Sri Ramakrishna Institute of Technology, Coimbatore-10

Department of Science and Humanities

Class Test - I

Part-A

1. State Dirichlet's condition.

Ans: A function $f(x)$ defined in $c \leq x \leq c + 2l$ can be expanded as an infinite trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

provided

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(c, c + 2l)$.

(ii) $f(x)$ is periodic in $(c, c + 2l)$.

(iii) $f(x)$ and $f^{'}(x)$ are piecewise continuous in $(c, c + 2l)$.

(iv) $f(x)$ has no or finite number of maxima or minima in $(c, c + 2l)$.

2. Define Fourier series expansion on $(c, c + 2l)$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

where

$$\mathrm{a}_0 = \frac{1}{l} \int_{c}^{c+2l} f(x) dx$$

$$\mathrm{a}_\mathrm{n} = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos\frac{n\pi x}{l} dx$$

$$\mathrm{b}_\mathrm{n} = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin\frac{n\pi x}{l} dx$$

3. If $f(x) = x \sin x$, find b_n in $(-\pi, \pi)$.

Solution:

$$f(-x) = (-x)(\sin(-x)) = (-x)(-\sin x) = x \sin x = f(x)$$

$\therefore f(x)$ is an even function

$$\therefore b_n = 0.$$

4. If $f(x) = x^3$, find a_n in $(-\pi, \pi)$.

Solution:

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

$\therefore f(x)$ is an odd function

$$\therefore a_n = 0.$$

Part –B

1. Find the Fourier series expansion of

i) $f(x) = (\pi - x)^2$ in $(0, 2\pi)$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\, dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(\pi - 2\pi)^3}{-3} - \frac{(\pi - 0)^3}{-3}\right] = \frac{1}{\pi}\left[\frac{(-\pi)^3}{-3} - \frac{\pi^3}{-3}\right] = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(\pi-2\pi)^2\left(\frac{\sin 2n\pi}{n}\right) - 2(\pi-2\pi)(-1)\left(-\frac{\cos 2n\pi}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin 2n\pi}{n^3}\right)\right]$$

$$- \frac{1}{\pi}\left[(\pi-0)^2\left(\frac{\sin 0}{n}\right) - 2(\pi-0)(-1)\left(-\frac{\cos 0}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{2\pi}{n^2} + 0 - 0 + \frac{2\pi}{n^2} + 0\right] = \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(\pi-2\pi)^2\left(-\frac{\cos 2n\pi}{n}\right) - 2(\pi-2\pi)(-1)\left(-\frac{\sin 2n\pi}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos 2n\pi}{n^3}\right)\right]$$

$$- \frac{1}{\pi}\left[(\pi-0)^2\left(-\frac{\cos 0}{n}\right) - 2(\pi-0)(-1)\left(-\frac{\sin 0}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n} - 0 + \frac{2}{n^3} - \frac{\pi^2}{n} - 0 - \frac{2}{n^3}\right] = 0$$

$$b_n = 0$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx$$

ii) $f(x) = x^2$ in $(-\pi, \pi)$.

Solution:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore f(x)$ is an even function.

$$\mathrm{b_n} = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - 2\pi\left(-\frac{\cos n\pi}{n^2}\right) + 0 - 0 + 0 + 0\right]$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

2. Find the Fourier series expansion of

i) $f(x) = \begin{cases} x, & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi}(2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[\frac{(2\pi - x)^2}{-2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0 + 0 - \left(\frac{\pi^2}{-2}\right)\right] = \frac{1}{\pi}[\pi^2] = \pi$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2} + 0 - \frac{\cos 2n\pi}{n^2} - 0 + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \pi\frac{\cos n\pi}{n}\right]$$

$$b_n = 0$$

$$f(x) = \frac{\pi}{2} - \sum_{n=1,3,5,\dots}^{\infty}\frac{4}{\pi n^2}\cos nx$$

ii) $f(x) = |x|$ in $-\pi < x < \pi$

Solution:

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore f(x)$ is an even function.

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{\cos n\pi}{n^2} + 0 - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

$$f(x) = \frac{\pi}{2} - \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{\pi n^2}\cos nx$$