B.E./B.TECH. DEGREE EXAMINATION, CHENNAI-APRIL/MAY 2010.

TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS(COMMON TO ALL BRANCHES)

1. Write the conditions for a function $f(x)$ to satisfy for the existence of a Fourier series.

i. $f(x)$ is defined, single valued and finite
ii. $f(x)$ and $f'(x)$ are piecewise continuous and $f(x)$ has finite number of finite discontinuities
iii. $f(x)$ has atmost finite number of maxima and minima.

2. If $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \; in \; (-\pi,\pi)$ deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$.

Given $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \; in \; (-\pi,\pi)$

Here $x=\pi$, an end point, is a point of discontinuity

Hence $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$

$$\frac{f(-\pi)+f(-\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$

$$\frac{\pi^2+\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

3. Find the Fourier Cosine transform of e^{-ax}, $x \geq 0$.

The Fourier Cosine transform of $f(x)$ is

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\, dx$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\;\frac{a}{a^2+s^2}$$

4. Find the Fourier transform of $f(x)$, show that $F(f(x-a)) = e^{ias}F(s)$.

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{iax}\, dx$$

Put $x-a=y \quad Then\; dx = dy$

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{ia(y+a)}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{ia(y+a)} \, dy$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{iay} \, dy$$

$$\therefore \boldsymbol{F\big(f(x-a)\big) = e^{ias} F(s)}.(since\ y\ is\ a\ dummy\ variable)$$

5. Form the partial differential equation by eliminating the constants a and b from

$\boldsymbol{z = (x^2 + a)(y^2 + b)}$.

Given $z = (x^2 + a)(y^2 + b) - - - - (1)$

Differentiating (1) partially with respect to $x\ and\ y$, we get

$$\frac{\partial z}{\partial x} = p = 2x\,(y^2 + b^2) \;=>\; \frac{p}{2x} = (y^2 + b^2) - - - (2)$$

$$\frac{\partial z}{\partial y} = q = 2y\,(x^2 + a^2) => \frac{q}{2y} = (x^2 + a^2) - - - (3)$$

Substituting (2) and (3) in (1), we get

$$z = \frac{p}{2x}\frac{q}{2y} => \boldsymbol{pq = 4xy}.$$

6. Solve the partial differential equation $pq = x$.

$$\frac{p}{x} = \frac{q}{1} = k \quad => p = kx\ and\ q = \frac{1}{k}$$

$$z = \int p\,dx + q\,dx) = \int kx\,dx + \int \frac{1}{k} dy$$

$$z = k\frac{x^2}{2} + \frac{1}{k}y + c$$

7. A tightly stretched string with fixed end points $x - 0\ and\ x = l$ is initially in a position given by $y(x, 0) = v_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest in this position, write the boundary conditions.

The boundary conditions are

i. $y(0, t) = 0, t > 0$

ii. $y(l, t) = 0, t > 0$

iii. $\frac{\partial y(x,0)}{\partial t} = 0$

iv. $y(x, 0) = v_0 \sin^3 \frac{\pi x}{l}, 0 < x < l.$

8. Write all three possible solutions of steady state two dimensional hear equation.

i. $u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

ii. $u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

iii. $u(x,y) = (Ax + B)(Cy + D)$

9. Find the Z- transform of $\sin\frac{n\pi}{2}$.

We know that $Z(\sin n\theta) = \frac{z\sin n\theta}{z^2 - 2z\cos\theta + 1}$

Put $\theta = \frac{\pi}{2}$,

$$Z(\sin\frac{n\pi}{2}) = \frac{z\sin\frac{n\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z}{z^2+1}$$

10. Find the difference equation generated by $y_n = an + b\,2^n$.

Given $y_n = an + b\,2^n$

$$y_{n+1} = a(n+1) + b\,2^{n+1} = a(n+1) + 2b2^n$$

$$y_{n+2} = a(n+2) + b\,2^{n+2} = a(n+2) + 4\,b\,2^n$$

Eliminating $a\ ans\ b$ we get

$$\begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix} = 0$$

$$y_n\big(4(n+1) - 2(n+2)\big) - n(4y_{n+1} - 2y_{n+2}) + \big((n+2)y_{n+1} - (n+1)y_{n+2}\big) = 0$$

$$\boldsymbol{(n-1)y_{n+2} + (-3n+2)y_{n+1} + 2ny_n = 0}$$

PART-B

11. a. (i) Find the Fourier series for $f(x) = 2x - x^2$ in the interval $0 < x < 2$.

The Fourier series of $f(x)$ in $(0,2)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x) ----\rightarrow (1)$$

$$a_0 = \frac{1}{1}\int_0^2 (2x - x^2)dx = \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3}$$

$$\boldsymbol{a_0 = \frac{4}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\,cos\frac{n\pi x}{l}\,dx = \frac{1}{1}\int_0^{2}(2x - x^2)\cos n\pi x\,dx$$

$$= \left[(2x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2 - 2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= (-2)\left(\frac{\cos 2n\pi}{n^2\pi^2}\right) - 2\left(\frac{\cos 0}{n^2\pi^2}\right) = \frac{-4}{n^2\pi^2}$$

$$\boldsymbol{a_n = \frac{-4}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\,\sin\frac{n\pi x}{l}\,dx = \int_0^{2} f(x)\,\sin n\pi x\,dx = \int_0^{2}(2x - x^2)\,\sin n\pi x\,dx$$

$$= \left[(2x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2 - x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[(-2)\left(\frac{\cos 2n\pi}{n^3\pi^3}\right) + \frac{2}{n^3\pi^3}\right]$$

$$\boldsymbol{b_n = 0}$$

The Fourier series is

$$\boldsymbol{f(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\cos n\pi x\right)}$$

11. (a). (ii). Find the half range cosine series for the function $f(x) = x\,(\pi - x)$ in $0 < x < \pi$.

Hence deduce $\boldsymbol{\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots}$.

Solution:

The half range cosine series for the function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx\ , \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad and \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \boldsymbol{\frac{\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\ \cos nx\ dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$a_n = \frac{-2}{n^2}\,[(-1)^n + 1]$$

$$b_n = \begin{cases} \dfrac{-4}{n^2} & if\ \ n\ is\ even \\ 0 & if\ \ n\ is\ odd \end{cases}$$

$$\therefore\quad \boldsymbol{(\pi x - x^2) = \frac{\pi^2}{6} + \sum_{n=evn}^{\infty} \left(\frac{-4}{n^2}\right)\ \sin nx}$$

By Parseval's Identity, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi x - x^2)^2\,dx = \frac{1}{4}\left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=even}^{\infty}\left(\frac{-4}{n^2}\right)^2$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=even}^{\infty}\frac{1}{n^4}$$

$$\boldsymbol{\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots}$$

11. (b). (i). Find the complex form of Fourier series $f(x) = e^{ax}\ \ in - \pi < x < \pi.$

Solution:

$$We\ know\ that\ C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\ e^{-inx}\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\ dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-in}\left(e^{(a-in)\pi} - e^{-(a-in)\pi}\right)\right] = \frac{1}{2\pi(a-in)}\left[e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{+in\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i\sin n\pi)\right]$$

$$= \frac{1}{2\pi(a-in)}[e^{a\pi}(\cos n\pi) - e^{-a\pi}(\cos n\pi)] = \frac{1}{2\pi(a-in)}[\cos n\pi\,(e^{a\pi} - e^{-a\pi})]$$

$$= \frac{(-1)^n}{2\pi(a-in)}[2\sinh a\pi]$$

$$\boldsymbol{C_n = \frac{(-1)^n}{\pi(a-in)}[\sinh a\pi]}$$

$$\therefore\quad f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$$

$$\therefore\quad e^{ax} = \sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{\pi(a-in)}[\sinh a\pi]\right] e^{inx}$$

11. (b). (ii). Find the Fourier series as for as the second harmonic to represent the function $f(x)$ with period 6, given the following table.

x:	0	1	2	3	4	5
$f(x)$:	9	18	24	28	26	20

Solution:

Refer Chennai Nov / Dec 2010, Q.no 11. (b). (ii)

12. (a). (i). Find the Fourier Transform of the function

$$f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$ **and hence find the value of** $\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt = \frac{\pi}{3}$.

Solution:

Refer Chennai Nov / Dec 2011, Q.no 12. (a). (ii)

12. (a). (ii). Show that the Fourier transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$.

Solution:

Refer Chennai Nov / Dec 2011, Q.no 12. (a). (i)

12.(b).(i) Find the Fourier sine transform of $f(x) = \begin{cases} \sin x, & 0 \le x \le a \\ 0, & x > a \end{cases}$

Solution:

The Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \; dx = \sqrt{\frac{2}{\pi}} \int_0^a \sin x \; \sin sx \; dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{(\cos(s+1)x - \cos(s-1)x)}{2} \; dx = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} - \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} - \frac{\sin(s-1)a}{s-1} \right]$$

12. (b). (ii). using Fourier cosine transform method, evaluate $\int_0^\infty \frac{dx}{(t^2+a^2)(t^2+b^2)}$.

Ans:

Refer Chennai Nov / Dec 2010, Q.no 12. (b). (ii)

13 (a)(i). Solve $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$

The subsidiary equations are

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - xz)} = \frac{dz}{(z^2 - xy)}$$

Choosing the multipliers as x, y, z

$$\frac{xdx}{x(x^2 - yz)} = \frac{ydy}{y(y^2 - xz)} = \frac{zdz}{z(z^2 - xy)} = \frac{dx + dy + dz}{1}$$

$$\frac{xdx + ydy + zdz}{x + y + z} = \frac{dx + dy + dz}{1}$$

$$xy + yz + zx = c_1$$

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)^2} = \frac{ddz}{(y^2 - zx) - (z^2 - xy)}$$

$$\log(x - y) = \log(y - z) + \log c_2$$

$$c_2 = \frac{x - y}{y - z}$$

The required solution is $\phi(u, v) = 0$

$$\boldsymbol{i.e.,\phi}\left(xy + yz + zx, \; \frac{x - y}{y - z} \right) = \mathbf{0}$$

13. (a) (ii) Solve $p(1+q) = qz$.

This is of the form $F(z,p,q) = 0$

Given $p(1+q) = qz - - - -(1)$

Let $z = f(x+ay)$ be the solution of (1)

$$\therefore z = f(u)$$

Partially differentiation with respect to $x \text{ and } y$,

$$p = \frac{dz}{du} \quad and \quad q = a\frac{dz}{du}$$

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$\left(1 + a\frac{dz}{du}\right) = az \quad \Rightarrow \quad \frac{dz}{du} = \frac{az-1}{a} \quad \Rightarrow \quad \frac{dz}{z - \frac{1}{a}} = du$$

$$\int \frac{dz}{z-\frac{1}{a}} = \int du \qquad i.e., \qquad \log\left(z - \frac{1}{a}\right) = u + b$$

$$\log\left(z - \frac{1}{a}\right) = x + ay + b$$

13. (b) (i) Find the partiasl differential equation of all planes which are at a constant distance a_1 from the origin.

The equation of a plane which is at a distance a_1 from origin is $x\cos\alpha + y\cos\beta + z\cos\gamma = a_1$

$$Let \cos\alpha = a \quad \cos\beta = b \quad \cos\gamma = c \quad Then\ ax + by + cz = a_1 \quad and\ x^2 + y^2 + z^2 = 1$$

$$ax + by + \sqrt{1 - a^2 - b^2}\,.z = a_1$$

Differentiating with respect to x and y,

$$a + \sqrt{1-a^2-b^2}\,.p = 0 \quad and \quad b + \sqrt{1-a^2-b^2}\,.q = 0$$

$$\frac{a}{p} = \frac{b}{q} = -\sqrt{1-a^2-b^2} = \lambda \quad \Rightarrow \quad a = p\lambda \text{ and } b = b\lambda$$

$$-\sqrt{1 - p^2\lambda^2 - q^2\lambda^2} = \lambda \quad \Rightarrow \quad 1 - p^2\lambda^2 - q^2\lambda^2 = \lambda^2$$

$$1 - \lambda^2(p^2+q^2) = \lambda^2 \quad \Rightarrow \quad \lambda^2 = \frac{1}{1+p^2+q^2} \quad \Rightarrow \lambda = \frac{1}{\sqrt{1+p^2+q^2}}$$

$$p\lambda x + q\lambda y - \lambda z = a_1 \quad \Rightarrow z = px + qy - \frac{a_1}{\lambda} \quad \Rightarrow \boldsymbol{z = px + qy - a_1\sqrt{1+p^2+q^2}}$$

13. (b). (ii). Solve $\left(D^2+2DD'+D'^2-2D-2D'\right)z=\sin(x+2y)$

$$D^2+2DD'+D'^2-2D-2D'=(D+D')(D+D'-2)$$

The auxiliary equation is $(m+1)(m-1)=0$

$\therefore$ the Complementary function is

$$C.F.=f_1(y-x)+f_2(y+x)$$

$$P.I.=\frac{1}{(D+D')(D+D'-2)}\sin(x+2y)$$

$$=I.P.of\ \frac{1}{(D+D')(D+D'-2)}e^{i(x+2y)}\ =I.P.of\ \frac{1}{(i+2i)(i+2i-2)}e^{ix+i2y}$$

$$=I.P.of\ \frac{1}{3i\,(3i-2)}e^{ix+i2y}=I.P.of\ -\frac{i}{3}\frac{(3i+2)}{(3i-2)(3i+2)}e^{ix+i2y}$$

$$=I.P.of\frac{(-3+2i)}{39}(\cos(x+2y)+i\sin(x+2y))$$

$$\therefore P.I.=\ -\frac{3}{39}\sin(x+2y)+\frac{2}{39}\cos(x+2y)$$

$$\therefore \boldsymbol{z=f_1(y-x)+f_2(y+x)-\frac{3}{39}\sin(x+2y)+\frac{2}{39}\cos(x+2y)}$$

14. (a). A tightly stretched string of length l is fastened at both ends. The midpoint of the string is displaced by a distance $'b'$ transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Ans:

Refer Chennai Nov / Dec 2010, Q.no 14. (a).

13. (a). A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $y=0$ is given by $u=\begin{cases}20x & for\ 0\le x\le 5\\ 20(10-x) & for\ 5\le x\le 10\end{cases}$ and all the other three edges are kept at 0°C. Find the steady state temperature at any point in the plate.

Solution:

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}=0$

The boundary conditions are

i. $u(0,y)=0\ for\ all\ y$

ii. $u(10, y) = 0 \; for\; all\; y$

iii. $u(x, \infty) = 0 \; 0 < x < 10$

iv. $u(x, 0) = \begin{cases} \mathbf{20x} & \mathbf{for\ 0 \le x \le 5} \\ \mathbf{20(10-x)} & \mathbf{for\ 5 \le x \le 10} \end{cases}$

The most general solution after applying the (i),(ii),(iii) boundary conditions is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

$$(iv) \Longrightarrow u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = \begin{cases} \mathbf{20x} & \mathbf{for\ 0 \le x \le 5} \\ \mathbf{20(10-x)} & \mathbf{for\ 5 \le x \le 10} \end{cases}$$

Using half range sine series

$$b_n = \frac{2}{10}\int_0^5 20\, x \sin\frac{n\pi x}{10}\, dx + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}\, dx$$

$$b_n = \begin{cases} \dfrac{800 \sin\frac{n\pi}{2}}{n^2\pi^2} & , n\ is\ odd \\ 0\ , & n\ is\ even \end{cases}$$

$$\boldsymbol{u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{800}{n^2\pi^2}\, \sin\frac{n\pi}{2}\, \sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}}$$

15.(a). (i) Solve by Z-transform $\boldsymbol{u_{n+2} - 2\, u_{n+1} + u_n = 2^n}$ **with** $\boldsymbol{u_0 = 2}$ $\boldsymbol{and}$ $\boldsymbol{u_1 = 1}$**.**

Given $u_{n+2} - 2\, u_{n+1} + u_n = 2^n$

Taking Z- transform, we get

$$Z[u_{n+2]} - 2Z[\, u_{n+1}\,] + Z[u_{n]} = Z[2^n]$$

$$z^2\bar{u} - z^2u_0 - zu_1 - 2z(\bar{u} - u_0) + \bar{u} = \frac{z}{z-2}$$

$$z^2\bar{u} - 2z^2 - z - 2z\bar{u} + 4z + \bar{u} = \frac{z}{z-2}$$

$$\bar{u}\,(z^2 - 2z + 1) - 2z^2 + 3z = \frac{z}{z-2}$$

$$\bar{u}\,(z-1)^2 = \frac{z}{z-2} + 2z^2 - 3z$$

$$\bar{u} = \frac{z + 2z^2(z-2) - 3z(z-2)}{(z-2)(z-1)^2}$$

$$\bar{u} = \frac{z(1 + 2z(z-2) - 3(z-2))}{(z-2)(z-1)^2}$$

$$\frac{\bar{u}}{z} = \frac{2z^2 - 7z + 7}{(z-2)(z-1)^2}$$

$$\frac{2z^2 - 7z + 7}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$2z^2 - 7z + 7 = A(z-1)^2 + B(z-2)(z-1) + C(z-2)$$

$$z = 1 \quad => \quad C = -2$$

$$z = 2 \quad => \quad A = 1$$

Equating coefficient of z^2, $\quad B = 1$

$$\frac{2z^2 - 7z + 7}{(z-2)(z-1)^2} = \frac{1}{z-2} + \frac{1}{z-1} + \frac{-2}{(z-1)^2}$$

$$\therefore \frac{\bar{u}}{z} = \frac{1}{z-2} + \frac{1}{z-1} + \frac{-2}{(z-1)^2}$$

$$z[u_n] = \frac{Z}{z-2} + \frac{Z}{z-1} - 2\frac{Z}{(z-1)^2}$$

Taking Z^{-1} on both sides, we get

$$u_n = Z^{-1}\left[\frac{Z}{z-2}\right] + Z^{-1}\left[\frac{Z}{z-1}\right] - 2Z^{-1}\left[\frac{Z}{(z-1)^2}\right]$$

$$\boldsymbol{u_n = 2^n + 1^n - 2n}$$

15. (a) (ii). Using convolution theorem, find the inverse $Z - transform\ of \left(\frac{z}{z-4}\right)^3$

$$Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] = Z^{-1}\left[\frac{z^3}{(z-4)^3}\right] = Z^{-1}\left[\frac{z^2}{(z-4)^2} \cdot \frac{z}{z-4}\right] = Z^{-1}\left[\frac{z^2}{(z-4)^2}\right] * Z^{-1}\left[\frac{z}{z-4}\right]$$

$$= (n+1)4^n * 4^n = \sum_{r=0}^{n} (r+1)\, 4^r\, 4^{n-r} = 4^n \sum_{r=0}^{n} (r+1) = 4^n\ [1 + 2 + 3 + \cdots + (n+1)]$$

$$\boldsymbol{Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] = 4^n \frac{(n+1)(n+2)}{2}}$$

15. (b) (i). Find $Z^{-1}\left[\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$ $\quad and \quad$ $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

$$(i)\quad Let \quad F(z) = \frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{(z^2 - z + 2)}{(z+1)(z-1)^2}$$

$$\frac{(z^2 - z + 2)}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$(z^2 - z + 2) = A(z-1)^2 B(z+1)(z-1) + C(z+1)$$

$z = 1 \quad => \quad \boldsymbol{C = 1}$

$z = -1 \quad => \quad \boldsymbol{A = 1}$

$z = 0 \quad => \quad \boldsymbol{B = 0}$

$$\frac{F(z)}{z} = \frac{(z^2 - z + 2)}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$F(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

Taking Z^{-1} on both sides

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z+1}\right] + Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$Z^{-1}\left[\frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}\right] = (-1)^n + n$$

$(ii) \quad Let\, F(z) = \dfrac{z}{(z-1)(z-2)}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$z = 2 \quad => \quad \boldsymbol{B = 1}$

$z = 1 \quad => \quad \boldsymbol{A = -1}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$F(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

Taking Z^{-1} on both sides

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-1}{z-1}\right] + Z^{-1}\left[\frac{1}{z-2}\right] = -1^n + 2^n$$

$$\boldsymbol{Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = 2^n - 1}$$

15(b) (ii). Find $\boldsymbol{Z(na^n \sin n\theta)}$

$$Z(na^n \sin n\theta) = -z\,\frac{d}{dz}(a^n \sin n\theta)$$

$$= -z\,\frac{d}{dz}\left[\frac{a\,z\sin\theta}{z^2 - 2az\cos\theta + a^2}\right]$$

$$= -z\,\left[\frac{(z^2 - 2az\cos\theta)\,a\ \sin\theta - a\,z\sin\theta\,(2z - 2a\cos\theta)}{(z^2 - 2az\cos\theta + a^2)^2}\right]$$

$$\boldsymbol{Z(na^n \sin n\theta) = \frac{az\sin\theta\,(z^2 - a^2)}{(z^2 - 2az\cos\theta + a^2)^2}}$$

B.E./B.TECH. DEGREE EXAMINATION, CHENNAI-NOVEMBER/DECEMBER 2009.

TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS(COMMON TO ALL BRANCHES)

1. State the conditions for a function $f(x)$ to be expressed as a Fourier series.

iv. $f(x)$ is defined, single valued and finite
v. $f(x)$ and $f'(x)$ are piecewise continuous and $f(x)$ has finite number of finite discontinuities
vi. $f(x)$ has at most finite number of maxima and minima.

2. Obtain the first term of the Fourier series for the function $f(x) = x^2, -\pi < x < \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2}{3}\pi^2$$

3. Find the Fourier transform of $\begin{cases} e^{ikx} & , a < x < b \\ 0 & , x \le a \text{ and } x > b \end{cases}$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{ikx}\ e^{isx}\,dx$$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)x}}{i(k+s)}\right]_b^a$$

$$\boldsymbol{F(f(x)) = \frac{1}{i(k+s)\sqrt{2\pi}}\left[e^{i(k+s)b} - e^{i(k+s)a}\right]}$$

4. Find the Fourier sine transform of $\frac{1}{x}$.

The Fourier Cosine transform of $f(x)$ is

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\ \frac{\pi}{2}\ = \sqrt{\frac{\pi}{2}}$$

5. Find the partial differential equation of all planes cutting equal intercepts from the x and y axes.

Equation of plane is $\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1$

Partially differentiating with respect to x and y,

$$\frac{1}{a}+\frac{p}{b}=0 \quad => \quad -\frac{1}{a}=\frac{p}{b}$$

$$\frac{1}{a}+\frac{q}{b}=0 \quad => -\frac{1}{a}=\frac{q}{b}$$

$$\therefore \frac{\boldsymbol{p}}{\boldsymbol{b}}=\frac{\boldsymbol{q}}{\boldsymbol{b}}$$

6. Solve $(D^3-2D^2D')z=0$.

A.E. is $m^3-2m^2=0 \quad => \quad m=0,0,2$

The solution is $\boldsymbol{z=f_1(y)+x\,f_2(y)+f_3(y+2x)}$

7. Classify the partial differential equation $4\frac{\partial^2 y}{\partial x^2}=\frac{\partial y}{\partial t}$.

Given $4\frac{\partial^2 y}{\partial x^2}-\frac{\partial y}{\partial t}=0$

Here $A=4, B=0, C=0 \quad and \quad B^2-4AC=0$

The given equation is **parabolic.**

8. Write down all possible solutions of one dimensional wave equation.

iv. $y(x,t)=(Ae^{px}+Be^{-px})(Ce^{pat}+De^{-pat})$

v. $u(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{A}\cos \boldsymbol{px}+\boldsymbol{B}\sin \boldsymbol{px})(\boldsymbol{C}\cos \boldsymbol{pat}+\boldsymbol{B}\sin \boldsymbol{pat})$

vi. $\boldsymbol{u(x,y)=(Ax+B)(Ct+D)}$

9. If $F(z)=\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$, find $f(0)$.

Given $Z[f(n)]=F(z)=\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$

By initial value theorem, $f(0)=\lim_{z\to\infty}F(z)$

$$=\lim_{z\to\infty}\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}=\lim_{z\to\infty}\frac{z^2}{z^3\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{4z}\right)\left(1-\frac{3}{4z}\right)}$$

$$\therefore \boldsymbol{f(0)=0}$$

10. Find the Z- transform of $x(n)=\begin{cases}\frac{a^n}{n!} & , for n \geq 0\\ o & , otherwise\end{cases}$

$$Z\left[\frac{a^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{a^n}{n!}z^{-n} = \sum_{n=0}^{\infty}\frac{(az^{-1})^n}{n!} = e^{\frac{a}{z}}$$

PART-B

11(a) (i). Obtain the Fourier series of the periodic function defined by

$$f(x) = \begin{cases} -\pi & in -\pi < x < 0 \\ x & in\ 0 < x < \pi \end{cases}$$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Here $f(x)$ is neither even nor odd.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\ dx + \int_{0}^{\pi} x\ dx\right] = \frac{1}{\pi}\left\{[-\pi x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\ dx + \int_{0}^{\pi} x\ \cos nx\ dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(-\pi)\left(\frac{\sin nx}{n}\right)\right]_{-\pi}^{0} + \left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left(\frac{\cos nx}{n^2}\right)_0^{\pi} = \frac{1}{\pi n^2}[(-)^n - 1]$$

$$\boldsymbol{a_n = \begin{cases} 0\ , & when\ n\ is\ even \\ -\dfrac{2}{\pi n^2}, & when\ n\ is\ odd \end{cases}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\ dx + \int_{0}^{\pi} x\ \sin nx\ dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(-\pi)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$\boldsymbol{b_n = \frac{1}{n}[1 - 2(-1)^n]}$$

The Fourier series is given by

$$\boldsymbol{f(x) = -\frac{\pi}{4} + \sum_{n=1,3,5}^{\infty} -\frac{2}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}[1 - 2(-1)^n]\sin nx \longrightarrow (1)}$$

Deduction :

Here $x=0,$ mid point is a point of discontinuity. Put $x=0$ in (1)

$$\frac{f(0+)+f(0-)}{2}=-\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

$$\left(-\frac{\pi}{2}+\frac{\pi}{4}\right)\times\frac{-\pi}{2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$\mathbf{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}}$$

11.(a).(ii). Compute upto first harmonics of the Fourier series of $f(x)$ given by the following table.

X	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

Refer Chennai-Nov./Dec.2011, Q.No. 11 (b) (ii)

11.(b)(i). Expand $f(x)=x-x^2$ as a Fourier series in $-L<x<L$ and using this series find the root mean square value of $f(x)$ in the interval.

The Fourier series of $f(x)$ in $(0,2l)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$f(x)$ is neither even nor odd.

$$a_0=\frac{1}{l}\int_{-l}^{l}(x-x^2)dx=\frac{1}{l}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-l}^{l}=\frac{1}{l}\left[\frac{l^2}{2}-\frac{l^3}{3}-\frac{l^2}{2}-\frac{l^3}{3}\right]=-\frac{\mathbf{2l^2}}{\mathbf{3}}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\,cos\frac{n\pi x}{l}\,dx=\frac{1}{l}\int_{l}^{l}(x-x^2)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{l}\left[(x-x^2)\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right)-(1-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)+(-2)\left(-\frac{\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[(1-2x)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{-l}^{l}=\frac{1}{l}\left[\frac{(1-2l)l^2(-1)^n}{n^2\pi^2}-\frac{(1+2l)l^2(-1)^n}{n^2\pi^2}\right]$$

$$\mathbf{a_n=\frac{-4l^2(-1)^n}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_{l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{l}^{l}(x-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)-(1-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[-(x-x^2)\left(\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right)-2\left(\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left(-\frac{(l-l^2)l(-1)^n}{n\pi}-\frac{2l^3(-1)^n}{n^3\pi^3}\right)-\left(-\frac{(-l-l^2)l(-1)^n}{n\pi}-\frac{2l^3(-1)^n}{n^3\pi^3}\right)\right]$$

$$\boldsymbol{b_n = -\frac{2l}{n\pi}(-1)^n}$$

$$\boldsymbol{f(x) = -\frac{l^2}{3}+\sum_{n=1}^{\infty}\left(\frac{-4l^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{l}-\frac{2l}{n\pi}(-1)^n\sin\frac{n\pi x}{l}\right)}$$

11.(b)(ii). Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$.

The complex form of the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,e^{\frac{-in\pi x}{l}}dx = \frac{1}{2}\int_{-1}^{1} e^{-x}\,e^{-in\pi x}dx = \frac{1}{2}\int_{-1}^{1} e^{-(in\pi+1)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(in\pi+1)x}}{-(in\pi+1)}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{e^{-(in\pi+1)}}{-(in\pi+1)}-\frac{e^{(in\pi+1)}}{-(in\pi+1)}\right] = \frac{1}{2(in\pi+1)}\left[-e^{-1}e^{-in\pi}+e^{1}e^{in\pi}\right]$$

$$= \frac{1}{2(in\pi+1)}\left[-e^{-1}(\cos n\pi - i\sin n\pi)+e^{1}(\cos n\pi + i\sin n\pi)\right]$$

$$= \frac{1}{2(in\pi+1)}\left[-e^{-1}(\cos n\pi)+e^{1}(\cos n\pi)\right] = \frac{(-1)^n \sin h1}{(in\pi+1)}$$

$$\boldsymbol{f(x) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n \sin h1}{(in\pi+1)}e^{in\pi x}}$$

12.(a)(i). Find the Fourier transform of

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$ **And hence find the value of $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$**

Refer Chennai-April/May.2010, Q.No. 12 (a) (ii)

12.(a)(ii). Evaluate $\int_0^\infty \frac{dx}{(4+x^2)(25+x^2)}$ using transforms.

We know that Fourier cosine transform of $f(x) = e^{-2x}$ is $\sqrt{\frac{2}{\pi}}\frac{2}{s^2+4}$ and

Fourier cosine transform of $f(x) = e^{-5x}$ is $\sqrt{\frac{2}{\pi}}\frac{5}{s^2+25}$

$$\int_0^\infty F_C[f(x)].F_c[g(x)]\,ds = \int_0^\infty f(x)\,g(x)\,dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{2}{s^2+4}.\sqrt{\frac{2}{\pi}}\frac{5}{s^2+25}\,ds = \int_0^\infty e^{-2x}\,e^{-5x}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{10}{(s^2+4)(s^2+25)}\,ds = \int_0^\infty e^{-7x}\,dx = \left[\frac{e^{-7x}}{-7}\right]_0^\infty = \frac{1}{7}$$

$$\int_0^\infty \frac{10}{(x^2+4)(x^2+25)}\,ds = \frac{\pi}{140}\ (since\ s\ is\ a\ dummy\ variable)$$

12(b)(i). Find the Fourier cosine transform of e^{-x^2}.

The Fourier sine transform of $f(x)$ is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = R.P.of\sqrt{\frac{2}{\pi}}\ 2\ \frac{1}{2}\int_0^\infty e^{-x^2}\,e^{isx}\,dx$$

$$= R.P.of\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}R.P.of\int_{-\infty}^{\infty} e^{-x^2}\,e^{isx}e^{-\frac{s^2}{4}}\ e^{\frac{s^2}{4}}\,dx$$

$$= \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}R.P.of\int_{-\infty}^{\infty} e^{-x^2}\,e^{isx}\,e^{\frac{s^2}{4}}\,dx = \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}R.P.of\int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2}\,dx$$

$$put\ x-\frac{is}{2} = y\ and\ dx = dy$$

$$= \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}R.P.of\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}R.P.of\ 2\int_0^\infty e^{-y^2}\,dy = \frac{e^{-\frac{s^2}{4}}}{\sqrt{2\pi}}R.P.of\ 2\frac{\sqrt{\pi}}{2}$$

$$\boldsymbol{F_c\left[e^{-x^2}\right] = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}}$$

12(b)(ii). Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

From Gamma function

$$\Gamma_n = \int_0^\infty e^{-x}\ x^{n-1}\,dx,\quad n > 0$$

Put $x = at\,,\ dx = a\,dt$

$$\Gamma_n = \int_0^\infty e^{-at}\ (at)^{n-1}\,dx = a^n\int_0^\infty e^{-at}\ t^{n-1}\,dx$$

$$\int_0^\infty e^{-ax}\ x^{n-1}\,dx = \frac{\Gamma_n}{a^n}$$

Put $a = is$, then

$$\int_0^\infty e^{-isx}\ x^{n-1}\,dx = \frac{\Gamma_n}{(is)^n} = \frac{\Gamma_n}{i^n s^n} = \frac{\Gamma_n}{(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^n s^n}$$

$$\int_0^\infty (\cos sx - i\sin sx)\ x^{n-1}\,dx = \frac{(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^{-n}\ \Gamma_n}{s^n} = \frac{(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^{n}\ \Gamma_n}{s^n}$$

Equating the real and imaginary parts,

$$\int_0^\infty \cos sx\ x^{n-1}\,dx = \frac{\cos\frac{n\pi}{2}\ \Gamma_n}{s^n} ----(1)$$

$$\int_0^\infty \sin sx\ x^{n-1}\,dx = \frac{\sin\frac{n\pi}{2}\ \Gamma_n}{s^n} ----(2)$$

$$From\ (1),\quad \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\ x^{n-1}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\cos\frac{n\pi}{2}\ \Gamma_n}{s^n}$$

$$\boldsymbol{F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\cos\frac{n\pi}{2}\ \Gamma_n}{s^n}}$$

$$From\ (1),\quad \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\ x^{n-1}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\sin\frac{n\pi}{2}\ \Gamma_n}{s^n}$$

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\sin\frac{n\pi}{2}\;\Gamma_n}{s^n}$$

Putting $n = \frac{1}{2}$, we get

$$F_c\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\cos\frac{\pi}{4}\,\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{2}\,s^{\frac{1}{2}}}$$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

$$F_s\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sin\frac{\pi}{4}\,\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{2}\,s^{\frac{1}{2}}}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

14(a). A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a initial velocity $3x\,(l - x)$, find the displacement.

The governing equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

i. $y(0,t) = 0\ for\ all\ t > 0$
ii. $y(l,t) = 0\ for\ all\ t > 0$
iii. $y(x,0) = 0\ for\ all\ x\ in\ (0,l)$
iv. $\frac{\partial y\,(x,0)}{\partial x} = 3x(l - x) for\ all\ x\ in\ (0,l)$

The correct solution is $u(x,y) = (A\cos px + B\sin px\,)(C\cos pat + B\sin pat\,)$

Applying the first three boundary conditions, the solution becomes

$$y(x,t) = (\quad)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty}(\quad)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

Partially differentiating with respect to t,

$$\frac{\partial y(x,t)}{\partial x} = \sum_{n=1}^{\infty} (\quad) \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

Putting t= 0,

$$\frac{\partial y(x,0)}{\partial x} = \sum_{n=1}^{\infty} (\quad) \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} = 3x(l-x)$$

From half-range sine series,

$$3x(l-x) = f(x) = \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{l}\right)$$

$$\sum_{n=1}^{\infty} (\quad) \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} = \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{l}\right)$$

$$(\quad) \frac{n\pi a}{l} = b_n \quad => \quad (\quad) = b_n \frac{l}{n\pi a}$$

$$b_n = \frac{2}{l}\int_0^l 3x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{l}\left[(3lx-3x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (3l-6x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-6)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{6}{l}\left[(-6)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l = \frac{12l^2}{n^3\pi^3}[1-(-1)^n]$$

$$\boldsymbol{b_n = \begin{cases} \dfrac{24l^2}{n^3\pi^3}, & n \text{ is odd} \\ 0 \quad , & n \text{ is even} \end{cases}}$$

$$(\quad) = \frac{24l^2}{n^3\pi^3}\frac{l}{n\pi a}, when\, n\, is\, odd$$

$$\boldsymbol{\therefore y(x,t) = \sum_{n=1,3,5}^{\infty} \left(\frac{24l^3}{an^4\pi^4}\right)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}}$$

13(b). A rod, 30 cm long has its ends A and B kept at $20℃$ *and* $80℃$ respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0℃$ and kept so. Find the resulting temperature distribution function $u(x,t)$ *taking* $x = 0$ *at* A.

The governing equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x}$

In steady state, $\frac{\partial u}{\partial t} = 0$ and $\frac{d^2 u}{dx^2} = 0$

The steady state solution is $u(x) = Ax + B$

Boundary conditions:

$$(i) u(0) = 20 \qquad (ii) u(l) = 80 \ , l = 30\ cm$$

Applying condition (i), $u(0) = B = 20 \ and \ u(x) = Bx + 20$

Applying (ii) condition, $u(l) = Al + 20 = 80 \ \ => \ \ A = \frac{60}{l}$

$$\therefore \boldsymbol{u(x) = \frac{60x}{l} + 20}$$

The temperature distribution reached at the steady state becomes initial temperature distribution for the unsteady state.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x}$$

Boundary condition:

i. $u(0,t) = 0 \ for \ all \ t > 0$
ii. $u(l,t) = 0 \ for \ all \ t > 0$
iii. $u(x,0) = \frac{60x}{l} + 20 \ \ for \ all \ x \ in \ (0,l)$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, e^{-\left(\frac{\alpha^2 n^2 \pi^2 t}{l^2}\right)}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{60x}{l} + 20$$

From the half- range sine series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{60x}{l} + 20 \ => B_n = b_n$$

$$b_n = \frac{2}{l} \int_0^l \left(\frac{60x}{l} + 20\right) \sin\frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\left[\left(\frac{60x}{l}+20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(\frac{60}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$b_n = B_n = \frac{40}{n\pi}[1-4(-1)^n]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}[1-4(-1)^n]\ \sin\frac{n\pi x}{l}\ e^{-\left(\frac{\alpha^2 n^2 \pi^2 t}{l^2}\right)}$$

14(a)(i).To find the inverse Z- transform of $\frac{10z}{z^2-3z+2}$

To find $Z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

$$F(z) = \frac{10z}{z^2-3z+2} \quad => \frac{F(z)}{10\,z} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2)+B(z-1)$$

$$z = 1 \quad => \quad \boldsymbol{A = -1} \quad and \quad z = 2 \quad => \quad \boldsymbol{B = 1}$$

$$\frac{F(z)}{10\,z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2} = \frac{-1}{z-1}+\frac{1}{z-2}$$

$$F(z) = \frac{-10z}{z-1}+\frac{10z}{z-2}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{-10z}{z-1}\right]+Z^{-1}\left[\frac{10z}{z-2}\right] = -10(1^n)+10(2^n)$$

$$\therefore \boldsymbol{Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1^n)+10(2^n)}$$

14(a)(ii).Solve by Z-transform $\boldsymbol{u_{n+2}+6\,u_{n+1}\,+9u_n = 2^n}$ with $\boldsymbol{u_0 = u_1 = 0}$.

Given $u_{n+2}+6\,u_{n+1}\,+9u_n = 2^n$

Taking Z- transform, we get

$$Z[u_{n+2}]+6Z[\,u_{n+1}\,]+9Z[u_n] = Z[2^n]$$

$$z^2\bar{u}-z^2u_0-zu_1+6z(\bar{u}-u_0)+9\bar{u} = \frac{z}{z-2}$$

$$\bar{u}\,(z^2+6z+9) = \frac{z}{z-2}$$

$$\bar{u} = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{u}}{z} = \frac{1}{(z-2)(z+3)^2}$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z+3)$$

$$z = 2 \quad => \quad A = \frac{1}{25} \quad and\, z = -3 \quad => \quad C = -\frac{1}{5}$$

Equating coefficient of z^2, $\quad A + B = 0 \quad => B = \frac{-1}{25}$

$$\frac{1}{(z-2)(z+3)^2} = \frac{\frac{1}{25}}{z-2} + \frac{\frac{-1}{25}}{z+3} + \frac{-\frac{1}{5}}{(z+3)^2}$$

$$\therefore \frac{\bar{u}}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$z[u_n] = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

Taking Z^{-1} on both sides, we get

$$u_n = Z^{-1}\left[\frac{1}{25}\frac{z}{z-2}\right] - Z^{-1}\left[\frac{1}{25}\frac{z}{z+3}\right] - Z^{-1}\left[\frac{1}{5}\frac{z}{(z+3)^2}\right]$$

$$\boldsymbol{u_n = \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}}$$

14(b)(i). Using convolution theorem, find the $Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{(z-4)}\;\frac{z}{(z-3)}\right]$$

$$Z^{-1}\left[\frac{z}{(z-4)}\right] * Z^{-1}\left[\frac{z}{(z-3)}\right] = 4^n * 3^n$$

$$= \sum_{r=0}^{n} 4^r 3^{n-1} = 3^n \sum_{r=0}^{n} 4^r\, 3^{-1}$$

$$= 3^n \sum_{r=0}^{n} \left(\frac{4}{3}\right)^r = 3^n[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots + \left(\frac{4}{3}\right)^n$$

$$= 3^n \left[\frac{\left(\frac{4}{3}\right)^{n+1} - 1}{\frac{4}{3} - 1} \right]$$

$$= 3^n\, 3 \left[\frac{4^{n+1} - 3^{n+1}}{3^{n+1}} \right]$$

$$\boldsymbol{Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = 4^{n+1} - 3^{n+1}}$$

14(b)(ii). Find the inverse Z-transform of $\frac{z^3-20z}{(z-2)^3(z-4)}$

$$F(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$$

$$\frac{F(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)}$$

$$\frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{z-4}$$

$$z^2 - 20 = A(z-4)(z-2)^2 + B(z-4)(z-2) + C(z-4) + D(z-2)^3$$

$$z = 4 \quad => \quad \boldsymbol{D = -\frac{1}{2}} \quad and \quad z = 2 \quad => \quad \boldsymbol{C = 8}$$

Equating the coefficient of z^3. $\quad A + D = 0 \quad => \quad \boldsymbol{A = \frac{1}{2}}$

$$z = 0 \quad => \quad -20 = -16A + 8B - 4C - 8D \quad => \quad \boldsymbol{B = 2}$$

$$\frac{F(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{1}{2}\frac{1}{z-2} + \frac{2}{(z-2)^2} + \frac{8}{(z-2)^3} - \frac{1}{2}\frac{1}{z-4}$$

$$F(z) = \frac{1}{2}\frac{z}{z-2} + \frac{z}{(z-2)^2} + \frac{z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

$$Z^{-1}[F(z)] = Z^{-1}\frac{1}{2}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^2}\right] + Z^{-1}\left[\frac{z}{(z-2)^3}\right] - Z^{-1}\left[\frac{1}{2}\frac{z}{z-4}\right]$$

$$Z^{-1}[F(z)] = \frac{1}{2}Z^{-1}\left[\frac{2z^2 + 4z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^3}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-4}\right]$$

$$\boldsymbol{z^{-1}\left[\frac{z^3 - 20z}{(z-2)^3(z-4)}\right] = \frac{1}{2}\, 2^n + n^2\, 2^n - \frac{1}{2} 4^n}$$

15(a)(i). Solve $\boldsymbol{z = px + qy + p^2 q^2}$

This is of the form $z = px + qy + f(p, q)$

The complete integral is $z = ax + by + a^2b^2$

<u>To find singular integral:</u>

$$\frac{\partial z}{\partial a} = x + 2ab^2 = 0 \quad => \quad x = -2ab^2$$

$$\frac{\partial z}{\partial b} = y + 2ba^2 = 0 \quad => \quad y = -2ba^2$$

Putting $a = -\frac{x}{2b^2}, \; we\ get\ b = -\left(\frac{x^2}{2y}\right)^{\frac{1}{3}} \; and\ hence\ a = -\left(\frac{y^2}{2x}\right)^{\frac{1}{3}}$

$$z = -x\left(\frac{y^2}{2x}\right)^{\frac{1}{3}} - y\left(\frac{x^2}{2y}\right)^{\frac{1}{3}} + \left(\frac{x^2y^2}{16}\right)^{\frac{1}{3}} = -\frac{3}{4}\,4^{\frac{1}{3}}\,x^{\frac{2}{3}}\,y^{\frac{2}{3}}$$

$$z^3 = -\frac{27}{16}\,x^2y^2$$

$$\mathbf{27x^2y^2 + 16z^3 = 0}$$

15(a) (ii). Solve $\left(D^2 + 2DD' + D'^2\right)z = \sin h(x+y) + e^{x+2y}$

A.E. is$m^2 + 2m + 1 = 0$

$$m = -1, -1$$

$$C..F. = f_1(y-x) + xf_2(y-x)$$

$$P.I_1. = \frac{1}{(D+D')^2}\sin h(x+y)\,. = \frac{1}{(D+D')^2}\left(\frac{e^{x+y} - e^{-(x+y)}}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{(D+D')^2}e^{x+y} - \frac{1}{(D+D')^2}e^{-x-y}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{(1+1)^2}e^{x+y} - \frac{1}{(-1-1)^2}e^{-x-y}\right\}$$

$$= \frac{1}{8}\left[e^{x+y} - e^{-(x+y)}\right]$$

$$P.I_1. = \frac{1}{4}\sin h(x+y)$$

$$P.I_2. = \frac{1}{(D+D')^2}e^{x+2y}$$

$$= \frac{1}{(1+4+4)^2}e^{x+2y} = \frac{1}{9}\,e^{x+2y}$$

The complete solution is $z = C.F. + P.I.$

$$z = f_1(y-x) + xf_2(y-x) + \frac{1}{4}\sinh(x+y) + \frac{1}{9}\, e^{x+2y}$$

15(b)(i). Solve $(\boldsymbol{y - xz})\boldsymbol{p} + (\boldsymbol{yz - x})\boldsymbol{q} = (\boldsymbol{x+y})(\boldsymbol{x-y})$

The subsidiary equations are

$$\frac{dx}{y - xz} = \frac{dy}{yz - x} = \frac{dz}{x^2 - y^2}$$

Choosing the multipliers as x, y, z

$$\frac{xdx}{x(y-xz)} = \frac{ydy}{y(yz-x)} = \frac{zdz}{z(x^2-y^2)} = \frac{xdx + ydy + zdz}{xy - x^2z + y^2z - xy + zx^2 - zy^2}$$

$$xdx + ydy + zdz = 0 \quad => \quad u = x^2 + y^2 + z^2 = c_1$$

Choosing the multipliers as $y, x, 1$, each ratios is equal to

$$= \frac{ydx + xdy + dz}{y^2 - xyz + xyz - x^2 + x^2 - y^2}$$

$$ydx + xdy + dz = 0 \;\; and \;\; d(xy) + d(z) = 0$$

$$v = xy + z = c_2$$

The required solution is $\phi(u, v) = 0$

$$\boldsymbol{i.e., \phi\left(x^2 + y^2 + z^2, xy + z\right) = 0}$$