Queueing theory and Replacement model

1. Trucks at a single platform weigh-bridge arrive according to Poisson probability distribution. The time required to weigh the truck follows an exponential probability distribution. The mean arrival rate is 12 trucks per day and mean service rate is 18 trucks per day. Determine

(i) What is the probability that no trucks are in the system?

(ii) What is the average number of trucks waiting for service?

(iii) What is the average time a truck waits for weighing service to begin?

(iv) What is the probability that an arriving truck will have to wait for service?

Solution:

Solution:

$$\delta = 12\, trucks\ \ per\, day, \mu = \ 18\, trucks\ \ per\, day$$

$$Utilization\, factor\, \phi = \frac{\delta}{\mu} = \frac{12}{18} = 0.67$$

The probability that no trucks are in the system is $\mathrm{P_0} = 1 - \phi = 1 - 0.67 = 0.33$

Average number of trucks in the system

$$L_s = \frac{\phi}{1-\phi} = \frac{0.67}{1-0.67} = 2.03$$

The average number of trucks waiting for service

$$\mathrm{L_q} = \mathrm{L_s} - \frac{\delta}{\mu} = 2.03 - 0.67 = 1.36$$

The average time a truck waits for weighing service to begin

$$W_q = \frac{L_q}{\delta} = \frac{1.36}{12} = 0.113\, days$$

The probability that an arriving truck will have to wait for service

$$Utilization\, factor\, \phi = \frac{\delta}{\mu} = \frac{12}{18} = 0.67$$

2. The city council of a small town has decided to build a tennis court in the central park. Players are expected to arrive on the average of 10 sets of players per 12 hour a day. Playing time is exponentially distributed with a mean of 1 hour. Arrivals are Poisson. What are the expected queueing statistics assuming the basic single server model?

Solution:

$$\delta = \frac{10}{12} \ per\ hour, \mu = 1\ per\ hour$$

$$Utilization\ factor\ \phi = \frac{\delta}{\mu} = \frac{10}{12} = 0.83$$

Average number of players in the park

$$L_s = \frac{\phi}{1-\phi} = \frac{\left(\frac{10}{12}\right)}{1-\frac{10}{12}} = \frac{0.83}{1-0.83} = 4.88$$

Average number of players in the queue

$$\mathrm{L_q} = \mathrm{L_s} - \frac{\delta}{\mu} = 4.88 - 0.83 = 4.05$$

Average waiting time of players in the park

$$W_s = \frac{L_s}{\delta} = \frac{4.88}{10} = 0.49\ hours$$

Average waiting time of players in the queue

$$W_q = \frac{L_q}{\delta} = \frac{4.05}{10} = 0.41\ hours$$

3. Customers arrive at a one window drive at a bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window including that for the serviced car can accommodate a maximum of 3 cars. Others can wait outside this space. Determine

(i) The probability that an arriving customer can drive directly to the space in front of the window.

(ii) The probability that an arriving customer will have to wait outside the indicated space.

(iii) The length of time an arriving customer is exposed to wait before starting service.

Solution:

$\delta = 10/\mathrm{hour}, \mu = \frac{60}{5} = 12/hour$

The probability that an arriving customer can drive directly to the space in front of the window is

$$P(x<3) = P(x=0) + P(x=1) + P(x=2)\ where\ P_n = \phi^n(1-\phi) = \left(\frac{\delta}{\mu}\right)^n\left(1-\frac{\delta}{\mu}\right)$$

$$= \left(\frac{\delta}{\mu}\right)^0 \left(1-\frac{\delta}{\mu}\right) + \left(\frac{\delta}{\mu}\right)^1 \left(1-\frac{\delta}{\mu}\right) + \left(\frac{\delta}{\mu}\right)^2 \left(1-\frac{\delta}{\mu}\right)$$

$$= \left(\frac{\delta}{\mu}\right)^0 \left(1-\frac{\delta}{\mu}\right) + \left(\frac{\delta}{\mu}\right)^1 \left(1-\frac{\delta}{\mu}\right) + \left(\frac{\delta}{\mu}\right)^2 \left(1-\frac{\delta}{\mu}\right) = \left(1-\frac{\delta}{\mu}\right)\left[1 + \left(\frac{\delta}{\mu}\right) + \left(\frac{\delta}{\mu}\right)^2\right]$$

$$= \left(1-\frac{10}{12}\right)\left[1 + \left(\frac{10}{12}\right) + \left(\frac{10}{12}\right)^2\right] = \frac{2}{12}\left(1 + \frac{10}{12} + \frac{100}{144}\right) = \frac{2}{12}(2.53) = 0.42$$

The probability that an arriving customer will have to wait outside the indicated space is

$$P(n \geq 3) = \left(\frac{\delta}{\mu}\right)^3 \ where \ P(n \geq k) = \left(\frac{\delta}{\mu}\right)^k$$

$$P(n \geq 3) = \left(\frac{10}{12}\right)^3 = 0.58$$

The length of time an arriving customer is exposed to wait before starting service is

$$W_q = \frac{L_q}{\delta}$$

$$L_s = \frac{\phi}{1-\phi} = \frac{\left(\frac{10}{12}\right)}{1-\frac{10}{12}} = \frac{0.83}{1-0.83} = 4.88$$

$$\mathrm{L_q} = \mathrm{L_s} - \frac{\delta}{\mu} = 4.88 - 0.83 = 4.05$$

$$W_q = \frac{L_q}{\delta} = \frac{4.05}{10} = 0.41 \ hours$$

4. Arrivals at telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of the phone call is assumed to be distributed exponentially, with mean 3 minutes.

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

(iii) What is the average length of the square that forms from time to time?

Solution:

$$\delta = \frac{1}{10}/minutes, \mu = \frac{1}{3}/minutes$$

The probability that a person arriving at the booth will have to wait

$$Utilization\,factor\ \phi = \frac{\delta}{\mu} = \frac{\frac{1}{10}}{\frac{1}{3}} = 0.3$$

The second booth will be installed, if the new waiting time $\mathrm{W_q'}$ is greater than 3. Let the corresponding arrival rate be λ_1.

$$\mathrm{W_q'} > 3$$

$$\frac{\delta_1}{\mu(\mu - \delta_1)} > 3 \Rightarrow \frac{\delta_1}{\frac{1}{3}\left(\frac{1}{3} - \delta_1\right)} > 3$$

$$\Rightarrow \delta_1 > 1 - 3\delta_1 \Rightarrow 4\delta_1 > 1 \Rightarrow \delta_1 > \frac{1}{4}$$

$$Increase\ in\ arrivals\ is\ \delta_1 - \delta = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

Average length of the square that forms from time to time

$$\mathrm{L_w} = \frac{\mu}{\mu - \delta} = \frac{\frac{1}{3}}{\frac{1}{3} - \frac{1}{10}} = \frac{\frac{1}{3}}{\frac{7}{30}} = \frac{10}{7} = 1.43$$

5. There are two clerks in a university to receive fees from the students. If the service time for each student is exponential with mean 4 minutes and if the boys arrive in a Poisson fashion at the counter at the ratio of 10 per hour, determine

(i) The probability of having to wait for service.

(ii) The expected percentage idle time for each clerk.

Solution:

$$C = 2, \delta = 10\ per\ hour, \mu = \frac{1}{4}\ per\ minutes = \frac{60}{4} = 15\ per\ hour$$

$$Utilization\,factor\ \phi = \frac{\delta}{\mu C} = \frac{10}{2(15)} = \frac{10}{30} = \frac{1}{3}$$

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{\left(\frac{\delta}{\mu}\right)^n}{n!} + \frac{\left(\frac{\delta}{\mu}\right)^c}{c!} . \frac{c\mu}{c\mu - \delta}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2-1} \frac{\left(\frac{10}{15}\right)^n}{n!} + \frac{\left(\frac{10}{15}\right)^2}{2!} \cdot \frac{2(15)}{2(15) - 10} \right]^{-1}$$

$$= \left[\frac{\left(\frac{10}{15}\right)^0}{0!} + \frac{\left(\frac{10}{15}\right)^1}{1!} + \frac{\left(\frac{10}{15}\right)^2}{2!} \cdot \frac{2(15)}{2(15) - 10} \right]^{-1}$$

$$= \left[1 + 0.67 + \frac{0.45}{2!} \cdot \frac{30}{30 - 10}\right]^{-1} = \left[1.67 + \frac{0.45}{2!} \cdot \frac{30}{20}\right]^{-1} = [1.67 + 0.34]^{-1}$$

$$= [20.1]^{-1} = 0.5$$

The probability of having to wait for service

$$P(n \geq c\,) = \frac{\mu . \left(\frac{\delta}{\mu}\right)^c}{(c-1)!\,(c\mu - \delta)} P_0$$

$$= \frac{15 . \left(\frac{10}{15}\right)^2}{(2-1)!\,(2(15) - 10)} 0.5 = 0.167$$

The expected percentage idle time for each clerk is

$$= (1 - \phi) \times 100 = \left(1 - \frac{1}{3}\right) \times 100 = \frac{2}{3} \times 100 = 0.67 \times 100 = 67\%$$

Replacement Problems

Replacement of items whose maintenance and repair costs increase with time and ignoring changes in the value of the money during the period

1. The cost of machine is Rs. 6100 and its scrap value is 100. The maintenance cost found from experience is as follows:

Year:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.):	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Solution:

Years n	Capital cost C	Scrap value S	Maintenance cost $f(t)$	$\sum f(t)$	Total cost $C - S + \sum f(t)$	Average annual cost $\frac{1}{n}(C - S + \sum f(t))$
1	6100	100	100	100	6100	6100
2	6100	100	250	350	6350	3175
3	6100	100	400	750	6750	2250
4	6100	100	600	1350	7350	1837.5
5	6100	100	900	2250	8250	1650
6	6100	100	1200	3450	9450	**1575**
7	6100	100	1600	5050	11050	1578.571
8	6100	100	2000	7050	13050	1631.25

Here the minimum average annual cost is Rs. 1575. Therefore the machine can be replaced at the end of sixth year.

2. Machine A cost Rs. 45000 and the operating costs are estimated at Rs. 1000 for the first year increasing by Rs. 10000 per year in the second and subsequent years. Machine B cost Rs. 50000 and operating cost are Rs 2000 for the first year, increasing by Rs. 4000 in the second and subsequent years. If we now have a machine of type A, should we replace it with B? If so when? Assume that both machines have no resale value and future costs are not discounted.

Solution:

Machine A:

Years n	Capital cost C	Scrap value S	Maintenance cost $f(t)$	$\sum f(t)$	Total cost $C - S + \sum f(t)$	Average annual cost $\frac{1}{n}(C - S + \sum f(t))$
1	45000	0	1000	1000	46000	46000
2	45000	0	11000	12000	57000	28500
3	45000	0	21000	33000	78000	**26000**
4	45000	0	31000	64000	109000	27250

Here the minimum average annual cost of Machine A is Rs. 26000. Therefore the Machine A can be replaced at the end of third year.

Machine B:

Years n	Capital cost C	Scrap value S	Maintenance cost $f(t)$	$\sum f(t)$	Total cost $C - S + \sum f(t)$	Average annual cost $\frac{1}{n}(C - S + \sum f(t))$
1	50000	0	2000	2000	52000	52000
2	50000	0	6000	8000	58000	29000
3	50000	0	10000	18000	68000	22666.67
4	50000	0	14000	32000	82000	20500
5	50000	0	18000	50000	100000	**20000**
6	50000	0	22000	72000	122000	20333.33

Here the minimum average annual cost of Machine B is Rs. 20000. Therefore the Machine B can be replaced at the end of third year.

Total cost of Machine A for the first year $= Rs. 46000$

Total cost of Machine A for the second year $= Rs. 57000 - 46000 = Rs. 11000 < 20000$

Total cost of Machine A for the third year $= Rs. 78000 - 57000 = Rs. 21000 > 20000$

∴ The Total cost of Machine A for the third year is greater than the minimum average annual cost of Machine B.

∴ Machine A is replaced by Machine B at the end of second year.

Replacement of items whose maintenance costs increases with time and value of the money also changes with Time

1. A manufacturer is offered two machines A and B. A has cost price of Rs. 2,500, its running cost is Rs. 400 for each of first years and increased by Rs. 100 every subsequent year, Machine B having the same capacity as A, costs Rs. 1250 and has running cost of Rs. 600 for 6 years, increasing by Rs. 100 per year thereafter. Taking money's value as 10% per year, which machine should be purchased? Scrap value of both the machines is negligibly small.
Solution:

$$r = 10\% = 0.1$$

$$v = \frac{1}{1+r} = \frac{1}{1+0.1} = 0.9091$$

Machine A

Capital cost C= Rs. 2500, Scrap Value S= 0,

Years n	C	Running cost R_n	Discount factor v^{n-1}	Discounted running cost $R_n v^{n-1}$	$C + \sum R_n v^{n-1}$	$\sum v^{n-1}$	$\frac{C+\sum R_n v^{n-1}}{\sum v^{n-1}}$
1	2500	400	1	400	2900	1	2900
2	2500	400	0.9091	363.64	3263.64	1.9091	1709.52
3	2500	400	0.8265	330.6	3594.24	2.7356	1313.88
4	2500	400	0.7513	300.52	3894.76	3.4869	1116.97
5	2500	400	0.683	273.2	4167.96	4.1699	999.53
6	2500	500	0.621	310.5	4478.46	4.7909	934.78
7	2500	600	0.5645	338.7	4817.16	5.3554	899.5
8	2500	700	0.5132	359.24	5176.4	5.8686	882.05
9	2500	800	0.4665	373.2	5549.6	6.3351	**876.01**
10	2500	900	0.4241	381.69	5931.29	6.7592	877.51

Hence machine A should be replaced after 9th year.

Machine B:

Capital cost C= Rs. 1250, Scrap Value S= 0

Years n	C	**Running cost** R_n	**Discount factor** v^{n-1}	**Discounted running cost** $R_n v^{n-1}$	$C + \sum R_n v^{n-1}$	$\sum v^{n-1}$	$\frac{C + \sum R_n v^{n-1}}{\sum v^{n-1}}$
1	1250	600	1	600	1850	1	1850
2	1250	600	0.9091	545.46	2395.46	1.9091	1254.76
3	1250	600	0.8265	495.9	2891.36	2.7356	1056.94
4	1250	600	0.7513	450.78	3342.14	3.4869	958.48
5	1250	600	0.683	409.8	3751.94	4.1699	899.77
6	1250	600	0.621	372.6	4124.54	4.7909	860.91
7	1250	700	0.5645	395.15	4519.69	5.3554	843.95
8	1250	800	0.5132	410.56	4930.25	5.8686	**840.11**
9	1250	900	0.4665	419.85	5350.1	6.3351	844.52

Hence machine A should be replaced after 8th year.

Since the weighted average cost in 9 years of machine A is Rs. 876 and weighted average cost in 8 years of machine B is Rs. 840, it is advisable to purchase machine B.

Replacement of items that fail suddenly

1. A plant has 6 numbers of cells and a study has been made on the nature of deterioration with time. The following is the probability of failure after replacement.

Months after replacement	1	2	3	4	5
Probability of failure	0.2	0.15	0.1	0.3	0.25

The cost of replacement as and when it fails has been found to be Rs. 200 per unit. Now it has been proposed to replace all the items at a specified frequency and in such a case the cost of such a replacement comes to Rs. 60 per unit. Can we accept this proposal? If so, at what frequency we have to undertake replacement of all the cells at a time?

Solution:

Let P_i be the probability that a cell fails during the ith month of its life.

Let N_i represent the number of replacements made at the end of ith month when all 6 cells are new initially. Then we have

Month	Expected number of failures	
0	N_0	$=\ 6$
1	$N_1 = N_0P_1 = 6 \times 0.2$	$= 1.2$
2	$N_2 = N_0P_2 + N_1P_1 = 6 \times 0.15 + 1.2 \times 0.2$	$= 1.14$
3	$N_3 = N_0P_3 + N_1P_2 + N_2P_1 = 6 \times 0.1 + 1.2 \times 0.15 + 1.14 \times 0.2$	$= 1.008$
4	$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$ $= 6 \times 0.3 + 1.2 \times 0.1 + 1.14 \times 0.15 + 1.008 \times 0.2$	$= 2.2926$
5	$N_4 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$ $= 6 \times 0.25 + 1.2 \times 0.3 + 1.14 \times 0.1 + 1.008 \times 0.15 + 2.2926 \times 0.2$	$= 2.5837$

<u>Individual replacement policy:</u>

$$Average\ life\ of\ cells = \sum_i iP_i = 1 \times 0.2 + 2 \times 0.15 + 3 \times 0.1 + 4 \times 0.3 + 5 \times 0.25 = 3.25$$

$$Average\ number\ of\ failures\ per\ month = \frac{Total\ number\ of\ cells}{aerage\ life\ of\ cells} = \frac{6}{3.25} = 1.85\ per\ month$$

$Cost\ of\ individual\ replacement$

$$= Average\ number\ of\ failures\ per\ month \times individual\ replacement\ cost\ per\ month$$

$$= 1.85 \times 200 = Rs.370\ per\ month$$

<u>Group replacement policy:</u>

End of month (n)	Total cost of replacing cells (T)	Average cost per month T/n
1	$6 \times 60 + 1.2 \times 200 = 600$	600
2	$6 \times 60 + (1.2 + 1.14) \times 200 = 812$	406
3	$6 \times 60 + (1.2 + 1.14 + 1.008) \times 200 = 1029.6$	**343.2**
4	$6 \times 60 + (1.2 + 1.14 + 1.008 + 2.2926) \times 200 = 1488.12$	372
5	$6 \times 60 + (1.2 + 1.14 + 1.008 + 2.2926 + 2.5837) \times 200 = 2004.86$	401

Minimum average cost per month is 343. So group replacement can be made at the end of third month.

Group replacement period = 3 months

Individual replacement cost/month = Rs. 370

Minimum group replacement cost/month = Rs. 343

Since the minimum group replacement cost/month is lesser than the individual replacement cost/month, the group replacement policy is best and hence all the cells are to be replaced once in three months and the cells which fail during this three month period are to be replaced individually.

2. The probability P_n of failure just before age n (in months) of 1000 light bulbs is given. If individual replacements costs Rs. 12.5 and group replacement costs Rs. 3.0 per bulb, find the optimal replacement policy.

n	1	2	3	4	5
P_n	0.1	0.2	0.25	0.3	0.15

Solution:

Let P_i be the probability that a light bulb fails during the ith month of its life.

Let N_i represent the number of replacements made at the end of ith month when all 1000 cells are new initially. Then we have

Month	Expected number of failures	
0	N_0	$= 1000$
1	$N_1 = N_0P_1 = 1000 \times 0.1$	$= 100$
2	$N_2 = N_0P_2 + N_1P_1 = 1000 \times 0.2 + 100 \times 0.1$	$= 210$
3	$N_3 = N_0P_3 + N_1P_2 + N_2P_1 = 1000 \times 0.25 + 100 \times 0.2 + 210 \times 0.1$	$= 291$
4	$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$ $= 1000 \times 0.3 + 100 \times 0.25 + 210 \times 0.2 + 291 \times 0.1$	$= 396$
5	$N_4 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$ $= 1000 \times 0.15 + 100 \times 0.3 + 210 \times 0.25 + 291 \times 0.2 + 396 \times 0.1$	$= 330$

Individual replacement policy:

$$Average\ life\ of\ light\ bulbs = \sum_i iP_i = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.25 + 4 \times 0.3 + 5 \times 0.15 = 3.2$$

$$Average\ number\ of\ failures\ per\ month = \frac{1000}{3.2} \cong 313$$

$Cost\ of\ individual\ replacement$

$= Average\ number\ of\ failures\ per\ month \times individual\ replacement\ cost\ per\ month$

$= 313 \times 12.5 \cong Rs. 3913\ per\ month$

Group replacement policy:

End of month (n)	Total cost of replacing cells (T)	Average cost per month T/n
1	$1000 \times 3 + 100 \times 12.5 = 4250$	4250
2	$1000 \times 3 + (100 + 210) \times 12.5 = 6875$	**3438**
3	$1000 \times 3 + (100 + 210 + 291) \times 12.5 = 10513$	3504
4	$1000 \times 3 + (100 + 210 + 291 + 396) \times 12.5 = 15463$	3866
5	$1000 \times 3 + (100 + 210 + 291 + 396 + 330) \times 12.5 = 19588$	3918

Minimum average cost per month is 3438. So group replacement can be made at the end of second month.

Group replacement period = 2 months

Individual replacement cost/month = Rs. 3913

Minimum group replacement cost/month = Rs. 3438

Since the minimum group replacement cost/month is lesser than the individual replacement cost/month, the group replacement policy is best and hence all the tube lights are to be replaced once in two months and the cells which fail during this two month period are to be replaced individually.

3. The probability P_n of failure just before age n shown below for 1000 bulbs. If the individual replacements costs Re. 1 and the group replacement costs Rs. 0.3 per item, find the optimal replacement policy.

n	1	2	3	4	5
P_n	0.3	0.1	0.1	0.2	0.3

Solution:

Let P_i be the probability that a bulbs fails during the ith month of its life.

Let N_i represent the number of replacements made at the end of ith month when all 1000 cells are new initially. Then we have

Month	Expected number of failures	
0	N_0	$= 1000$
1	$N_1 = N_0P_1 = 1000 \times 0.3$	$= 300$
2	$N_2 = N_0P_2 + N_1P_1 = 1000 \times 0.1 + 300 \times 0.3$	$= 190$
3	$N_3 = N_0P_3 + N_1P_2 + N_2P_1 = 1000 \times 0.1 + 300 \times 0.1 + 190 \times 0.3$	$= 187$
4	$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$ $= 1000 \times 0.2 + 300 \times 0.1 + 190 \times 0.1 + 187 \times 0.3$	$= 305$
5	$N_4 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$ $= 1000 \times 0.3 + 300 \times 0.2 + 190 \times 0.1 + 187 \times 0.1 + 305 \times 0.3$	$= 489$

Individual replacement policy:

$$Average\ life\ of\ bulbs = \sum_i iP_i = 1 \times 0.3 + 2 \times 0.1 + 3 \times 0.1 + 4 \times 0.2 + 5 \times 0.3 = 3.1$$

$$Average\ number\ of\ failures\ per\ month = \frac{1000}{3.1} \cong 323$$

$$Cost\ of\ individual\ replacement$$

$$= Average\ number\ of\ failures\ per\ month \times individual\ replacement\ cost\ per\ month$$

$$= 323 \times 1 \cong Rs. 323\ per\ month$$

Group replacement policy:

End of month (n)	**Total cost of replacing cells (T)**	**Average cost per month T/n**
1	$1000 \times 0.3 + 300 \times 1 = 600$	600
2	$1000 \times 0.3 + (300 + 190) \times 1 = 790$	395
3	$1000 \times 0.3 + (300 + 190 + 187) \times 1 = 977$	326
4	$1000 \times 0.3 + (300 + 190 + 187 + 305) \times 1 = 1282$	**321**
5	$1000 \times 0.3 + (300 + 190 + 187 + 305 + 489) \times 1 = 1771$	354

Minimum average cost per month is 321. So group replacement can be made at the end of fourth month.

Group replacement period = 4 months

Individual replacement cost/month = Rs. 323

Minimum group replacement cost/month = Rs. 321

Since the minimum group replacement cost/month is lesser than the individual replacement cost/month, the group replacement policy is best and hence all the bulbs are to be replaced once in four months and the cells which fail during this four month period are to be replaced individually.