Solve the difference equation

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \text{ given that } y(0) = 4, y(1) = 0 \text{ and } y(2) = 8.$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+3) - 3y(n+1) + 2y(n) = 0 \ldots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+3) - 3y(n+1) + 2y(n)\} = Z\{0\}$$

$$Z\{y(n+3)\} - 3Z\{y(n+1)\} + 2Z\{y(n)\} = 0 \ldots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+3)\} = z^3F(z) - z^3y(0) - z^2y(1) - zy(2) \ldots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$z^3F(z) - z^3y(0) - z^2y(1) - zy(2) - 3\big(zF(z) - zy(0)\big) + 2F(z) = 0$$

$$z^3F(z) - 4z^3 - z^2(0) - 8z - 3(zF(z) - 4z) + 2F(z) = 0$$

$$(z^3 - 3z + 2)F(z) - 4z^3 - 8z + 12z = 0$$

$$(z^3 - 3z + 2)F(z) = 4z^3 - 4z$$

$$F(z) = Z\{y(n)\} = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$y(n) = Z^{-1}\left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right]$$

$$Let\ F(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$z^{n-1}F(z) = z^{n-1}\frac{4z^3 - 4z}{z^3 - 3z + 2} = \frac{4z^n(z^2 - 1)}{z^3 - 3z + 2}$$

To find poles:

$$z^3 - 3z + 2 = 0 \Rightarrow z = -2,1,1$$

The poles are $z = -2,1$

$$z^{n-1}F(z) = \frac{4z^n(z^2 - 1)}{(z-1)^2(z+2)} = \frac{4z^n(z+1)(z-1)}{(z-1)^2(z+2)}$$

$$z^{n-1}F(z) = \frac{4z^n(z+1)}{(z-1)(z+2)}$$

Residue at the pole $z = 1$ of order one $R_1$

$$= \lim_{z\to 1}(z-1)z^{n-1}F(z)$$

$$= \lim_{z\to 1}(z-1)\frac{4z^n(z+1)}{(z-1)(z+2)}$$

$$= \lim_{z\to 1}\frac{4z^n(z+1)}{(z+2)} = \frac{4(1)^n(1+1)}{(1+2)} = \frac{8}{3}$$

Residue at the pole $z = -2$ of order one $R_2$

$$= \lim_{z\to -2}(z+2)z^{n-1}F(z)$$

$$= \lim_{z\to -2}(z+2)\frac{4z^n(z+1)}{(z-1)(z+2)}$$

$$= \lim_{z\to -2}\frac{4z^n(z+1)}{(z-1)}$$

$$= \frac{4(-2)^n(-2+1)}{(-2-1)} = \frac{4(-2)^n}{3}$$

$$y(n) = Z^{-1}\left[\frac{4z^3-4z}{z^3-3z+2}\right] = Sum\ of\ all\ residues$$

$$y(n) = \frac{8}{3} + \frac{4(-2)^n}{3}$$

Solve the difference equation

$$y(n+2) + 6y(n+1) + 9y(n) = 2^n\ given\ that\ y(0) = y(1) = 0$$

Solution:

Let $Z\{y(n)\} = F(z)$

$$y(n+2) + 6y(n+1) + 9y(n) = 2^n \ldots (1)$$

Taking Z-transform on both sides in (1), we get

$$Z\{y(n+2) + 6y(n+1) + 9y(n)\} = Z\{2^n\}$$

$$Z\{y(n+2)\} + 6Z\{y(n+1)\} + 9Z\{y(n)\} = \frac{z}{z-2} \ldots (2) \qquad [By\ linear\ property]$$

$$Z\{y(n+2)\} = z^2F(z) - z^2y(0) - zy(1) \ldots (3)$$

$$Z\{y(n+1)\} = zF(z) - zy(0) \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$z^2F(z) - z^2y(0) - zy(1) + 6[zF(z) - zy(0)] + 9F(z) = \frac{z}{z-2}$$

$$z^2F(z) + 6[zF(z)] + 9F(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)F(z) = \frac{z}{z-2}$$

$$F(z) = Z\{y(n)\} = \frac{z}{(z^2 + 6z + 9)(z-2)}$$

$$y(n) = Z^{-1}\left[\frac{z}{(z+3)^2(z-2)}\right]$$

$$Let\ F(z) = \frac{z}{(z+3)^2(z-2)}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z}{(z+3)^2(z-2)} = \frac{z^n}{(z+3)^2(z-2)}$$

To find poles:

$$(z+3)^2(z-2) = 0 \Rightarrow z = -3, -3, 2$$

The poles are $z = -3, 2$

Residue at the pole $z = -3$ of order two $R_1$

$$= \lim_{z \to -3} \frac{d}{dz}(z+3)^2 z^{n-1}F(z)$$

$$= \lim_{z \to -3} \frac{d}{dz}(z+3)^2 \frac{z^n}{(z+3)^2(z-2)}$$

$$= \lim_{z \to -3} \frac{d}{dz}\frac{z^n}{(z-2)}$$

$$= \lim_{z \to -3} \frac{(z-2)nz^{n-1} - z^n}{(z-2)}$$

$$= \frac{(-3-2)n(-3)^{n-1} - (-3)^n}{(-3-2)}$$

$$= \frac{-5n(-3)^{n-1} - (-3)^n}{-5} = \frac{5n(-3)^{n-1} + (-3)^n}{5}$$

Residue at the pole $z = 2$ of order one $R_2$

$$= \lim_{z \to 2} (z-2) z^{n-1} F(z)$$

$$= \lim_{z \to 2} (z-2) \frac{z^n}{(z+3)^2(z-2)}$$

$$= \lim_{z \to 2} \frac{z^n}{(z+3)^2} = \frac{2^n}{(2+3)^2} = \frac{2^n}{25}$$

$$y(n) = Z^{-1}\left[\frac{4z^3 - 4z}{z^3 - 3z + 2}\right] = Sum\ of\ all\ residues$$

$$y(n) = \frac{5n(-3)^{n-1} + (-3)^n}{5} + \frac{2^n}{25}$$

$Find\ Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

Solution:

$$Let\ F(z) = \frac{z}{(z+1)(z+2)}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z}{(z+1)(z+2)} = \frac{z^n}{(z+1)(z+2)}$$

To find poles:

$$(z+1)(z+2) = 0 \Rightarrow z = -1, -2$$

The poles are $z = -1, -2$

Residue at the pole $z = -1$ of order one $R_1$

$$= \lim_{z \to -1} (z+1) z^{n-1} F(z)$$

$$= \lim_{z \to -1} (z+1) \frac{z^n}{(z+1)(z+2)}$$

$$= \lim_{z \to -1} \frac{z^n}{(z+2)} = \frac{(-1)^n}{-1+2} = (-1)^n$$

Residue at the pole $z = -2$ of order one $R_2$

$$= \lim_{z \to -2} (z+2) z^{n-1} F(z)$$

$$= \lim_{z \to -2} (z+2) \frac{z^n}{(z+1)(z+2)}$$

$$= \lim_{z \to -2} \frac{z^n}{(z+1)} = \frac{(-2)^n}{-2+1} = -(-2)^n$$

$$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = (-1)^n - (-2)^n$$

$Find\ Z^{-1}\left[\frac{10z}{z^2 - 3z + 2}\right]$

Solution:

$$Let\ F(z) = \frac{10z}{z^2 - 3z + 2}$$

$$z^{n-1}F(z) = z^{n-1}\frac{10z}{z^2 - 3z + 2} = \frac{10z^n}{(z-1)(z-2)}$$

To find poles:

$$(z-1)(z-2) = 0 \Rightarrow z = 1,2$$

The poles are $z = 1,2$

Residue at the pole $z = 1$ of order one $R_1$

$$= \lim_{z \to 1} (z-1) z^{n-1} F(z)$$

$$= \lim_{z \to 1} (z-1) \frac{10z^n}{(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{10z^n}{(z-2)} = \frac{10(1)^n}{1-2} = -10$$

Residue at the pole $z = 2$ of order one $R_2$

$$= \lim_{z \to 2} (z-2) z^{n-1} F(z)$$

$$= \lim_{z \to 2} (z-2) \frac{10z^n}{(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{10z^n}{(z-1)} = \frac{10(2)^n}{2-1} = 10(2)^n$$

$$Z^{-1}\left[\frac{10z}{z^2 - 3z + 2}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{10z}{z^2 - 3z + 2}\right] = -10 + 10(2)^n$$

$$Find\ Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

Solution:

$$Let\ F(z) = \frac{z^2}{(z-a)(z-b)}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z^2}{(z-a)(z-b)} = \frac{z^{n+1}}{(z-a)(z-b)}$$

To find poles:

$$(z-a)(z-b) = 0 \Rightarrow z = a, b$$

The poles are $z = a, b$

Residue at the pole $z = a$ of order one $R_1$

$$= \lim_{z \to a}(z-a)z^{n-1}F(z)$$

$$= \lim_{z \to a}(z-a)\frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \lim_{z \to a}\frac{z^{n+1}}{(z-b)} = \frac{a^{n+1}}{a-b}$$

Residue at the pole $z = 2$ of order one $R_2$

$$= \lim_{z \to b}(z-a)z^{n-1}F(z)$$

$$= \lim_{z \to b}(z-b)\frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \lim_{z \to b}\frac{z^{n+1}}{(z-a)} = \frac{b^{n+1}}{b-a}$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Sum\ of\ all\ residues$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}$$

$$Find\ Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]\ using\ convolution\ theorem.$$

Solution:

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right]$$

$$Let\ F(z) = \frac{z}{\left(z-\frac{1}{2}\right)}\ and\ G(z) = \frac{z}{\left(z+\frac{1}{4}\right)}$$

$$f(n) = Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] = \left(\frac{1}{2}\right)^n$$

$$g(n) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{z}{\left(z+\frac{1}{4}\right)}\right] = \left(-\frac{1}{4}\right)^n$$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}[F(z)G(z)] = f(n) * g(n)$$

$$= \sum_{k=0}^{n} f(k)g(n-k)$$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{4}\right)^{n-k} = \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{4}\right)^n\left(-\frac{1}{4}\right)^{-k}$$

$$= \left(-\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{4}\right)^{-k} = \left(-\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}(-4)\right)^k = \left(-\frac{1}{4}\right)^n\sum_{k=0}^{n}(-2)^k$$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = \left(-\frac{1}{4}\right)^n\ [1+(-2)+(-2)^2+(-2)^3+\cdots+(-2)^n]$$

$$= \left(-\frac{1}{4}\right)^n\frac{((-2)^{n+1}-1)}{-2-1}$$

$$= \left(-\frac{1}{4}\right)^n \frac{((-2)^{n+1} - 1)}{-3} = \frac{1}{3}\left(-\frac{1}{4}\right)^n - \frac{1}{3}\left(-2\left(-\frac{2}{-4}\right)^n\right)$$

$$= \frac{1}{3}\left(-\frac{1}{4}\right)^n + \frac{2}{3}\left(\frac{1}{2}\right)^n$$

*Find* $Z^{-1}\left[\frac{6z^2}{(2z-1)(3z+1)}\right]$ *using convolution theorem.*

Solution:

$$Z^{-1}\left[\frac{6z^2}{(2z-1)(3z+1)}\right] = Z^{-1}\left[\frac{6z^2}{2\left(z-\frac{1}{2}\right)3\left(z+\frac{1}{3}\right)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}\right]$$

*Let* $F(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$ *and* $G(z) = \frac{z}{\left(z+\frac{1}{3}\right)}$

$$f(n) = Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] = \left(\frac{1}{2}\right)^n$$

$$g(n) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{z}{\left(z+\frac{1}{3}\right)}\right] = \left(-\frac{1}{3}\right)^n$$

$$Z^{-1}\left[\frac{6z^2}{(2z-1)(3z+1)}\right] = Z^{-1}[F(z)G(z)] = f(n) * g(n)$$

$$= \sum_{k=0}^{n} f(k)g(n-k)$$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{3}\right)^{n-k} = \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{3}\right)^n\left(-\frac{1}{3}\right)^{-k}$$

$$= \left(-\frac{1}{3}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{3}\right)^{-k} = \left(-\frac{1}{3}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}(-3)\right)^k = \left(-\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(-\frac{3}{2}\right)^k$$

$$Z^{-1}\left[\frac{6z^2}{(2z-1)(3z+1)}\right] = \left(-\frac{1}{3}\right)^n\left[1 + \left(-\frac{3}{2}\right) + \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^3 + \cdots + \left(-\frac{3}{2}\right)^n\right]$$

$$= \left(-\frac{1}{3}\right)^n \frac{\left(\left(-\frac{3}{2}\right)^{n+1} - 1\right)}{\left(-\frac{3}{2}\right) - 1}$$

$$= \left(-\frac{1}{3}\right)^n \frac{\left(\left(-\frac{3}{2}\right)^{n+1} - 1\right)}{\left(-\frac{5}{2}\right)} = \frac{2}{5}\left(-\frac{1}{3}\right)^n - \frac{2}{5}\left(-\frac{3}{2}\left(-\frac{3}{2} \times -\frac{1}{3}\right)^n\right)$$

$$= \frac{2}{5}\left(-\frac{1}{3}\right)^n + \frac{3}{5}\left(\frac{1}{2}\right)^n$$

State and prove second shifting theorem in Z-transform:

Statement:

$Let\ F(z) = Z\{f(n)\}$ then

$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - z^{-1}f(1) - z^{-2}f(2) - \cdots - f(k-1)z^{-(k-1)}\right]$

Proof:

$Z\{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)\, z^{-n}$ ...(1)

Multiplying and dividing by $z^k$ in R.H.S of (1), we get

$$= z^k \sum_{n=0}^{\infty} f(n+k)\, z^{-(n+k)}$$

$$= z^k\left(f(k)z^{-(k)} + f(k+1)z^{-(k+1)} + f(k+2)z^{-(k+2)} + \cdots\right)$$

$$= z^k\left(\sum_{n=0}^{\infty} f(n)\, z^{-n} - f(0) - f(1)z^{-1} - f(2)z^{-2} - \cdots - f(k-1)z^{-(k-1)}\right)$$

$$\left[Since \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + f(1)z^{-1} \ldots + f(k-1)z^{-(k-1)} + f(k)z^{-(k)} + \cdots\right]$$

$$Z\{f(n+k)\} = z^k\left[F(z) - f(0) - z^{-1}f(1) - z^{-2}f(2) - \cdots - f(k-1)z^{-(k-1)}\right]$$

Solve $x\,(y^2 - z^2)\,p + y\,(z^2 - x^2)q = z(x^2 - y^2)$.

Solution:

$$x\,(z^2 - y^2)\,p + y\,(x^2 - z^2)q = z(y^2 - x^2).$$

The equation is of the form $Pp + Qq = R$

Here $P = x\,(z^2 - y^2)\,, Q = y\,(x^2 - z^2), R = z(y^2 - x^2)$

$$\frac{dx}{x\,(z^2 - y^2)} = \frac{dy}{y\,(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy}{z^2 - y^2 + x^2 - z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy}{-(y^2 - x^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{1}{x}dx + \frac{1}{y}dy = -\frac{dz}{z}$$

Integrating on both sides, we get

$$\log x + \log y = -\log z + \log c_1$$

$$\log x + \log y + \log z = \log c_1$$

$$\log xyz = \log c_1 \Rightarrow c_1 = xyz$$

$$\frac{xdx + ydy}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{xdx + ydy}{x^2z^2 - y^2z^2} = \frac{dz}{z(y^2 - x^2)}$$

$$\frac{xdx + ydy}{-z^2(y^2 - x^2)} = \frac{dz}{z(y^2 - x^2)}$$

$$xdx + ydy = -zdz$$

Integrating on both sides, we get

$\frac{x^2}{2} + \frac{y^2}{2} = -\frac{z^2}{2} + c_2 \Rightarrow c_3 = x^2 + y^2 + z^2$

The general solution is

$$\Phi(c_1, c_3) = 0$$

$$\Phi(xyz, x^2 + y^2 + z^2) = 0$$

Solve $(mz - ny)\,p + (nx - lz)q = (ly - mx)$.

Solution:

The equation is of the form $Pp + Qq = R$

Here $P = mz - ny\,, Q = nx - lz, R = ly - mx$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

$$\frac{ldx + mdy + ndz}{lmz - lny + mnx - lmz + lny - mnx} = \frac{ldx + mdy + ndz}{0}$$

$$ldx + mdy + ndz = 0$$

Integrating on both sides, we get

$$lx + my + nz = c_1$$

$$\frac{xdx + ydy + zdz}{mxz - nxy + nyx - lyz + lzy - mzx} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

Integrating on both sides, we get

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 \Rightarrow c_3 = x^2 + y^2 + z^2$

The general solution is

$$\Phi(c_1, c_3\ ) = 0$$

$$\Phi(lx + my + nz, x^2 + y^2 + z^2\ ) = 0$$

Solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(2x + y)$

Solution:

To find complementary function:

Put $D = m \ and \ D' = 1$,

$$m^3 - 7m - 6 = 0 \Rightarrow m^3 - m - 6m - 6 = 0$$

$$m(m^2 - 1) - 6(m + 1) = 0 \Rightarrow m(m + 1)(m - 1) - 6(m + 1) = 0$$

$$(m + 1)(m^2 - m - 6) = 0$$

$$m = -1, -2, 3$$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x)$$

To find Particular Integral:

$$P.I = \frac{\sin(2x+y)}{D^3 - 7DD'^2 - 6D'^3}$$

$$= \frac{\sin(2x+y)}{-4D - 7D(-1) - 6(-1)D'} = \frac{\sin(2x+y)}{-4D + 7D + 6D'}$$

$$= \frac{\sin(2x+y)}{3D + 6D'} = \frac{(3D - 6D')\sin(2x+y)}{(3D + 6D')(3D - 6D')}$$

$$= \frac{(3D\sin(2x+y) - 6D'\sin(2x+y))}{9D^2 - 36D'^2}$$

$$= \frac{6\cos(2x+y) - 6\sin(2x+y)}{9(-4) - 36(-1)}$$

$$= x\frac{6\cos(2x+y) - 6\sin(2x+y)}{18D}$$

$$= \frac{x}{18}\int (6\cos(2x+y) - 6\sin(2x+y))dx$$

$$= \frac{6x}{18}\left[\frac{1}{2}\sin(2x+y) + \frac{1}{2}\cos(2x+y)\right]$$

$$P.I = \frac{x}{6}[\sin(2x+y) + \cos(2x+y)]$$

The general solution is

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) + \frac{x}{6}[\sin(2x+y) + \cos(2x+y)]$$

Solve $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$

Solution:

To find complementary function:

Put $D = m \; and \; D' = 1$,

$$m^3 - m^2 = 0 \Rightarrow m^2(m-2) = 0$$

$$m = 0, 0, 2$$

$$z = f_1(y) + xf_2(y) + f_3(y+2x)$$

To find Particular Integral:

$$P.I_1 = \frac{2e^{2x}}{D^3 - 2D^2D'} = \frac{2e^{2x}}{2^3 - 2(2)^2(0)} = \frac{2e^{2x}}{2^3} = \frac{e^{2x}}{8}$$

$$P.I_2 = \frac{3x^2y}{D^3 - 2D^2D'} = \frac{3x^2y}{D^3\left(1 - \frac{2D'}{D}\right)}$$

$$= \frac{3\left(1 - \frac{2D'}{D}\right)^{-1} x^2y}{D^3} = \frac{3\left(1 + \frac{2D'}{D}\right)x^2y}{D^3} = \frac{3x^2y + \frac{6}{D}x^2}{D^3}$$

$$= \frac{3}{D^3}(x^2y) + \frac{6}{D^4}(x^2) = \frac{yx^5}{20} + \frac{x^6}{60}$$

The general solution is

$$z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{e^{2x}}{8} + \frac{yx^5}{20} + \frac{x^6}{60}$$

Solve $(D^3 - 7DD'^2 - 6D'^3)z = x^2y + e^{2x+y}$

Solution:

To find complementary function:

Put $D = m \; and \; D' = 1$,

$$m^3 - 7m - 6 = 0 \Rightarrow m^3 - m - 6m - 6 = 0$$

$$m(m^2 - 1) - 6(m + 1) = 0 \Rightarrow m(m + 1)(m - 1) - 6(m + 1) = 0$$

$$(m + 1)(m^2 - m - 6) = 0$$

$$m = -1, -2, 3$$

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$$

To find Particular Integral:

$$P.I_1 = \frac{x^2y}{D^3 - 7DD'^2 - 6D'^3} = \frac{x^2y}{D^3\left(1 - \frac{7D'^2}{D^2} - \frac{6D'^3}{D^3}\right)} = \frac{x^2y}{D^3\left(1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right)}$$

$$= \frac{\left(1 - \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right)^{-1} x^2y}{D^3} = \frac{\left(1 + \left(\frac{7D'^2}{D^2} + \frac{6D'^3}{D^3}\right)\right)x^2y}{D^3} = \frac{1}{D^3}(x^2y) = \frac{x^5y}{60}$$

$$P.I_2 = \frac{e^{2x+y}}{D^3 - 7DD'^2 - 6D'^3} = \frac{e^{2x+y}}{2^3 - 7(2)(1)^2 - 6(1)^3} = \frac{e^{2x+y}}{8 - 14 - 6} = -\frac{e^{2x+y}}{12}$$

The general solution is

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x^5 y}{60} - \frac{e^{2x+y}}{12}$$

Solve $\left(D^2 - 6DD' + 5D'^2\right)z = xy + e^x \sin hy$

Solution:

The auxiliary equation is $m^2 - 6m + 5 = 0$

$$m = 1,5$$

The complementary function CF is $z = \phi_1(y + x) + \phi_2(y + 5x)$

To find particular integral:

$$P.I_1 = \frac{xy}{D^2 - 6DD' + 5D'^2}$$

$$= \frac{xy}{D^2\left(1 - 6\frac{D'}{D} + 5\frac{D'^2}{D^2}\right)} = \frac{xy}{D^2\left(1 - \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right)}$$

$$= \frac{\left(1 - \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right)^{-1} xy}{D^2}$$

$$= \frac{\left(1 + \left(6\frac{D'}{D} - 5\frac{D'^2}{D^2}\right)\right) xy}{D^2} \qquad [(1 - x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \frac{\left(xy + \left(6\frac{D'}{D} xy - 5\frac{D'^2}{D^2} xy\right)\right)}{D^2}$$

$$= \frac{\left(xy + \left(6\frac{x}{D}\right)\right)}{D^2} = \frac{1}{D^2}xy + \frac{6}{D^3}x$$

$$P.I_1 = \frac{x^3y}{6} + \frac{6x^4}{24} = \frac{x^3y}{6} + \frac{x^4}{4} \qquad \left[\frac{1}{D^n}x^m = \frac{m!\,x^{m+n}}{(m+n)!}\right]$$

$$P.I_2 = \frac{e^x \sin hy}{D^2 - 6DD' + 5D'^2} = \frac{e^x\left(\frac{e^y - e^{-y}}{2}\right)}{D^2 - 6DD' + 5D'^2}$$

$$= \frac{e^{x+y}}{2\left(D^2 - 6DD' + 5D'^2\right)} - \frac{e^{x-y}}{2\left(D^2 - 6DD' + 5D'^2\right)}$$

$$= \frac{e^{x+y}}{2(1^2 - 6(1)(1) + 5(1)^2)} - \frac{e^{x-y}}{2(1^2 - 6(1)(-1) + 5(-1)^2)}$$

$$= \frac{xe^{x+y}}{2(2D - 6D')} - \frac{e^{x-y}}{2(11)}$$

$$\begin{bmatrix} if\ denominator\ becomes\ zero\ then\ muliply\ numerator\ by\ x \\ and\ differentiating\ denominator\ partially\ with\ respect\ to\ D \end{bmatrix}$$

$$P.I_2 = \frac{xe^{x+y}}{2(2(1) - 6(1))} - \frac{e^{x-y}}{22} = \frac{xe^{x+y}}{-8} - \frac{e^{x-y}}{22}$$

The general solution is

$$CF\ + P.I_1 + P.I_2$$

$$z = f_1(y + x) + f_2(y + 5x) + \frac{x^3y}{6} + \frac{x^4}{4} - \frac{xe^{x+y}}{8} - \frac{e^{x-y}}{22}$$

Two marks:

Define unit impulse step function and its Z transform.

Define convolution of two sequences.

If $F(z) = \frac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim_{t\to\infty} f(t)$

Form a difference equation by eliminating the arbitrary constants A from $y_n = A\,3^n$

Define unit step function and its Z transform.

State Final value theorem for Z transforms.

If $F(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ find $f(0)$.

Find P.I of $(D^2 + 4DD')z = e^x$.

Solve $(D^2 - 4DD' - 4D'^2)z = 0$.

Solve $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = 0$