\frac{x-0}{\frac{l}{3}-0} = \frac{y-0}{h-0}$$

$$y = \frac{3hx}{l}$$

The equation of line AB

$$\frac{x - \frac{l}{3}}{l - \frac{l}{3}} = \frac{y-h}{0-h}$$

$$y - h = -\frac{hx - h\frac{l}{3}}{2\frac{l}{3}}$$

$$y = h - \frac{3hx - hl}{2l}$$

$$y = \frac{2hl - 3hx + hl}{2l} = \frac{3h(l-x)}{2l}$$

The boundary conditions are

$i)\ y(0,t) = 0, t > 0$

$ii)\ y(l,t) = 0, t > 0$

$iii)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, 0 < x < l$

$$iv)\ y(x,0) = \begin{cases} \dfrac{3hx}{l} & for\ 0 < x < \dfrac{l}{3} \\ \dfrac{3h(l-x)}{2l} & for\ \dfrac{l}{3} < x < l \end{cases}$$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t) = (At + B)(Cx + D)$

$II)\ y(x,t) = (Ae^{-\alpha at} + Be^{\alpha at})(Ce^{-\alpha x} + De^{\alpha x})$

$III)\ y(x,t) = (A\ cos\ \alpha at + B\ sin\ \alpha at)(C\ cos\ \alpha x + D\ sin\ \alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)C = 0$

$\therefore C = 0 \qquad because\ A \cos \alpha at + B \sin \alpha at \neq 0$

$\left[If\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ \Rightarrow y(x,t) = 0\right]$

Now applying the boundary condition (i) and put $C = 0$ for solution III, we get

$y(l,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)D\ sin\ \alpha l = 0$

$$sin\ \alpha l = 0 \Rightarrow \alpha l = n\pi \Rightarrow \alpha = \frac{n\pi}{l}$$

$$\left[\begin{array}{c} A\cos\alpha at + B\sin\alpha at \neq 0\ and\ D \neq 0 \\ Since\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ and\ \ D = 0\ \Rightarrow\ y(x,t) = 0 \end{array}\right]$$

Substituting $\alpha = \frac{n\pi}{l}$ $\ and\ C = 0$ in solution III, we get

$$y(x,t) = \left(A\ cos\ \frac{n\pi at}{l} + B\ sin\ \frac{n\pi at}{l}\right) D\ sin\ \frac{n\pi x}{l} \ldots (2)$$

By putting $AD = A_n\ and\ BD = B_n\ in\ (2),\ we\ get$

$$y(x,t) = \left(A_n\ cos\ \frac{n\pi at}{l} + B_n\ sin\ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (3)$$

The most general form of (3) is

$$y(x,t) = \sum_{n=1}^{\infty}\left(A_n\ cos\ \frac{n\pi at}{l} + B_n\ sin\ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (4)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-A_n\ \frac{n\pi a}{l} sin\ \frac{n\pi at}{l} + B_n\ \frac{n\pi a}{l} cos\ \ \frac{n\pi at}{l}\right)\ sin\ \frac{n\pi x}{l} \ldots (5)$$

Now applying the boundary condition (iii) for (5), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0 \Rightarrow \sum_{n=1}^{\infty} B_n\ \frac{n\pi a}{l}\ sin\ \frac{n\pi x}{l} = 0$$

$\therefore B_n = 0$

Substituting $B_n = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} A_n\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l} \ldots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$y(x,0) = \begin{cases} \dfrac{3hx}{l} & for\ 0 < x < \dfrac{l}{3} \\ \dfrac{3h(l-x)}{2l} & for\ \dfrac{l}{3} < x < l \end{cases}$$

$$\sum_{n=1}^{\infty} A_n\ \sin\frac{n\pi x}{l} = \begin{cases} \dfrac{3hx}{l} & for\ 0 < x < \dfrac{l}{3} \\ \dfrac{3h(l-x)}{2l} & for\ \dfrac{l}{3} < x < l \end{cases}$$

The above equation is Fourier Sine series.

$$A_n = \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3hx}{l} \sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{l} \frac{3h(l-x)}{2l} \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\int_0^{\frac{l}{3}} x \sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{l} \frac{(l-x)}{2} \sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{\frac{l}{3}} + \left[\frac{(l-x)}{2}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \frac{(-1)}{2}\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{\frac{l}{3}}^{l}\right]$$

$$= \frac{6h}{l^2}\left[\left[\frac{l}{3}\left(\frac{-\cos\frac{n\pi l}{3l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi l}{3l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right] - \left[\frac{(l-\frac{l}{3})}{2}\left(\frac{-\cos\frac{n\pi l}{3l}}{\frac{n\pi}{l}}\right) - \frac{(-1)}{2}\left(\frac{-\sin\frac{n\pi l}{3l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right]$$

$$= \frac{6h}{l^2}\left[\frac{l}{3}\left(\frac{-\cos\frac{n\pi}{3}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi}{3}}{\left(\frac{n\pi}{l}\right)^2}\right) - \frac{l}{3}\left(\frac{-\cos\frac{n\pi}{3}}{\frac{n\pi}{l}}\right) + \frac{(-1)}{2}\left(\frac{-\sin\frac{n\pi}{3}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{6h}{l^2}\left[\frac{\sin\frac{n\pi}{3}}{\left(\frac{n\pi}{l}\right)^2} + \frac{1}{2}\frac{\sin\frac{n\pi}{3}}{\left(\frac{n\pi}{l}\right)^2}\right] = \frac{9hl^2}{l^2 n^2 \pi^2}\sin\frac{n\pi}{3}$$

$$A_n = \frac{9h}{n^2\pi^2} sin\ \frac{n\pi}{3}$$

Substituting the value of A_n in (6), we get

$$y(x,t) = \sum_{n=1}^{\infty} \frac{9h}{n^2\pi^2} sin\ \frac{n\pi}{3}\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

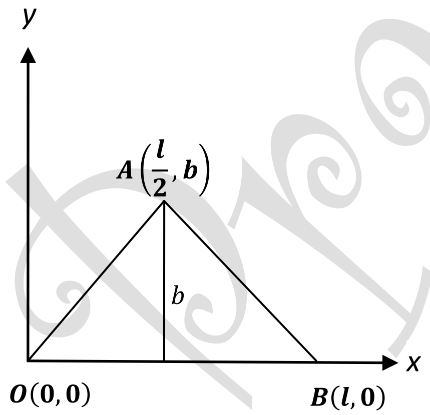
$$y(x,t) = \frac{9h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\ sin\ \frac{n\pi}{3}\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

4. A string is tightly stretched and its ends are fastened at two points $x = 0$ and $x = l$. The point of the string is displaced transversely through a small distance 'b' and the string is released from rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \dots (1)$$



The equation of line OA

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

$$\frac{x - 0}{\frac{l}{2} - 0} = \frac{y - 0}{b - 0}$$

$$y = \frac{2bx}{l}$$

The equation of line AB

$$\frac{x - \frac{l}{2}}{l - \frac{l}{2}} = \frac{y - b}{0 - b}$$

$$y - b = -\frac{bx - b\frac{l}{3}}{\frac{l}{2}}$$

$$y = b - \frac{2bx - bl}{l}$$

$$y = \frac{bl - 2bx + bl}{l} = \frac{2b(l - x)}{l}$$

The boundary conditions are

$i)\ y(0,t) = 0, t > 0$

$ii)\ y(l,t) = 0, t > 0$

$iii)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \qquad 0 < x < l$

$$iv)\ y(x,0)=\begin{cases}\dfrac{2bx}{l} & for\ 0<x<\dfrac{l}{2}\\ \dfrac{2b(l-x)}{l} & for\ \dfrac{l}{2}<x<l\end{cases}$$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t)=(At+B)(Cx+D)$

$II)\ y(x,t)=(Ae^{-\alpha at}+Be^{\alpha at})(Ce^{-\alpha x}+De^{\alpha x})$

$III)\ y(x,t)=(A\cos\alpha at+B\sin\alpha at)(C\cos\alpha x+D\sin\alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t)=0\Rightarrow(A\cos\alpha at+B\sin\alpha at)C=0$

$\therefore C=0 \qquad\qquad because\ A\cos\alpha at+B\sin\alpha at\neq 0$

$\left[If\ A\cos\alpha at+B\sin\alpha at=0\ \Rightarrow y(x,t)=0\right]$

Now applying the boundary condition (i) and put $C=0$ for solution III, we get

$y(l,t)=0\Rightarrow(A\cos\alpha at+B\sin\alpha at)D\sin\alpha l=0$

$\sin\alpha l=0\Rightarrow\alpha l=n\pi\Rightarrow\alpha=\dfrac{n\pi}{l}$

$$\begin{bmatrix}A\cos\alpha at+B\sin\alpha at\neq 0\ and\ D\neq 0\\ Since\ A\cos\alpha at+B\sin\alpha at=0\ and\ D=0\Rightarrow y(x,t)=0\end{bmatrix}$$

Substituting $\alpha=\frac{n\pi}{l}$ and $C=0$ in solution III, we get

$$y(x,t)=\left(A\cos\frac{n\pi at}{l}+B\sin\frac{n\pi at}{l}\right)D\sin\frac{n\pi x}{l}\ \ldots(2)$$

By putting $AD=A_n$ and $BD=B_n$ in (2), we get

$$y(x,t)=\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}\ \ldots(3)$$

The most general form of (3) is

$$y(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}\ \ldots(4)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \ \frac{n\pi a}{l} sin \ \frac{n\pi at}{l} + B_n \ \frac{n\pi a}{l} cos \ \ \frac{n\pi at}{l} \right) \ sin \ \frac{n\pi x}{l} \dots (5)$$

Now applying the boundary condition (iii) for (5), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0 \Rightarrow \sum_{n=1}^{\infty} B_n \ \frac{n\pi a}{l} \ sin \ \frac{n\pi x}{l} = 0$$

$$\therefore B_n = 0$$

Substituting $B_n = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} A_n \ cos \ \frac{n\pi at}{l} \ sin \ \frac{n\pi x}{l} \dots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$y(x,0) = \begin{cases} \dfrac{2bx}{l} & for \ 0 < x < \dfrac{l}{2} \\ \dfrac{2b(l-x)}{l} & for \ \dfrac{l}{2} < x < l \end{cases}$$

$$\sum_{n=1}^{\infty} A_n \ \ sin \ \frac{n\pi x}{l} = \begin{cases} \dfrac{2bx}{l} & for \ 0 < x < \dfrac{l}{2} \\ \dfrac{2b(l-x)}{l} & for \ \dfrac{l}{2} < x < l \end{cases}$$

The above equation is Fourier Sine series.

$$A_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}} \frac{2bx}{l} sin \ \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^{l} \frac{2b(l-x)}{l} sin \ \frac{n\pi x}{l} dx \right]$$

$$= \frac{4b}{l^2}\left[\int_0^{\frac{l}{3}} x \ sin \ \frac{n\pi x}{l} dx + \int_{\frac{l}{2}}^{l} (l-x) \ sin \ \frac{n\pi x}{l} dx \right]$$

$$= \frac{4b}{l^2}\left[\left[x\left(\frac{-cos \ \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 1\left(\frac{-sin \ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_0^{\frac{l}{2}} + \left[\frac{(l-x)}{2}\left(\frac{-cos \ \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (-1)\left(\frac{-sin \ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) \right]_{\frac{l}{2}}^{l} \right]$$

$$= \frac{4b}{l^2}\left[\left[\frac{l}{2}\left(\frac{-cos\ \frac{n\pi l}{2l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-sin\ \frac{n\pi l}{2l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right.$$

$$\left. - \left[(l - \frac{l}{2})\left(\frac{-cos\ \frac{n\pi l}{2l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-sin\ \frac{n\pi l}{2l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]\right]$$

$$= \frac{4b}{l^2}\left[\frac{l}{2}\left(\frac{-cos\ \frac{n\pi}{2}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2}\right) - \frac{l}{2}\left(\frac{-cos\ \frac{n\pi}{2}}{\frac{n\pi}{l}}\right) + (-1)\left(\frac{-sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{4b}{l^2}\left[\frac{sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2} + \frac{sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2}\right] = \frac{8bl^2}{l^2n^2\pi^2} sin\ \frac{n\pi}{2}$$

$$A_n = \frac{8b}{n^2\pi^2} sin\ \frac{n\pi}{2}$$

Substituting the value of A_n in (6), we get

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2} sin\ \frac{n\pi}{2}\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\ sin\ \frac{n\pi}{2}\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

5. A rod of length 20 cm has its ends A and B kept at temperature $30^o\ C$ and $90^o\ C$ respectively until steady state conditions prevail. If the temperature at each end is then suddenly reduced to $0^o\ C$ and maintained so, find the temperature distribution at a distance from A at time t.

Solution:

The equation of heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \dots (1)$$

When the steady state conditions prevail the temperature function $u(x,t)$ is independent of time t. Hence (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \left[\because \frac{\partial u}{\partial t} = 0\right]$$

The solution for this equation is

$u(x) = ax + b \dots (2)$

When the steady state condition prevails the boundary conditions are

$a)\ u(0) = 30$

$b)\ u(20) = 90$

Applying boundary condition (a) in (2), we get

$u(0) = 30 \Rightarrow b = 30$

Applying boundary condition (b) and put b=30 in (2), we get

$u(20) = 90 \Rightarrow a20 + 30 = 90 \Rightarrow a = 3$

Substituting the value of a and b in (2), we get

$$u(x) = 3x + 30$$

In the steady state, the temperature function is

$$u(x) = 3x + 30$$

Here the steady state is changed to unsteady state. For this unsteady state the initial temperature is given by

$$u(x, 0) = 3x + 30$$

The boundary conditions for the unsteady state are

$i)\ u(0, t) = 0, t > 0$

$ii)\ u(20, t) = 0, t > 0$

$iii)\ u(x, 0) = 3x + 30, 0 < x < 20$

The solution for the equation (1) is

$$u(x, t) = Ae^{-a^2\alpha^2 t}(B \cos ax + C \sin ax) \dots (3)$$

Applying boundary condition (i) for (3), we get

$$u(0, t) = 0 \Rightarrow Ae^{-a^2\alpha^2 t}B = 0$$

$B = 0$ $\left[Since\ A \neq 0\ because\ if\ A = 0 \Rightarrow u(x, t) = 0\right]$

Applying boundary condition (ii) and the value of B for (3), we get

$$u(20, t) = 0 \Rightarrow Ae^{-a^2\alpha^2 t}C \sin a20 = 0$$

$\sin a20 = 0$ $\left[Since\ A \neq 0, C \neq 0\ because\ if\ C = 0 \Rightarrow u(x,t) = 0\right]$

$$20\ a = n\pi$$

$$a = \frac{n\pi}{20}$$

Substituting the value of B and a in (3), we get

$$u(x,t) = Ae^{-\frac{n^2\pi^2\alpha^2 t}{400}} C \sin\frac{n\pi x}{20} \dots (4)$$

By putting $AC = A_n$ in (4), we get

$$u(x,t) = A_n e^{-\frac{n^2\pi^2\alpha^2 t}{400}} \sin\frac{n\pi x}{20} \dots (5)$$

The most general form of (5) is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha^2 t}{400}} \sin\frac{n\pi x}{20} \quad \dots (6)$$

Applying boundary condition (iii) and the value of B for (6), we get

$$u(x,0) = 3x + 30 \Rightarrow \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{20} = 3x + 30\ \ for\ \ 0 < x < 20$$

The above equation is Fourier Sine series.

$$A_n = \frac{2}{20}\left[\int_0^{20} (3x+30)\ \ sin\ \frac{n\pi x}{20}\, dx\right]$$

$$= \frac{1}{10}\left[(3x+30)\left(\frac{-cos\ \frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - 3\left(\frac{-sin\ \frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[(3(20)+30)\left(\frac{-cos\ \frac{n\pi 20}{20}}{\frac{n\pi}{20}}\right) - 3\left(\frac{-sin\ \frac{n\pi 20}{20}}{\left(\frac{n\pi}{20}\right)^2}\right) - (30)\left(\frac{-cos\ 0}{\frac{n\pi}{20}}\right)\right]$$

$$= \frac{1}{10}\left[90\left(\frac{-cos\ n\pi}{\frac{n\pi}{20}}\right) - (30)\left(\frac{-1}{\frac{n\pi}{20}}\right)\right] = \frac{30(20)}{10n\pi}[-3(-1)^n + 1]$$

$$A_n = \frac{60}{n\pi}[1 - 3(-1)^n]$$

Substituting the value of A_n in (6), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{60}{n\pi}[1-3(-1)^n]e^{-\frac{n^2\pi^2\alpha^2 t}{400}} \sin\frac{n\pi x}{20}$$

$$u(x,t) = \frac{60}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}[1-3(-1)^n]e^{-\frac{n^2\pi^2\alpha^2 t}{400}} \sin\frac{n\pi x}{20}$$

6. A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where

$$v = \begin{cases} \dfrac{cx}{l} & in\ 0 < x < l \\ \dfrac{c}{l}(2l - x) & in\ l < x < 2l \end{cases}$$

x being the distance from an end point. Find the displacement of the string at any time.

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \ldots (1)$$

The boundary conditions are

$i)\ y(0,t) = 0, t > 0$

$ii)\ y(2l,t) = 0, t > 0$

$iii)\ y(x,0) = 0, 0 < x < 2l$

$$iv)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \begin{cases} \dfrac{cx}{l} & in\ 0 < x < l \\ \dfrac{c}{l}(2l - x) & in\ l < x < 2l \end{cases}$$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t) = (At + B)(Cx + D)$

$II)\ y(x,t) = (Ae^{-\alpha at} + Be^{\alpha at})(Ce^{-\alpha x} + De^{\alpha x})$

$III)\ y(x,t) = (A\ cos\ \alpha at + B\ sin\ \alpha at)(C\ cos\ \alpha x + D\ sin\ \alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)C = 0$

$\therefore C = 0$ $\qquad$ $because\ A \cos \alpha at + B \sin \alpha at \neq 0$

$\left[If\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ \Rightarrow y(x,t) = 0\right]$

Now applying the boundary condition (i) and put $C = 0$ for solution III, we get

$y(2l,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)D\ sin\ 2\alpha l = 0$

$sin\ 2\alpha l = 0 \Rightarrow 2\alpha l = n\pi \Rightarrow \alpha = \dfrac{n\pi}{2l}$

$$\left[\begin{matrix} A \cos \alpha at + B \sin \alpha at \neq 0\ and\ D \neq 0 \\ Since\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ and\ \ D = 0\ \Rightarrow\ y(x,t) = 0 \end{matrix}\right]$$

Substituting $\alpha = \frac{n\pi}{2l}$ $\ and\ C = 0$ in solution III, we get

$y(x,t) = \left(A\ cos\ \frac{n\pi at}{2l} + B\ sin\ \frac{n\pi at}{2l}\right) D\ sin\ \frac{n\pi x}{2l}$ ... (2)

By putting $AD = A_n\ and\ BD = B_n\ in\ (2),\ we\ get$

$y(x,t) = \left(A_n\ cos\ \dfrac{n\pi at}{2l} + B_n\ sin\ \dfrac{n\pi at}{2l}\right)\ sin\ \dfrac{n\pi x}{2l}$ ... (3)

The most general form of (3) is

$y(x,t) = \displaystyle\sum_{n=1}^{\infty} \left(A_n\ cos\ \frac{n\pi at}{2l} + B_n\ sin\ \frac{n\pi at}{2l}\right)\ sin\ \frac{n\pi x}{2l}$ ... (4)

Now applying the boundary condition (iii) for (4), we get

$$y(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n\ \ sin\ \frac{n\pi x}{2l} = 0$$

$$\therefore A_n = 0$$

Substituting $A_n = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n\ sin\ \frac{n\pi at}{2l}\ sin\ \frac{n\pi x}{2l} \ldots (5)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \ \frac{n\pi a}{2l} cos \ \frac{n\pi at}{2l} \ sin \ \frac{n\pi x}{2l} \ldots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \begin{cases} \dfrac{cx}{l} & in \ 0 < x < l \\ \dfrac{c}{l}(2l - x) & in \ l < x < 2l \end{cases}$$

$$\sum_{n=1}^{\infty} B_n \ \frac{n\pi a}{2l} \ sin \ \frac{n\pi x}{2l} = \begin{cases} \dfrac{cx}{l} & in \ 0 < x < l \\ \dfrac{c}{l}(2l - x) & in \ l < x < 2l \end{cases}$$

The above equation is Fourier Sine series.

$$B_n \ \frac{n\pi a}{2l} = \frac{2}{2l}\left[\int_0^l \frac{cx}{l} sin \ \frac{n\pi x}{2l} dx + \int_l^{2l} \frac{c}{l}(2l - x) \ sin \ \frac{n\pi x}{2l} dx\right]$$

$$= \frac{c}{l^2}\left[\int_0^l x \ sin \ \frac{n\pi x}{2l} dx + \int_l^{2l} (2l - x) \ sin \ \frac{n\pi x}{2l} dx\right]$$

$$= \frac{c}{l^2}\left[\left[x\left(\frac{-cos \ \frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-sin \ \frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l + \left[(2l - x)\left(\frac{-cos \ \frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-sin \ \frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_l^{2l}\right]$$

$$= \frac{c}{l^2}\left[\left[l\left(\frac{-cos \ \frac{n\pi l}{2l}}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-sin \ \frac{n\pi l}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right] - \left[(2l - l)\left(\frac{-cos \ \frac{n\pi l}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-sin \ \frac{n\pi l}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]\right]$$

$$= \frac{c}{l^2}\left[\left[l\left(\frac{-cos \ \frac{n\pi}{2}}{\frac{n\pi}{2l}}\right) - 1\left(\frac{-sin \ \frac{n\pi}{2}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right] - l\left(\frac{-cos \ \frac{n\pi l}{2l}}{\frac{n\pi}{2l}}\right) + (-1)\left(\frac{-sin \ \frac{n\pi}{2}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]$$

$$= \frac{c}{l^2}\left[\frac{sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{2l}\right)^2} + \frac{sin\ \frac{n\pi}{2}}{\left(\frac{n\pi}{2l}\right)^2}\right] = \frac{8cl^2}{l^2n^2\pi^2} sin\ \frac{n\pi}{2}$$

$$B_n\ \frac{n\pi a}{2l} = \frac{8c}{n^2\pi^2} sin\ \frac{n\pi}{2}$$

$$B_n\ = \frac{16cl}{an^3\pi^3} sin\ \frac{n\pi}{2}$$

Substituting the value of B_n in (5), we get

$$y(x,t) = \frac{16cl}{an^3\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} sin\ \frac{n\pi}{2}\ \ \sin\frac{n\pi x}{2l}\ \sin\frac{na\pi t}{2l}$$

7. Solve the problem of heat conduction in a rod given that

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$(i)\ u\ is\ finite\ as\ t \to \infty$

$(ii)\ \frac{\partial u}{\partial x} = 0\ for\ x = 0\ and\ x = l, t > 0$

$(iii)\ u = lx - x^2, for\ t = 0, 0 \leq x \leq l.$

Solution:

The equation of heat flow is

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \dots (1)$$

By the method of separation of variables we get three possible solutions. They are

$I)\ u(x,t) = A(Bx + C)$

$II)\ u(x,t) = Ae^{a^2\alpha^2 t}(Be^{-ax} + Ce^{ax})$

$III)\ u(x,t) = Ae^{-a^2\alpha^2 t}(B\ cos\ ax + C\ sin\ ax)$

Applying the boundary condition (i) for all the possible solutions we get solution III is suitable because it only satisfies the boundary condition (i)

Differentiating solution III partially with respect to x, we get

$$\frac{\partial u}{\partial x} = Ae^{-a^2\alpha^2 t}(-aB\ sin\ ax + aC\ cos\ ax) \dots (2)$$

Now applying boundary condition (ii) when $x = 0$, we get

$$\frac{\partial u}{\partial x} = 0\ for\ x = 0$$

$$Ae^{-a^2\alpha^2 t} aC\ cos\ ax = 0$$

$$C = 0 \qquad [Since\ A \neq 0]$$

Now applying boundary condition (ii) when $x = l\ and\ put\ C = 0\ in\ (2)$, we get

$$Ae^{-a^2\alpha^2 t}\ aB\ sin\ al = 0$$

$$sin\ al = 0 \Rightarrow al = n\pi$$

$$a = \frac{n\pi}{l}$$

Substituting the values of a and C in solution III, we get

$$u(x,t) = Ae^{-\frac{n^2\pi^2\alpha^2 t}{l^2}}\left(B\ cos\ \frac{n\pi x}{l}\right) \dots (3)$$

By putting $AB = B_n$

$$u(x,t) = B_n e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} cos\ \frac{n\pi x}{l} \dots (4)$$

The most general form of (4) is

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} cos\ \frac{n\pi x}{l} \dots (5)$$

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} cos\ \frac{n\pi x}{l} \dots (6)$$

Applying boundary condition (iii) in (6), we get

$$u(x,0) = lx - x^2, \qquad 0 \le x \le l.$$

$$lx - x^2 = B_0 + \sum_{n=1}^{\infty} B_n cos\ \frac{n\pi x}{l}$$

This is Half range Cosine series where $B_0 = \frac{a_0}{2}$

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)dx$$

$$= \frac{2}{l}\left(l\frac{x^2}{2} - \frac{x^3}{3}\right) = \frac{2}{l}\left(\frac{l^3}{2} - \frac{l^3}{3}\right) = \frac{l^2}{3}$$

$$B_0 = \frac{a_0}{2} = \frac{l^2}{6}$$

$$B_n = \frac{2}{l}\int_0^l (lx - x^2)cos\ \frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{sin\ \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-cos\ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{-sin\ \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-(l - 2l)\left(\frac{-cos\ \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (l)\left(\frac{-1}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{2}{l}\left[l\left(\frac{-cos\ n\pi}{\left(\frac{n\pi x}{l}\right)^2}\right) + (l)\left(\frac{-1}{\left(\frac{n\pi x}{l}\right)^2}\right)\right] = \frac{2l^2}{n^2\pi^2}[-(-1)^n - 1]$$

$$B_n = \begin{cases} -\frac{4l^2}{n^2\pi^2} & if\ n\ is\ even \\ 0 & if\ n\ is\ odd \end{cases}$$

Substituting the values of B_0 and B_n in (6), we get

$$u(x,t) = \frac{l^2}{6} + \sum_{n=2,4,\ldots}^{\infty} -\frac{4l^2}{n^2\pi^2}e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}}cos\ \frac{n\pi x}{l}$$

$$u(x,t) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^2}e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}}cos\ \frac{n\pi x}{l}$$

8. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = kx^2(l - x)$, where $'k'$ is a constant, and then released from rest. Find the displacement of any point 'x' of the string at any time $t > 0$.

Solution:

The displacement $y(x,t)$ from one end is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \dots (1)$$

The boundary conditions are

$i)\ y(0,t) = 0, t > 0$

$ii)\ y(l,t) = 0, t > 0$

$iii)\ \left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \qquad 0 < x < l$

$iv)\ y(x,0) = kx^2(l - x), \qquad 0 < x < l$

By the method of separation of variables we get three possible solutions for (1). They are

$I)\ y(x,t) = (At + B)(Cx + D)$

$II)\ y(x,t) = (Ae^{-\alpha at} + Be^{\alpha at})(Ce^{-\alpha x} + De^{\alpha x})$

$III)\ y(x,t) = (A\ cos\ \alpha at + B\ sin\ \alpha at)(C\ cos\ \alpha x + D\ sin\ \alpha x)$

The solution III is suitable for (1). Since the solution III is periodic in t and it also satisfies all the boundary conditions.

Now applying the boundary condition (i) for solution III, we get

$y(0,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)C = 0$

$\therefore C = 0 \qquad\qquad because\ A\cos\alpha at + B\sin\alpha at \neq 0$

$\left[If\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ \Rightarrow y(x,t) = 0\right]$

Now applying the boundary condition (i) and put $C = 0$ for solution III, we get

$y(l,t) = 0 \Rightarrow (A\ cos\ \alpha at + B\ sin\ \alpha at)D\ sin\ \alpha l = 0$

$sin\ \alpha l = 0 \Rightarrow \alpha l = n\pi \Rightarrow \alpha = \frac{n\pi}{l}$

$$\begin{bmatrix} A\cos\alpha at + B\sin\alpha at \neq 0\ and\ D \neq 0 \\ Since\ A\ cos\ \alpha at + B\ sin\ \alpha at = 0\ and\ \ D = 0\ \Rightarrow\ y(x,t) = 0 \end{bmatrix}$$

Substituting $\alpha = \frac{n\pi}{l}\ \ and\ C = 0$ in solution III, we get

$$y(x,t) = \left(A\ cos\ \frac{n\pi at}{l} + B\ sin\ \frac{n\pi at}{l}\right) D\ sin\ \frac{n\pi x}{l} \dots (2)$$

By putting $AD = A_n\ and\ BD = B_n\ in\ (2),\ we\ get$

$$y(x,t) = \left(A_n \cos \frac{n\pi at}{l} + B_n \sin \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \ldots (3)$$

The most general form of (3) is

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi at}{l} + B_n \sin \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \ldots (4)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + B_n \frac{n\pi a}{l} \cos \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \ldots (5)$$

Now applying the boundary condition (iii) for (5), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0 \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0$$

$\therefore B_n = 0$

Substituting $B_n = 0$ in (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi at}{l} \sin \frac{n\pi x}{l} \ldots (6)$$

Now applying the boundary condition (iv) for (6), we get

$$y(x,0) = k(lx^2 - x^3) \Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = k(lx^2 - x^3)$$

The above equation is Fourier Sine series.

$$A_n = \frac{2}{l} \int_0^l k(lx^2 - x^3) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l} \left[(lx^2 - x^3) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (2lx - 3x^2) \left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right) + (2l - 6x) \left(\frac{\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3} \right) - (-6) \left(\frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^4} \right) \right]_0^l$$

$$= \frac{2k}{l}\left[(l^3 - l^3)\left(\frac{-cos\ \frac{n\pi l}{l}}{\frac{n\pi}{l}}\right) - (2l^2 - 3l^2)\left(\frac{-sin\ \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (2l - 6l)\left(\frac{cos\ \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^3}\right) - (-6)\left(\frac{sin\ \frac{n\pi l}{l}}{\left(\frac{n\pi}{l}\right)^4}\right)\right] - \frac{2k}{l}\left[0 - 0 + (2l - 0)\left(\frac{cos\ 0}{\left(\frac{n\pi}{l}\right)^3}\right) - (-6)\left(\frac{sin\ 0}{\left(\frac{n\pi}{l}\right)^4}\right)\right]$$

$$= \frac{2k}{l}\left[(-4l)\left(\frac{cos\ n\pi}{\left(\frac{n\pi}{l}\right)^3}\right) - (2l)\left(\frac{1}{\left(\frac{n\pi}{l}\right)^3}\right)\right]$$

$$A_n = -\frac{4kl^3}{n^3\pi^3}[2(-1)^n + 1]$$

Substituting the value of A_n in (6), we get

$$y(x,t) = \sum_{n=1}^{\infty} -\frac{4kl^3}{n^3\pi^3}[2(-1)^n + 1]\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

$$y(x,t) = -\frac{4kl^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}[2(-1)^n + 1]\ cos\ \frac{n\pi at}{l}\ sin\ \frac{n\pi x}{l}$$

9. The ends A and B of a rod 10 cm long have the temperature $20^o\ C$ and $40^o\ C$ until steady state prevails. The temperature at A is suddenly raised to $50^o\ C$ and at the same time that at B is lowered to$10^o\ C$. Find the temperature at the midpoint of the rod remains for all time, regardless of the material of the rod.

Solution:

The equation of heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \dots (1)$$

When the steady state conditions prevail the temperature function $u(x,t)$ is independent of time t. Hence (1) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \left[\because \frac{\partial u}{\partial t} = 0\right]$$

The solution for this equation is

$u(x) = ax + b \dots (2)$

When the steady state condition prevails the boundary conditions are

$a)\ u(0) = 20$

$b)\ u(10) = 40$

Applying boundary condition (a) in (2), we get

$u(0) = 20 \Rightarrow b = 20$

Applying boundary condition (b) and put b=30 in (2), we get

$u(10) = 40 \Rightarrow a10 + 20 = 40 \Rightarrow a = 2$

Substituting the value of a and b in (2), we get

$$u(x) = 2x + 20$$

In the steady state, the temperature function is

$$u(x) = 2x + 20$$

Here the steady state is changed to unsteady state. For this unsteady state the initial temperature is given by

$$u(x, 0) = 2x + 20$$

The boundary conditions for the unsteady state are

$i)\ u(0, t) = 50, t > 0$

$ii)\ u(10, t) = 10, t > 0$

$iii)\ u(x, 0) = 2x + 20, 0 < x < 10$

The boundary condition (i) and (ii) are non zero, the solution for (i) is not applicable until the boundary condition (i) and (ii) becomes zero.

For,

Let $u(x, t) = u_s(x) + u_t(x, t) \ldots (3)$

where $u_s(x)$ is the solution of (1) and is a function of x alone and satisfying conditions $u_s(0) = 50\ and\ u_s(10) = 10\ and\ u_t(x, t)$ is a transient solution satisfying (3) which decreases as t increases.

$u_s(x) = cx + d \ldots (4)$

$u_s(0) = d = 50$

$u_s(10) = 10c + d = 10 \ldots (5)$

Substituting the value of $d = 50$ in (5) we get

$10c + 50 = 10 \Rightarrow c = -4$

Substituting the value of c and d in (4) we get

$u_s(x) = -4x + 50 \dots (6)$

$u_t(x,t) = u(x,t) - u_s(x) \dots (7) \quad from\ (3)$

Now applying boundary conditions (i), (ii), (iii) and $u_s(0) = 50, u_s(10) = 10$ and (6) in (7), we get the boundary conditions for $u_t(x,t)$ which is the transient solution of (1).

$iv)\ u_t(0,t) = u(0,t) - u_s(0) = 50 - 50 = 0$

$v)\ u_t(10,t) = u(10,t) - u_s(10) = 10 - 10 = 0$

$vi)\ u_t(x,0) = u(x,0) - u_s(x) = 2x + 20 - (-4x + 50) = 6x - 30, 0 < x < 10.$

The solution for the equation (1) for $u_t(x,t)$ is

$$u_t(x,t) = Ae^{-a^2\alpha^2 t}(B\cos ax + C\sin ax) \dots (8)$$

Applying boundary condition (iv) for (8), we get

$$u_t(0,t) = 0 \Rightarrow\ Ae^{-a^2\alpha^2 t}B = 0$$

$B = 0 \qquad \left[Since\ A \neq 0\ because\ if\ A = 0 \Rightarrow u(x,t) = 0\right]$

Applying boundary condition (v) and the value of B for (8), we get

$$u_t(10,t) = 0 \Rightarrow Ae^{-a^2\alpha^2 t}C\sin a20 = 0$$

$\sin a10 = 0 \qquad \left[Since\ A \neq 0, C \neq 0\ because\ if\ C = 0 \Rightarrow u(x,t) = 0\right]$

$$10\ a = n\pi$$

$$a = \frac{n\pi}{10}$$

Substituting the value of B and a in (8), we get

$$u_t(x,t) = Ae^{-\frac{n^2\pi^2\alpha^2 t}{100}}C\sin\frac{n\pi x}{10} \dots (9)$$

By putting $AC = A_n$ in (9), we get

$$u_t(x,t) = A_n e^{-\frac{n^2\pi^2\alpha^2 t}{100}}\sin\frac{n\pi x}{10} \dots (10)$$

The most general form of (10) is

$$u_t(x,t)=\sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha^2 t}{100}} \sin\frac{n\pi x}{10} \quad \dots(11)$$

Applying boundary condition (vi) and the value of B for (11), we get

$$u_t(x,0)=6x-30 \Rightarrow \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10} = 6x-30 \ \ for\ \ 0<x<10$$

The above equation is Fourier Sine series.

$$A_n=\frac{2}{10}\left[\int_0^{20}(6x-30)\ \ sin\ \frac{n\pi x}{10}dx\right]$$

$$=\frac{1}{5}\left[(6x-30)\left(\frac{-cos\ \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-6\left(\frac{-sin\ \frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$=\frac{1}{5}\left[(6(10)-30)\left(\frac{-cos\ \frac{n\pi 10}{10}}{\frac{n\pi}{10}}\right)-6\left(\frac{-sin\ \frac{n\pi 10}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)-(-30)\left(\frac{-cos\ 0}{\frac{n\pi}{10}}\right)\right]$$

$$=\frac{1}{5}\left[30\left(\frac{-cos\ n\pi}{\frac{n\pi}{10}}\right)-(-30)\left(\frac{-1}{\frac{n\pi}{10}}\right)\right]=\frac{30(10)}{5n\pi}[-(-1)^n-1]$$

$$A_n=-\frac{60}{n\pi}[1+(-1)^n]=\begin{cases}-\dfrac{120}{n\pi} & if\ n\ is\ even\\ 0 & if\ n\ is\ odd\end{cases}$$

Substituting the value of A_n in (11), we get

$$u_t(x,t)=\sum_{n=2,4,\dots}^{\infty} -\frac{120}{n\pi} e^{-\frac{n^2\pi^2\alpha^2 t}{100}} \sin\frac{n\pi x}{10}$$

$$u_t(x,t)=-\frac{120}{\pi}\sum_{n=2,4,\dots}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2\alpha^2 t}{100}} \sin\frac{n\pi x}{10} \dots(12)$$

Substituting the value of $u_t(x,t)\ and\ u_s(x)$ in (3), we get

$$u(x,t)=-4x+50-\frac{120}{\pi}\sum_{n=2,4,\dots}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2\alpha^2 t}{100}} \sin\frac{n\pi x}{10}$$

10. The boundary value problem governing the steady state temperature distribution in a flat, thin, square plate is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < a, 0 < y < a.$$

$$u(x, 0) = 0, u(x, a) = 4\, sin^3\left(\frac{\pi x}{a}\right), 0 < x < a, u(0, y) = 0, u(a, y) = 0, 0 < y < a$$

Find the steady state temperature distribution in the plate.

Solution:

The two dimensional heat flow is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ldots (1)$$

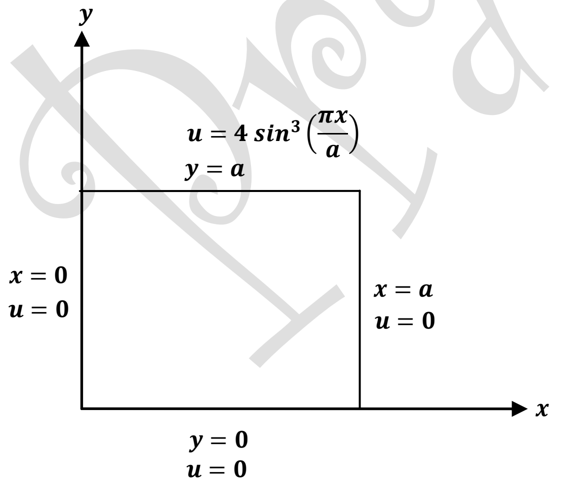
The boundary conditions are

$i)\ u(0, y) = 0, 0 < y < a$

$ii)\ u(a, y) = 0, 0 < y < a$

$iii)\ u(x, 0) = 0, 0 < x < a$

$iv)\ u(x, a) = 4\, sin^3\left(\frac{\pi x}{a}\right), 0 < x < a$



The possible solutions are

$I)\ u(x, y) = (Ax + B)(Cy + D)$

$II)\ u(x, y) = (A\cos \lambda x + B\sin \lambda x)(Ce^{-\lambda y} + De^{\lambda y})$

$III)\ u(x, y) = (Ae^{-\lambda x} + Be^{\lambda x})(C\cos \lambda y + D\sin \lambda y)$

The suitable solution for (1) which satisfies all the given boundary conditions is

$$u(x,y) = (A\cos \lambda x + B\sin \lambda x)\left(Ce^{-\lambda y} + De^{\lambda y}\right) \dots (2)$$

Applying boundary condition (i) in (2), we get

$$u(0,y) = (A\cos \lambda 0 + B\sin \lambda 0)\left(Ce^{-\lambda y} + De^{\lambda y}\right) = 0$$

$$A\left(Ce^{-\lambda y} + De^{\lambda y}\right) = 0 \Rightarrow A = 0\left[\because \left(Ce^{-\lambda y} + De^{\lambda y}\right) \neq 0\right]$$

Applying boundary condition (ii) and substituting $A = 0$ in (2), we get

$$u(a,y) = B\sin \lambda a\left(Ce^{-\lambda y} + De^{\lambda y}\right) = 0$$

$$\sin \lambda a = 0\left[\because \left(Ce^{-\lambda y} + De^{\lambda y}\right) \neq 0, B \neq 0\right]$$

$$\lambda a = n\pi \Rightarrow \lambda = \frac{n\pi}{a}$$

Substituting $A = 0$ *and* $\lambda = \frac{n\pi}{a}$ *in* (2), *we get*

$$u(x,y) = B\sin \frac{n\pi x}{a}\left(Ce^{-\frac{n\pi y}{a}} + De^{\frac{n\pi y}{a}}\right) \dots (3)$$

Applying boundary condition (iii) in (3), we get

$$u(x,0) = B\sin \frac{n\pi x}{a}\left(Ce^{-\frac{n\pi 0}{a}} + De^{\frac{n\pi 0}{a}}\right) = 0$$

$$B\sin \frac{n\pi x}{a}(C + D) = 0 \Rightarrow C + D = 0\left[\because \sin \frac{n\pi x}{a} \neq 0, B \neq 0\right] \Rightarrow C = -D$$

Substituting $C = -D$ *in* (3), *we get*

$$u(x,y) = B\sin \frac{n\pi x}{a}\left(-De^{-\frac{n\pi y}{a}} + De^{\frac{n\pi y}{a}}\right) = BD\sin \frac{n\pi x}{a}\left(-e^{-\frac{n\pi y}{a}} + e^{\frac{n\pi y}{a}}\right)$$

$$u(x,y) = 2BD\sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \dots (4)$$

Put $B_n = BD$ *in* (4), *we get*

$$u(x,y) = 2B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \dots (5)$$

The most general form of (5) is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \dots (6)$$

Applying boundary condition (iv) in (6), we get

$$u(x,a) = \sum_{n=1}^{\infty} B_n sin\,\frac{n\pi x}{a} sinh\,\frac{n\pi a}{a} = 4\,sin^3\left(\frac{\pi x}{a}\right)$$

$$\sum_{n=1}^{\infty} B_n sinh\,n\pi\,sin\,\frac{n\pi x}{a} = 4\,sin^3\left(\frac{\pi x}{a}\right), 0 < x < a$$

$$B_1 sinh\,\pi sin\,\frac{\pi x}{a} + B_2 sinh\,2\pi sin\,\frac{2\pi x}{a} + B_3 sinh\,3\pi\,sin\,\frac{3\pi x}{a} + \cdots = 3sin\,\frac{\pi x}{a} - sin\,\frac{3\pi x}{a}$$

Equating coefficients of $sin\,\frac{n\pi x}{a}, n = 1,2, \ldots on\ both\ sides\ of\ above\ equation$

$$B_1 sinh\,\pi = 3 \Rightarrow B_1 = \frac{3}{sinh\,\pi}$$

$$B_3 sinh\,3\pi = -1 \Rightarrow B_3 = -\frac{1}{sinh\,3\pi}$$

$$B_n = 0, n \neq 1,3$$

Substituting the values of $B_1, B_3, B_n\ in\ (6), we\ get$

$$u(x,y) = \frac{3}{sinh\,\pi} sin\,\frac{\pi x}{a} sinh\,\frac{\pi y}{a} - \frac{1}{sinh\,3\pi}\,sin\,\frac{3\pi x}{a} sinh\,\frac{3\pi y}{a}$$

UNIT-V

Z-transform

Z-transform:

The two sided Z-transform $F(z)$ for a sequence $f(x)$ is defined as

$$F(z) = \sum_{n=-\infty}^{\infty} f(x) z^{-n}$$

The one sided Z-transform $F(z)$ for a sequence $f(x)$ is defined as

$$F(z) = \sum_{n=0}^{\infty} f(x) z^{-n}$$

Z-transform of $\mathbf{1}$ $\boldsymbol{or}$ $\boldsymbol{u(n)}$:

$$Z[1] = \sum_{n=0}^{\infty} 1.z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \qquad if\ \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}\ if\ |z| > 1.$$

Z-transform of $\boldsymbol{a^n f(x)}$:

$$Z[a^n f(x)] = \sum_{n=0}^{\infty} a^n f(x) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(x) \left(\frac{z}{a}\right)^{-n}$$

$$= F\left[\frac{z}{a}\right] \qquad \left[Since\ F(z) = \sum_{n=0}^{\infty} f(x) z^{-n}\right]$$

Z-transform of $\boldsymbol{\frac{1}{n!}}$:

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} (z^{-1})^n$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \frac{(z^{-1})^3}{3!} + \cdots = e^{z^{-1}}$$

$$Z\left[\frac{1}{n!}\right] = e^{1/z}$$

Z-transform of $\boldsymbol{\frac{1}{n}, n \geq 1}$:

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right) \; if \; \left|\frac{1}{z}\right| < 1$$

$$= \log\left(\frac{z}{z-1}\right) \; if \; |z| > 1$$

First shifting theorem in Z-transform:

Statement:

If $\boldsymbol{Z[f(t)] = F(z)}$ ***then prove that*** $\boldsymbol{Z[e^{-at} f(t)] = F(ze^{aT})}$

Proof:

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(ze^{aT})^{-n}$$

$$= F(ze^{aT}) \qquad \left[Since\ F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}\right]$$

Z-transform of $\boldsymbol{nf(t)}$ ***if*** $\boldsymbol{Z[f(t)] = F(z)}$:

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

Differentiate both sides w.r.t. z^{-1}.

$$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} -nf(nT) z^{-n-1}$$

$$z\frac{dF(z)}{dz} = -\sum_{n=0}^{\infty} nf(nT) z^{-n} = -Z[nf(t)]$$

$$Z[nf(t)] = -z\frac{dF(z)}{dz}$$

Second shifting theorem in Z-transform:

Statement:

$Let\ F(z) = Z\{f(n)\}$ then

$$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - z^{-1}f(1) - z^{-2}f(2) - \cdots - f(k-1)z^{-(k-1)}\right]$$

Proof:

$Z\{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)\, z^{-n}$...(1)

Multiplying and dividing by z^k in R.H.S of (1), we get

$$= z^k \sum_{n=0}^{\infty} f(n+k)\, z^{-(n+k)}$$

$$= z^k\left(f(k)z^{-(k)} + f(k+1)z^{-(k+1)} + f(k+2)z^{-(k+2)} + \cdots\right)$$

$$= z^k\left(\sum_{n=0}^{\infty} f(n)\, z^{-n} - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right)$$

$$\left[Since \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + f(1)z^{-1} \ldots + f(k-1)z^{-(k-1)} + f(k)z^{-(k)} + \cdots\right]$$

$$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - z^{-1}f(1) - z^{-2}f(2) - \cdots - f(k-1)z^{-(k-1)}\right]$$

Initial and final value theorem in Z transform:

Initial value theorem:

Statement:

If $Z\{f(t)\} = F(z)\ then$

$$\lim_{z\to\infty} F(z) = f(0) = \lim_{t\to 0} f(t)$$

Proof:

$$F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$F(z) = f(0) + z^{-1}f(1T) + z^{-2}f(2T) + z^{-3}f(3T) + \cdots$$

Taking limit $z \to \infty$ on both sides, we get

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + z^{-1}f(1T) + z^{-2}f(2T) + z^{-3}f(3T) + \cdots\right]$$

$$\lim_{z\to\infty} F(z) = f(0)$$

Final value theorem:

Statement:

If $Z\{f(t)\} = F(z)\ then$

$$\lim_{z\to 1}(z-1)F(z) = f(\infty) = \lim_{t\to\infty} f(t)$$

Proof:

$$Z\{f(t+T) - f(t)\} = \sum_{n=0}^{\infty}\big(f(nT+T) - f(nT)\big)\,z^{-n}$$

$$Z\{f(t+T)\} - Z\{f(t)\} = \sum_{n=0}^{\infty}\big(f((n+1)T) - f(nT)\big)\,z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty}\big(f((n+1)T) - f(nT)\big)\,z^{-n}$$

$$[Since\ Z\{f(t+T)\} = zF(z) - zf(0)]$$

Taking limit $z \to 1$ on both sides, we get

$$\lim_{z\to 1}\ [(z-1)F(z) - zf(0)] = \lim_{z\to 1}\left[\sum_{n=0}^{\infty}\big(f((n+1)T) - f(nT)\big)\,z^{-n}\right]$$

$$\lim_{z\to 1}\ [(z-1)F(z) - zf(0)] = \sum_{n=0}^{\infty}\lim_{z\to 1}\big(f((n+1)T) - f(nT)\big)\,z^{-n}$$

$$\lim_{z\to 1}\ (z-1)F(z) - f(0) = \sum_{n=0}^{\infty}\big(f((n+1)T) - f(nT)\big)$$

$$\lim_{z\to 1}\ (z-1)F(z) - f(0) = f(T) - f(0) + f(2T) - f(T) + f(3T) - f(2T) + \cdots$$

$$\lim_{z\to 1}\ (z-1)F(z) - f(0) = f(\infty) - f(0)$$

$$\lim_{z\to 1}(z-1)F(z) = f(\infty) = \lim_{t\to\infty} f(t)$$

1. Prove that

$$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$$

Solution:

$$Z\left[\frac{1}{n+1}\right]=\sum_{n=0}^{\infty}\frac{1}{(n+1)}z^{-n}$$

$$=\left(1+\frac{z^{-1}}{2}+\frac{z^{-2}}{3}+\frac{z^{-3}}{4}+\cdots\right)$$

$$=z\left(z^{-1}+\frac{z^{-2}}{2}+\frac{z^{-3}}{3}+\frac{z^{-4}}{4}+\cdots\right)$$

$$=-z\,log(1-z^{-1}),\left|\frac{1}{z}\right|<1$$

$$\left[Since\ x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots=-\log(1-x),|x|<1\right]$$

$$=-z\,log\left(1-\frac{1}{z}\right)=-z\,log\left(\frac{z-1}{z}\right),\left|\frac{1}{z}\right|<1$$

$$Z\left[\frac{1}{n+1}\right]=z\log\left(\frac{z}{z-1}\right),|z|>1$$

$$\left[Since\ -\log\left(\frac{a}{b}\right)=\log\left(\frac{b}{a}\right)\right]$$

2. Find the Z-transform of

$$f(n)=\frac{2n+3}{(n+1)(n+2)}$$

Solution:

$$\frac{2n+3}{(n+1)(n+2)}=\frac{A}{(n+1)}+\frac{B}{(n+2)}\ \ldots(1)$$

$$2n+3=A(n+2)+B(n+1)\ \ldots(2)$$

Put $n=-2\ in\ (2), we\ get$

$$2(-2)+3=A(-2+2)+B(-2+1)$$

$$-1=-B\Rightarrow B=1$$

Put $n=-1\ in\ (2), we\ get$

$$2(-1)+3=A(-1+2)+B(-1+1)$$

$$A=1$$

Substituting $A=1\ and\ B=1\ in\ (1), we\ get$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{(n+1)} + \frac{1}{(n+2)}$$

$$Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = Z\left\{\frac{1}{(n+1)} + \frac{1}{(n+2)}\right\}$$

$$= Z\left\{\frac{1}{(n+1)}\right\} + Z\left\{\frac{1}{(n+2)}\right\} \qquad [By\ linear\ property]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{(n+2)} z^{-n}$$

$$= \left(1 + \frac{z^{-1}}{2} + \frac{z^{-2}}{3} + \frac{z^{-3}}{4} + \cdots\right) + \left(\frac{1}{2} + \frac{z^{-1}}{3} + \frac{z^{-2}}{4} + \frac{z^{-3}}{5} + \cdots\right)$$

$$= z\left(z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \cdots\right) + z^2\left(\frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \frac{z^{-5}}{5} + \cdots\right)$$

$$= -z\, log(1 - z^{-1}) + z^2(-log(1 - z^{-1}) - z^{-1}), \left|\frac{1}{z}\right| < 1$$

$$\left[Since\ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = -\log(1-x)\,, |x| < 1\right]$$

$$= -z\, log\left(1 - \frac{1}{z}\right) + z^2\left(-log\left(1 - \frac{1}{z}\right) - z^{-1}\right), |z| > 1$$

$$= -z\, log\left(\frac{z-1}{z}\right) - z^2\left(log\left(\frac{z-1}{z}\right) + z^{-1}\right), |z| > 1$$

$$Z\left\{\frac{2n+3}{(n+1)(n+2)}\right\} = -z\left[1 + (1+z)\, log\left(\frac{z-1}{z}\right)\right], |z| > 1$$

3. Find $Z\{n\ \ (n-1)a^n\ u(n)\}$

Solution:

$$Z[n\ \ (n-1)a^n\ u(n)] = Z[\ n^2 a^n\ u(n) - na^n\ u(n)]$$

$$= Z\{\ n^2 a^n\ u(n)\} - Z\{\ na^n\ u(n)\} \qquad [By\ linear\ property]$$

$$= [Z\{\ n^2\}]_{z\to\frac{z}{a}} - [Z\{\ n\ \}]_{z\to\frac{z}{a}} \qquad \left[Since\ Z\{\ a^n\ f(n)\} = [Z\{\ f(n)\ \}]_{z\to\frac{z}{a}}\right]$$

$$= \left[\frac{z(z+1)}{(z-1)^3}\right]_{z\to\frac{z}{a}} - \left[\frac{z}{(z-1)^2}\right]_{z\to\frac{z}{a}}$$

$$\left(Since\ Z\{\ n^2\} = -z\frac{d}{dz}Z\{\ n\ \}\right.$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = -z\left[\frac{(z-1)^2.1 - 2z(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1) - 2z}{(z-1)^3}\right] = -z\left[\frac{-1-z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}\Big)$$

$$= \left[\frac{\frac{z}{a}\left(\frac{z}{a}+1\right)}{\left(\frac{z}{a}-1\right)^3}\right] - \left[\frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}\right]$$

$$= \frac{az(z+a)}{(z-a)^3} - \frac{az}{(z-a)^2} = \frac{az^2 + a^2z - az(z-a)}{(z-a)^3}$$

$$= \frac{az^2 + a^2z - az^2 + a^2z}{(z-a)^3}$$

$$= \frac{2a^2z}{(z-a)^3}$$

4. Using convolution theorem find

$$Z^{-1}\left[\frac{z^2}{(z+2)^2}\right]$$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z+2)^2}\right] = Z^{-1}\left[\frac{z}{(z+2)}\frac{z}{(z+2)}\right]$$

$$Let\ F(z) = \frac{z}{(z+2)}\ and\ G(z) = \frac{z}{(z+2)}$$

$$f(n) = Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{(z+2)}\right] = (-2)^n$$

$$g(n) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{z}{(z+2)}\right] = (-2)^n$$

$$Z^{-1}[F(z).G(z)] = \sum_{k=0}^{n} f(n-k)g(k)$$

$$= \sum_{k=0}^{n} (-2)^{n-k}(-2)^k = \sum_{k=0}^{n} (-2)^n$$

$$= (-2)^n \sum_{k=0}^{n} 1 = (-2)^n[1 + 1 + 1 + \cdots (n+1)times]$$

$$Z^{-1}\left[\frac{z^2}{(z+2)^2}\right] = (-2)^n(n+1)$$

5. By the method of partial fraction find

$$Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$$

Solution:

$$Let\ F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$z = A(z^2+4) + (Bz+C)(z+2)\ldots(1)$$

Put $z = -2$ in (1),we get

$$-2 = A(4+4) \Rightarrow A = -\frac{1}{4}$$

Equating coefficients of z^2 in (1) on both sides

$$A + B = 0 \Rightarrow B = -A \Rightarrow B = \frac{1}{4}$$

Put $z = 0$ in (1),we get

$$4A + 2C = 0 \Rightarrow 2C = -4A \Rightarrow C = \frac{1}{2}$$

Substituting the values of A,B and C in (i), we get

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{-\frac{1}{4}}{z+2} + \frac{\frac{1}{4}z+\frac{1}{2}}{z^2+4}$$

$$F(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}$$

$$Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right] = Z^{-1}\left[-\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}\right]$$

$$= -\frac{1}{4}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4}Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{z^2+4}\right]$$

$$= -\frac{1}{4}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4}Z^{-1}\left[\frac{z^2}{z^2+2^2}\right] + \frac{1}{4}Z^{-1}\left[\frac{2z}{z^2+2^2}\right]$$

$$= -\frac{1}{4}(-2)^n + \frac{1}{4}(2)^n \cos\frac{n\pi}{2} + \frac{1}{4}(2)^n \sin\frac{n\pi}{2}$$

$$\left[Since\ Z^{-1}\left[\frac{z^2}{z^2+a^2}\right] = a^n \cos\frac{n\pi}{2} \quad and\ Z^{-1}\left[\frac{az}{z^2+a^2}\right] = a^n \sin\frac{n\pi}{2}\right]$$

6. By the method of residues find

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$$

Solution:

$$Let\ F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$To\ find\ poles$:

$$(z-1)^2(z-2) = 0 \Rightarrow z = 1, 1, 2$$

The poles are $z = 1$ which is of order two and $z = 2$ which is of order one

$$z^{n-1}F(z) = \frac{z^{n-1}z^3}{(z-1)^2(z-2)} = \frac{z^{n+2}}{(z-1)^2(z-2)}$$

Residue at the pole $z = 1$ of order two R_1

$$= \lim_{z\to 1}\frac{d}{dz}(z-1)^2 z^{n-1}F(z)$$

$$= \lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{z^{n+2}}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 1}\frac{d}{dz}\frac{z^{n+2}}{(z-2)} = \lim_{z\to 1}\left(\frac{(z-2)(n+2)z^{n+1} - z^{n+2}}{(z-2)^2}\right)$$

$$= \frac{(1-2)(n+2)1^{n+1} - 1^{n+2}}{(1-2)^2} = -(n+2) - 1 = -n - 3$$

Residue at the pole $z = 2$ of order one R_2

$$= \lim_{z\to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z^{n+2}}{(z-1)^2(z-2)}$$

$$= \lim_{z\to 2}\frac{z^{n+2}}{(z-1)^2} = 2^{n+2}$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = sum\ of\ residues$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = -n-3+\ 2^{n+2}$$

7. Using convolution theorem find

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{(z-1)}\frac{z}{(z-3)}\right]$$

$$Let\ F(z) = \frac{z}{z-1}\ and\ G(z) = \frac{z}{z-3}$$

$$f(n) = Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z-1}\right] = 1\ or\ u(n)$$

$$g(n) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{z}{z-3}\right] = 3^n$$

$$Z^{-1}[F(z).G(z)] = \sum_{k=0}^{n} f(n-k)g(k)$$

$$= \sum_{k=0}^{n} 1.3^k = 1+3+3^2+3^3+\cdots+3^n$$

$$= \frac{3^{n+1}-1}{3-1} \qquad \left[Since\ 1+x+x^2+x^3+\cdots+x^n = \frac{x^{n+1}-1}{x-1}\ or\ \frac{1-x^{n+1}}{1-x}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1}-1}{2}$$

8. Find the inverse Z-transform of

$$\frac{z(z+1)}{(z-1)^3}$$

Solution:

$$Let\ F(z) = \frac{z(z+1)}{(z-1)^3}$$

$$z^{n-1}F(z) = \frac{z^n(z+1)}{(z-1)^3}$$

To find poles:

$$(z-1)^3 = 0 \Rightarrow z = 1$$

The pole is $z = 1$which is of order three

Residue at $z = 1$is of order three R_1

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3 z^{n-1}F(z)$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3\frac{z^n(z+1)}{(z-1)^3}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}z^n(z+1) = \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{n+1}+z^n\right) = \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left((n+1)z^n + nz^{n-1}\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\left(n(n+1)z^{n-1} + n(n-1)z^{n-2}\right) = \frac{1}{2}\left(n(n+1)1^{n-1} + n(n-1)1^{n-2}\right)$$

$$= \frac{1}{2}(n(n+1) + n(n-1)) = n^2$$

$$Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = n^2$$

9. Find the inverse Z-transform of

$$\frac{z(z+2)}{z^2+2z+4}$$

Solution:

$$Let\ F(z) = \frac{z(z+2)}{z^2+2z+4}$$

$$z^{n-1}F(z) = \frac{z^n(z+2)}{z^2+2z+4}$$

The poles are $z = -1 + i\sqrt{3}\,, -1 - i\sqrt{3}$ are of order one

Residue at the pole $z = -1 + i\sqrt{3}\ of\ order\ one\ R_1$

$$= \lim_{z \to -1+i\sqrt{3}} (z-(-1+i\sqrt{3}))z^{n-1}F(z)$$

$$= \lim_{z \to -1+i\sqrt{3}} (z-(-1+i\sqrt{3})) \frac{z^n(z+2)}{(z-(-1+i\sqrt{3}))(z-(-1-i\sqrt{3}))}$$

$$= \lim_{z \to -1+i\sqrt{3}} \frac{z^n(z+2)}{(z-(-1-i\sqrt{3}))} = \frac{(-1+i\sqrt{3})^n(-1+i\sqrt{3}+2)}{(-1+i\sqrt{3}-(-1-i\sqrt{3}))}$$

$$= \frac{(-1+i\sqrt{3})^n(i\sqrt{3}+1)}{2i\sqrt{3}} = -i\frac{(-1+i\sqrt{3})^n(i\sqrt{3}+1)}{2\sqrt{3}}$$

$$= \frac{\left(-1+i\sqrt{3}\right)^n(\sqrt{3}-i)}{2\sqrt{3}}$$

Residue at the pole $z = -1 - i\sqrt{3}\ of\ order\ one\ R_2$

$$= \lim_{z \to -1-i\sqrt{3}} (z-(-1-i\sqrt{3}))z^{n-1}F(z)$$

$$= \lim_{z \to -1-i\sqrt{3}} (z-(-1-i\sqrt{3})) \frac{z^n(z+2)}{(z-(-1+i\sqrt{3}))(z-(-1-i\sqrt{3}))}$$

$$= \lim_{z \to -1-i\sqrt{3}} \frac{z^n(z+2)}{(z-(-1+i\sqrt{3}))} = \frac{\left(-1-i\sqrt{3}\right)^n(-1-i\sqrt{3}+2)}{(-1-i\sqrt{3}-(-1+i\sqrt{3}))}$$

$$= \frac{\left(-1-i\sqrt{3}\right)^n(-i\sqrt{3}+1)}{-2i\sqrt{3}} = i\frac{\left(-1-i\sqrt{3}\right)^n(-i\sqrt{3}+1)}{2\sqrt{3}}$$

$$= \frac{\left(-1-i\sqrt{3}\right)^n(\sqrt{3}+i)}{2\sqrt{3}}$$

$$Z^{-1}\left[\frac{z(z+2)}{z^2+2z+4}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{z(z+2)}{z^2+2z+4}\right] = \frac{\left(-1+i\sqrt{3}\right)^n(\sqrt{3}-i)}{2\sqrt{3}} + \frac{\left(-1-i\sqrt{3}\right)^n(\sqrt{3}+i)}{2\sqrt{3}}$$

10. Find the inverse Z-transform of

$$\frac{z^2-3z}{(z+2)(z-5)}$$

Solution:

$$Let\ F(z) = \frac{z^2-3z}{(z+2)(z-5)}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z^2-3z}{(z+2)(z-5)} = \frac{z^n(z-3)}{(z+2)(z-5)}$$

$To\ find\ poles$:

$$(z+2)(z-5) = 0 \Rightarrow z = -2, 5$$

The poles are $z = -2, 5$

Residue at the pole $z = -2$ of order one R_1

$$= \lim_{z\to -2}(z+2)z^{n-1}F(z)$$

$$= \lim_{z\to -2}(z+2)\frac{z^n(z-3)}{(z+2)(z-5)}$$

$$= \lim_{z\to -2}\frac{z^n(z-3)}{(z-5)} = \frac{(-2)^n(-2-3)}{(-2-5)} = \frac{5}{7}(-2)^n$$

Residue at the pole $z = 5$ of order one R_2

$$= \lim_{z\to 5}(z-5)z^{n-1}F(z)$$

$$= \lim_{z\to 5}(z-5)\frac{z^n(z-3)}{(z+2)(z-5)}$$

$$= \lim_{z\to 5}\frac{z^n(z-3)}{(z+2)} = \frac{5^n(5-3)}{(5+2)} = \frac{2}{7}5^n$$

$$Z^{-1}\left[\frac{z^2-3z}{(z+2)(z-5)}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{z^2-3z}{(z+2)(z-5)}\right] = \frac{5}{7}(-2)^n + \frac{2}{7}5^n$$

11. Find the inverse Z-transform of

$$\frac{z(z^2-1)}{(z^2+1)^2}$$

Solution:

$$Let\ F(z) = \frac{z(z^2-1)}{(z^2+1)^2}$$

$$z^{n-1}F(z) = \frac{z^n(z^2-1)}{(z^2+1)^2}$$

To find poles:

$$(z^2+1)^2 = 0 \Rightarrow z = -i, i$$

The poles are $z = -i, i$ of order two

$$z^{n-1}F(z) = \frac{z^n(z^2-1)}{(z+i)^2(z-i)^2}$$

Residue at the pole $z = -i$ of order two R_1

$$= \lim_{z\to -i}\frac{d}{dz}(z+i)^2 z^{n-1}F(z)$$

$$= \lim_{z\to -i}\frac{d}{dz}(z+i)^2\frac{z^n(z^2-1)}{(z+i)^2(z-i)^2}$$

$$= \lim_{z\to -i}\frac{d}{dz}\frac{z^n(z^2-1)}{(z-i)^2} = \lim_{z\to -i}\frac{d}{dz}\frac{z^{n+2}-z^n}{(z-i)^2}$$

$$= \lim_{z\to -i}\frac{(z-i)^2\left((n+2)z^{n+1}-nz^{n-1}\right)-(z^{n+2}-z^n)2(z-i)}{(z-i)^4}$$

$$= \frac{(-i-i)^2\left((n+2)(-i)^{n+1}-n(-i)^{n-1}\right)-((-i)^{n+2}-(-i)^n)2(-i-i)}{(-i-i)^4}$$

$$= \frac{(-2i)^2\left((n+2)(-i)^{n+1}-n(-i)^{n-1}\right)-((-i)^{n+2}-(-i)^n)2(-2i)}{(-2i)^4}$$

$$= \frac{-4\left((n+2)(-i)^{n+1}-n(-i)^{n-1}\right)-((-i)^{n+2}-(-i)^n)2(-2i)}{16}$$

$$= \frac{-\left((n+2)(-i)^{n+1}-n(-i)^{n-1}\right)-((-i)^{n+3}-(-i)^{n+1})}{4}$$

$$= \frac{-\left((n+1)(-i)^{n+1}-n(-i)^{n-1}\right)-(-i)^{n+3}}{4}$$

$$= -(-i)^n\frac{\left((n+1)(-i)-n(-i)^{-1}\right)+(-i)^3}{4}$$

$$= -\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)\frac{(-ni-i-ni)+i}{4}$$

$$= -\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)\frac{(-2ni)}{4} = i\frac{n}{2}\cos\frac{n\pi}{2}+\frac{n}{2}\sin\frac{n\pi}{2}$$

Residue at the pole $z = i$ of order two R_2

$$= \lim_{z\to i} \frac{d}{dz}(z-i)^2 z^{n-1} F(z)$$

$$= \lim_{z\to i} \frac{d}{dz}(z-i)^2 \frac{z^n(z^2-1)}{(z+i)^2(z-i)^2}$$

$$= \lim_{z\to i} \frac{d}{dz}\frac{z^n(z^2-1)}{(z+i)^2} = \lim_{z\to i} \frac{d}{dz}\frac{z^{n+2}-z^n}{(z+i)^2}$$

$$= \lim_{z\to i} \frac{(z+i)^2\left((n+2)z^{n+1}-nz^{n-1}\right)-(z^{n+2}-z^n)2(z+i)}{(z+i)^4}$$

$$= \frac{(i+i)^2\left((n+2)(i)^{n+1}-n(i)^{n-1}\right)-((i)^{n+2}-(i)^n)2(i+i)}{(i+i)^4}$$

$$= \frac{(2i)^2\left((n+2)(i)^{n+1}-n(i)^{n-1}\right)-((i)^{n+2}-(i)^n)2(2i)}{(2i)^4}$$

$$= \frac{-4\left((n+2)(i)^{n+1}-n(i)^{n-1}\right)-((i)^{n+2}-(i)^n)2(2i)}{16}$$

$$= \frac{-\left((n+2)(i)^{n+1}-n(i)^{n-1}\right)-((i)^{n+3}-(i)^{n+1})}{4}$$

$$= \frac{-\left((n+1)(i)^{n+1}-n(i)^{n-1}\right)-(i)^{n+3}}{4}$$

$$= -(i)^n \frac{\left((n+1)(i)-n(i)^{-1}\right)+(i)^3}{4}$$

$$= -\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)\frac{(ni+i+ni)-i}{4}$$

$$= -\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)\frac{(2ni)}{4} = -i\frac{n}{2}\cos\frac{n\pi}{2}+\frac{n}{2}\sin\frac{n\pi}{2}$$

$$Z^{-1}\left[\frac{z(z^2-1)}{(z^2+1)^2}\right] = Sum\ of\ the\ residues$$

$$= i\frac{n}{2}\cos\frac{n\pi}{2}+\frac{n}{2}\sin\frac{n\pi}{2} - i\frac{n}{2}\cos\frac{n\pi}{2}+\frac{n}{2}\sin\frac{n\pi}{2}$$

$$Z^{-1}\left[\frac{z(z^2-1)}{(z^2+1)^2}\right] = n\sin\frac{n\pi}{2}$$

$$12. Find\ Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] by\ partial\ fraction\ method$$

Solution:

$$Let\ F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)} \dots (1)$$

$$A(z-1)(z-2) + B(z-2) + C(z-1)^2 = z^2 \dots (2)$$

Put $z = 1$ in (2), we get

$$B(1-2) = 1 \Rightarrow B = -1$$

Put $z = 2$ in (2), we get

$$C(2-1)^2 = 2^2 \Rightarrow C = 4$$

Put $z = 0$ in (2), we get

$$A(-1)(-2) + B(-2) + C(-1)^2 = 0$$

$$2A - 2B + C = 0 \Rightarrow 2A + 2 + 4 = 0 \Rightarrow 2A = -6 \Rightarrow A = -3$$

Substituting the values of $A, B\ and\ C$ in (1), we get

$$\frac{F(z)}{z} = \frac{-3}{(z-1)} + \frac{-1}{(z-1)^2} + \frac{4}{(z-2)}$$

$$F(z) = -\frac{3z}{(z-1)} - \frac{z}{(z-1)^2} + \frac{4z}{(z-2)}$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = Z^{-1}\left[-\frac{3z}{(z-1)} - \frac{z}{(z-1)^2} + \frac{4z}{(z-2)}\right]$$

$$= -3Z^{-1}\left[\frac{z}{(z-1)}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4\,Z^{-1}\left[\frac{z}{(z-2)}\right]$$

$$= -3(1)^n - n + 4\,(2)^n$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = -3 - n + 4\,(2)^n$$

13. Solve the difference equation

$$y(n+3) - 3y(n+1) + 2y(n) = 0\ given\ that\ y(0) = 4, y(1) = 0\ and\ y(2) = 8.$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \dots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+3) - 3y(n+1) + 2y(n)\} = Z\{0\}$$

$$Z\{y(n+3)\} - 3Z\{y(n+1)\} + 2Z\{y(n)\} = 0 \dots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+3)\} = z^3F(z) - z^3y(0) - z^2y(1) - zy(2) \dots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \dots (4)$$

Substituting (3) and (4) in (2), we get

$$z^3F(z) - z^3y(0) - z^2y(1) - zy(2) - 3\big(zF(z) - zy(0)\big) + 2F(z) = 0$$

$$z^3F(z) - 4z^3 - z^2(0) - 8z - 3(zF(z) - 4z) + 2F(z) = 0$$

$$(z^3 - 3z + 2)F(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2)F(z) = 4z^3 - 4z$$

$$F(z) = Z\{y(n)\} = \frac{4z^3 - 4z}{z^3 - 3z^2 + 2}$$

$$y(n) = Z^{-1}\left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right]$$

$$Let\ F(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$z^{n-1}F(z) = z^{n-1}\frac{4z^3 - 4z}{z^3 - 3z + 2} = \frac{4z^n(z^2 - 1)}{z^3 - 3z + 2}$$

To find poles:

$$z^3 - 3z + 2 = 0 \Rightarrow z = -2,1,1$$

The poles are $z = -2,1$

$$z^{n-1}F(z) = \frac{4z^n(z^2 - 1)}{(z-1)^2(z+2)} = \frac{4z^n(z+1)(z-1)}{(z-1)^2(z+2)}$$

$$z^{n-1}F(z) = \frac{4z^n(z+1)}{(z-1)(z+2)}$$

Residue at the pole $z = 1$ of order one R_1

$$= \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}(z-1)\frac{4z^n(z+1)}{(z-1)(z+2)}$$

$$= \lim_{z\to 1}\frac{4z^n(z+1)}{(z+2)} = \frac{4(1)^n(1+1)}{(1+2)} = \frac{8}{3}$$

Residue at the pole $z = -2$ of order one R_2

$$= \lim_{z\to -2}(z+2)z^{n-1}F(z)$$

$$= \lim_{z\to -2}\frac{4z^n(z+1)}{(z-1)} = \frac{4(-2)^n(-2+1)}{(-2-1)} = \frac{4(-2)^n}{3}$$

$$y(n) = Z^{-1}\left[\frac{4z^3-4z}{z^3-3z+2}\right] = Sum\ of\ all\ residues$$

$$y(n) = \frac{8}{3} + \frac{4(-2)^n}{3}$$

14. Solve $y_{n+2} + y_n = 2$, $y_0 = 0$, $y_1 = 0$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+2) + y(n) = 2 \ldots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+2) + y(n)\} = Z\{2\}$$

$$Z\{y(n+2)\} - 4Z\{y(n+1)\} + 4Z\{y(n)\} = \frac{2z}{z-1} \ldots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \ldots (3)$$

Substituting (3) in (2), we get

$$z^2F(z) - z^2y(0) - zy(1) + F(z) = \frac{2z}{z-1}$$

$$z^2F(z) - z^2(0) - z(0) + F(z) = \frac{2z}{z-1}$$

$$(z^2+1)F(z) = \frac{2z}{z-1}$$

$$F(z) = Z\{y(n)\} = \frac{2z}{(z-1)(z^2+1)}$$

$$y(n) = Z^{-1}\left[\frac{2z}{(z-1)(z^2+1)}\right]$$

$$Let\ F(z) = \frac{2z}{(z-1)(z^2+1)}$$

By Residue method:

$$z^{n-1}F(z) = z^{n-1}\frac{2z}{(z-1)(z^2+1)} = \frac{2z^n}{(z-1)(z^2+1)}$$

$To\ find\ poles$:

$$(z-1)(z^2+1) = 0 \Rightarrow z = 1, i, -i$$

The poles are $z = 1, i, -i$

$$z^{n-1}F(z) = \frac{2z^n}{(z-1)(z^2+1)} = \frac{2z^n}{(z-1)(z+i)(z-i)}$$

Residue at the pole $z = 1$ of order one R_1

$$= \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}(z-1)\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to 1}\frac{2z^n}{(z+i)(z-i)} = \frac{2(1)^n}{(1+i)(1-i)} = \frac{2(1)^n}{(1-i^2)} = \frac{2(1)^n}{(1+1)} = 1$$

Residue at the pole $z = i$ of order one R_2

$$= \lim_{z\to i}(z-i)z^{n-1}F(z)$$

$$= \lim_{z\to i}(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{2z^n}{(z-1)(z+i)} = \frac{2i^n}{(i-1)(i+i)} = \frac{2i^n}{(i-1)2i}$$

$$= \frac{i^{n-1}}{(i-1)} = \frac{i^{n-1}(-i-1)}{(i-1)(-i-1)} = \frac{-i^n - i^{n-1}}{2} = -i^n\frac{1+i^{-1}}{2} = -i^n\frac{1-i}{2}$$

$$= -\frac{\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)(1-i)}{2} = -\frac{1}{2}\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) + \frac{i}{2}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)$$

Residue at the pole $z = -i$ of order one R_3

$$= \lim_{z\to -i}(z+i)z^{n-1}F(z)$$

$$= \lim_{z \to -i} (z+i) \frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z \to -i} \frac{2z^n}{(z-1)(z-i)} = \frac{2(-i)^n}{(-i-1)(-i-i)} = \frac{2(-i)^n}{(-i-1)(-2i)}$$

$$= \frac{(-i)^{n-1}}{(-i-1)} = \frac{(-i)^{n-1}(i-1)}{(i-1)(-i-1)} = \frac{-(-i)^n - (-i)^{n-1}}{2}$$

$$= -(-i)^n \frac{1+(-i)^{-1}}{2} = -(-i)^n \frac{1+i}{2}$$

$$= -\frac{\left(cos\frac{n\pi}{2} - i\,sin\frac{n\pi}{2}\right)(1+i)}{2} = -\frac{1}{2}\left(cos\frac{n\pi}{2} + sin\frac{n\pi}{2}\right) - \frac{i}{2}\left(cos\frac{n\pi}{2} - sin\frac{n\pi}{2}\right)$$

$$y(n) = Z^{-1}\left[\frac{2z}{(z-1)(z^2+1)}\right] = Sum\ of\ all\ residues$$

$$y(n) = 1 - \frac{1}{2}\left(cos\frac{n\pi}{2} + sin\frac{n\pi}{2}\right) + \frac{i}{2}\left(cos\frac{n\pi}{2} - sin\frac{n\pi}{2}\right) - \frac{1}{2}\left(cos\frac{n\pi}{2} + sin\frac{n\pi}{2}\right) - \frac{i}{2}\left(cos\frac{n\pi}{2} - sin\frac{n\pi}{2}\right)$$

$$y(n) = 1 - cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$$

By Partial fraction method:

$$\frac{F(z)}{z} = \frac{2}{(z-1)(z^2+1)}$$

$$\frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1} \ldots (i)$$

$$2 = A(z^2+1) + (Bz+C)(z-1) \ldots (1)$$

Put $z = 1$ in (1),we get

$$2 = A(1+1) \Rightarrow A = 1$$

Equating coefficients of z^2 in (1) on both sides

$$A + B = 0 \Rightarrow B = -A \Rightarrow B = -1$$

Put $z = 0$ in (1),we get

$$A - C = 2 \Rightarrow C = A - 2 \Rightarrow C = -1$$

Substituting the values of A,B and C in (i), we get

$$\frac{F(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{1}{z-1} + \frac{-z-1}{z^2+1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$y(n) = Z^{-1}\left[\frac{2z}{(z-1)(z^2+1)}\right] = Z^{-1}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right]$$

$$y(n) = 1 - cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$$

15. Solve the difference equation

$$y(n+2) - 4y(n+1) + 4y(n) = 0 \; given \; that \; y(0) = 1, y(1) = 0.$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+2) - 4y(n+1) + 4y(n) = 0 \dots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+2) - 4y(n+1) + 4y(n)\} = Z\{0\}$$

$$Z\{y(n+2)\} - 4Z\{y(n+1)\} + 4Z\{y(n)\} = 0 \dots (2) \qquad [By \; linear \; property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \dots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \dots (4)$$

Substituting (3) and (4) in (2), we get

$$z^2F(z) - z^2y(0) - zy(1) - 4\big(zF(z) - zy(0)\big) + 4F(z) = 0$$

$$z^2F(z) - z^2 - z(0) - 4(zF(z) - z) + 4F(z) = 0$$

$$(z^2 - 4z + 4)F(z) - z^2 + 4z = 0$$

$$(z^2 - 4z + 4)F(z) = z^2 - 4z$$

$$F(z) = Z\{y(n)\} = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$y(n) = Z^{-1}\left[\frac{z^2 - 4z}{z^2 - 4z + 4}\right]$$

$$Let \; F(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z^2 - 4z}{z^2 - 4z + 4} = \frac{z^n(z-4)}{z^2 - 4z + 4}$$

$To\ find\ poles$:

$$z^2 - 4z + 4 = 0 \Rightarrow z = 2,2$$

The poles are $z = 2$ which is of order two

$$z^{n-1}F(z) = \frac{z^n(z-4)}{z^2 - 4z + 4} = \frac{z^n(z-4)}{(z-2)^2}$$

Residue at the pole $z = 2$ of order two R_1

$$= \lim_{z\to 2}\frac{d}{dz}(z-2)^2 z^{n-1}F(z)$$

$$= \lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{z^n(z-4)}{(z-2)^2}$$

$$= \lim_{z\to 2}\frac{d}{dz}z^n(z-4) = \lim_{z\to 2}\left(z^n + nz^{n-1}(z-4)\right)$$

$$= \left(2^n + n2^{n-1}(2-4)\right) = 2^n + n2^{n-1}(-2)$$

$$= 2^n - n2^n = 2^n(1-n)$$

$$y(n) = Z^{-1}\left[\frac{z^2 - 4z}{z^2 - 4z + 4}\right] = Sum\ of\ all\ residues$$

$$y(n) = 2^n(1-n)$$

16. Solve the difference equation

$$y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2, given\ that\ y(0) = 3, y(1) = -2.$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2 \ldots (1)$$

Replacing $n\ by\ \ n + 2\ in\ (1), we\ get$

$$y(n+2) + 3y(n+1) - 4y(n) = 0, n \geq 0 \ldots (2)$$

Taking Z-transform on both sides in (2), we get

$$Z\{y(n+2) + 3y(n+1) - 4y(n)\} = Z\{0\}$$

$$Z\{y(n+2)\} + \ 3\,Z\{y(n+1)\} - 4Z\{y(n)\} = 0 \ldots (4) \qquad [By\ linear\ property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \ldots (4)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \ldots (5)$$

Substituting (4) and (5) in (3), we get

$$z^2F(z) - z^2y(0) - zy(1) + 3\big(zF(z) - zy(0)\big) - 4F(z) = 0$$

$$z^2F(z) - 3z^2 - z(-2) + 3(zF(z) - 3z) - 4F(z) = 0$$

$$(z^2 + 3z - 4)F(z) - 3z^2 + 2z - 9z = 0$$

$$(z^2 + 3z - 4)F(z) = 3z^2 + 7z$$

$$F(z) = Z\{y(n)\} = \frac{3z^2 + 7z}{z^2 + 3z - 4}$$

$$y(n) = Z^{-1}\left[\frac{3z^2 + 7z}{z^2 + 3z - 4}\right]$$

$$z^{n-1}F(z) = z^{n-1}\frac{3z^2 + 7z}{z^2 + 3z - 4} = \frac{z^n(3z+7)}{(z+4)(z-1)}$$

$To\ find\ poles$:

$$(z+4)(z-1) = 0 \Rightarrow z = -4,1$$

The poles are $z = -4,1$

Residue at the pole $z = -4$ of order one R_1

$$= \lim_{z\to-4}(z+4)z^{n-1}F(z)$$

$$= \lim_{z\to-4}(z+4)\frac{z^n(3z+7)}{(z+4)(z-1)}$$

$$= \lim_{z\to-4}\frac{z^n(3z+7)}{(z-1)} = \frac{(-4)^n(3(-4)+7)}{(-4-1)} = (-4)^n$$

Residue at the pole $z = 1$ of order one R_2

$$= \lim_{z\to1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to1}(z-1)\frac{z^n(3z+7)}{(z+4)(z-1)}$$

$$= \lim_{z\to1}\frac{z^n(3z+7)}{(z+4)} = \frac{1^n(3+7)}{(1+4)} = 2$$

$$y(n) = Z^{-1}\left[\frac{3z^2 + 7z}{z^2 + 3z - 4}\right] = sum\ of\ residues$$

$$y(n) = (-4)^n + 2$$

17. Solve the difference equation

$$y(n+2) - 7y(n+1) + 12y(n) = 2^n \ given\ that\ y(0) = 0, y(1) = 0.$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+2) - 7y(n+1) + 12y(n) = 2^n \ldots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+2) - 7y(n+1) + 12y(n)\} = Z\{2^n\}$$

$$Z\{y(n+2)\} - 7\,Z\{y(n+1)\} + 12\,Z\{y(n)\} = \frac{z}{z-2} \ldots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \ldots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$z^2F(z) - z^2y(0) - zy(1) - 7\big(zF(z) - zy(0)\big) + 12F(z) = \frac{z}{z-2}$$

$$z^2F(z) - z^2(0) - z(0) - 7(zF(z) - z(0)) + 12F(z) = \frac{z}{z-2}$$

$$(z^2 - 7z + 12)F(z) = \frac{z}{z-2}$$

$$F(z) = Z\{y(n)\} = \frac{z}{(z-2)(z^2-7z+12)}$$

$$y(n) = Z^{-1}\left[\frac{z}{(z-2)(z^2-7z+12)}\right]$$

$$z^{n-1}F(z) = z^{n-1}\frac{z}{(z-2)(z^2-7z+12)} = \frac{z^n}{(z-2)(z-3)(z-4)}$$

$To\ find\ poles$:

$$(z-2)(z-3)(z-4) = 0 \Rightarrow z = 2, 3, 4$$

The poles are $z = 2, 3, 4$

Residue at the pole $z = 2$ of order one R_1

$$= \lim_{z \to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z \to 2}(z-2)\frac{z^n}{(z-2)(z-3)(z-4)}$$

$$= \lim_{z \to 2} \frac{z^n}{(z-3)(z-4)} = \frac{2^n}{(2-3)(2-4)} = 2^{n-1}$$

Residue at the pole $z = 3$ of order one R_2

$$= \lim_{z \to 3} (z-3) z^{n-1} F(z)$$

$$= \lim_{z \to 3} (z-3) \frac{z^n}{(z-2)(z-3)(z-4)}$$

$$= \lim_{z \to 3} \frac{z^n}{(z-2)(z-4)} = \frac{3^n}{(3-2)(3-4)} = -\,3^n$$

Residue at the pole $z = 4$ of order one R_3

$$= \lim_{z \to 4} (z-4) z^{n-1} F(z)$$

$$= \lim_{z \to 4} (z-4) \frac{z^n}{(z-2)(z-3)(z-4)}$$

$$= \lim_{z \to 4} \frac{z^n}{(z-2)(z-3)} = \frac{4^n}{(4-2)(4-3)} = \frac{4^n}{2}$$

$$y(n) = Z^{-1}\left[\frac{z}{(z-2)(z^2-7z+12)}\right] = sum\ of\ all\ residues$$

$$y(n) = 2^{n-1} - 3^n + \frac{1}{2}\,4^n$$

18. Solve $y_{n+2} - 5y_{n+1} + 6\,y_n = 1\,, y_0 = 1\ ,\ y_1 = 1$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+2) - 5y(n+1) + 6y(n) = 1 \ldots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+2) - 5y(n+1) + 6y(n)\} = Z\{1\}$$

$$Z\{y(n+2)\} - 5\,Z\{y(n+1)\} + 6\,Z\{y(n)\} = \frac{z}{z-1} \ldots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \ldots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$z^2F(z) - z^2y(0) - zy(1) - 5\big(zF(z) - zy(0)\big) + 6F(z) = \frac{z}{z-1}$$

$$z^2F(z) - z^2 - z - 5(zF(z) - z) + 6F(z) = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)F(z) - \ z^2 - z + 5z = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)F(z) = \frac{z}{z-1} + \ \ z^2 - 4z$$

$$(z^2 - 5z + 6)F(z) = \frac{z + (z^2 - 4z)(z-1)}{z-1}$$

$$F(z) = Z\{y(n)\} = \frac{z + (z^2 - 4z)(z-1)}{(z-1)(z^2 - 5z + 6)}$$

$$y(n) = Z^{-1}\left[\frac{z + (z^2 - 4z)(z-1)}{(z-1)(z^2 - 5z + 6)}\right]$$

$$z^{n-1}F(z) = z^{n-1}\frac{z + (z^2 - 4z)(z-1)}{(z-1)(z^2 - 5z + 6)} = \frac{z^n\big(1 + (z-4)(z-1)\big)}{(z-1)(z-2)(z-3)}$$

$To\ find\ poles$:

$$(z-1)(z-2)(z-3) = 0 \Rightarrow z = 1,2,3$$

The poles are $z = 1,2,3$

Residue at the pole $z = 1$ of order one R_1

$$= \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}(z-1)\frac{z^n\big(1 + (z-4)(z-1)\big)}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to 1}\frac{z^n\big(1 + (z-4)(z-1)\big)}{(z-2)(z-3)} = \frac{1^n\big(1 + (1-4)(1-1)\big)}{(1-2)(1-3)} = \frac{1}{2}$$

Residue at the pole $z = 2$ of order one R_2

$$= \lim_{z\to 2}(z-2)z^{n-1}F(z)$$

$$= \lim_{z\to 2}(z-2)\frac{z^n\big(1 + (z-4)(z-1)\big)}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z\to 2}\frac{z^n\big(1 + (z-4)(z-1)\big)}{(z-1)(z-3)} = \frac{2^n\big(1 + (2-4)(2-1)\big)}{(2-1)(2-3)} = \ 2^n$$

Residue at the pole $z = 3$ of order one R_3

$$= \lim_{z\to 3}(z-3)z^{n-1}F(z)$$

$$= \lim_{z \to 3} (z-3) \frac{z^n \left(1 + (z-4)(z-1)\right)}{(z-1)(z-2)(z-3)}$$

$$= \lim_{z \to 3} \frac{z^n \left(1 + (z-4)(z-1)\right)}{(z-1)(z-2)} = \frac{3^n \left(1 + (3-4)(3-1)\right)}{(3-1)(3-2)} = -\frac{3^n}{2}$$

$$y(n) = Z^{-1}\left[\frac{z + (z^2 - 4z)(z-1)}{(z-1)(z^2 - 5z + 6)}\right] = sum\ of\ all\ residues$$

$$y(n) = \frac{1}{2} + 2^n - \frac{1}{2}\, 3^n$$