STATISTICS AND NUMERICAL METHODS

QUESTION I APRIL / MAY 2010

1. **Mention various steps involved in testing of hypothesis.**

Solution:

(i). Set up the null hypothesis.

(ii). Choose the appropriate level of significance (e.g 5% or 1% or 10%)

(iii). Calculate the test statistic $z = \frac{t-E(t)}{s\,E(t)}$.

(iv). Draw conclusion. If $|Calc\ value| < Table\ vale$ then Accept H_0.

If $|Calc\ value| > Table\ vale$ then Reject H_0.

2. **Define Chi-square test for goodness of fit.**

Solution:

Karl Pearson developed a test for testing the significance of similarity between experimental values and the theoretical values obtained under some theory or hypothesis. This test is known as $\boldsymbol{\psi^2}$ test of goodness of fit. Karl Pearson proved that the statistic

$$\boldsymbol{\psi^2} = \sum \frac{(\boldsymbol{O}-\boldsymbol{E})^2}{\boldsymbol{E}}, \quad \boldsymbol{where\ \ O - Observed\ frequency,\ \ E - Expected\ frequency}$$

$\boldsymbol{\psi^2}$ is used to test whether differences between observed and expected frequencies are significant.

3. **Discuss the advantages and disadvantages of randomized block design.**

Solution:

a. Evaluation and comparison of basic design of configurations.

b. Evaluation of material alternatives.

c. Determination of key product design parameters that affect performance.

Also it uses to improve manufacturing a product, field performance, reliability and lower product cost, etc.

4. **State the advantages of a factorial experiment over a simple experiment.**

Solution:

Factorial designs are frequently used in experiments involving several factors where it is necessary to the joint effect of the factors on a response.

5. **Using Newton – Raphson method, find the iteration formula to compute $\sqrt{N}$.**

Solution: $f(x) = x^2 - N\ \ and\ \ f'(x) = 2\,x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow\ x_n - \frac{({x_n}^2 - N)}{2\ x_n} = \left(\frac{1}{2}\right)\left(x_n + \frac{N}{x_n}\right), n = 0,1,2, \ldots$$

6. **Write down the Lagrange's interpolation formula.**

Solution: The Lagrange's interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\ y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\ y_1 + \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)}\ y_2 + \cdots$$

8. **Write down Simpson's $\frac{1}{3}$ rule in numerical integration.**

Solution:

If (x_i, y_i) $i = 0,1,2, \dots n$ where $x_i = x_0 + ih,$ $then$

Simpson's $\frac{1}{3}$ rule :

$$\int_{x_0}^{x_n} y\, dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots +) + 2(y_2 + y_4 + y_6 + \dots)\right]$$

9. **Using Euler's method, solve the following differential equation $y' = -y$ subject to $y(0) = 1$.**

Answer :

The Euler's formula is $y_{n+1}(x_n + h) = y_n + hf(x_n, y_n)$

$$y_1 = 1 + 0.1[-1] = 1 - 0.1 \quad \Rightarrow \quad y(0.1) = 0.99$$

10. **Write down the Milne's Predictor-Corrector formula for solving initial value problem in first order differential equation.**

Solution:

Predictor : $y_{n+1}^{p} = y_{n-3} + \frac{4h}{3}\,[2\, y'_{n-2} - y'_{n-1} + 2\, y'_n]$

Corrector : $y_{n+1}^{c} = y_{n-1} + \frac{h}{3}\,[\, y'_{n-2} + 4y'_{n-1} + \, y'_{n+1}]$

Part-B

11. a. (i). A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cms and standard deviation of 2.61 cms? (Test at 5% level of significance. The value of z at 5% level is $|z_\propto| = 1.96$)

Solution:

Given that $n = 900,\quad \bar{x} = 3.4,\quad s = 2.61\quad and\quad \mu = 3.25\quad \sigma = 2.61$

Null Hypothesis : $\boldsymbol{H_0}$: $\boldsymbol{\mu = 3.25}$ i.e., there is no difference b/w sample mean and pop mean.

Alternative Hypothesis : $\boldsymbol{H_1: \mu \neq 3.25}$ (Two tailed alternative)

The test statistic is given by

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.4 - 3.25}{\frac{2.61}{\sqrt{900}}} = 1.724$$

$$\therefore \quad z = 1.724 \qquad [Calculated\ value]$$

At 5% significance level the tabulated value for $z_\propto$ is 1.96.

$$|Calculated\ value\ | \leq Tabulated\ value \quad then \quad Accept\ H_0$$

$$But \quad |1.724| < 1.96 \quad So\ we\ Accept\ H_0$$

Conclusion:

$|z| < z_\propto$, we accept Null Hypothesis. That is there is no significant difference between the sample mean and population means.

11. (a). (ii). Before an increase in excise duty on tea, 800 persons out of a sample of 1000 persons were found to be tea drinkers. After an increase in duty, 800 people were tea drinkers in a sample of 1200 people. State whether there is a significant decrease in the consumption of tea after the increase in excise duty?

Solution:

$$Given\ that\quad n_1 = 1000,\quad x_1 = 800 \qquad and \qquad n_2 = 1200,\quad x_2 = 800$$

$$p_1 = Prop\ of\ tea\ drinkers\ before\ excise \qquad p_1 = \frac{x_1}{n_1} = \frac{800}{1000} = 0.80 \quad \& \quad q_1 = 0.20$$

$$p_2 = Prop\ of\ tea\ drinkers\ after\ excise \qquad p_2 = \frac{x_2}{n_2} = \frac{800}{1200} = 0.677 \quad \& \quad q_2 = 0.33$$

Null Hypothesis:

$$H_0 : P_1 = P_2 \quad i.e., \quad there\ is\ no\ difference\ before\ \&\ after\ excise$$

Alternative hypothesis:

$$H_1 : P_1 > P_2 \quad (Right\ Tailed)$$

Here Population Proportion P is not known.

$$\therefore \quad P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$P = \frac{0.8 * 1000 + 0.67 * 1200}{1000 + 1200} = 0.729$$

$$\therefore \quad P = 0.729 \quad and \quad Q = 0.271$$

$$The\ test\ statistic\ is\ given\ by\ z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$z = \frac{0.8 - 0.67}{\sqrt{[0.729 * 0.271]\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = \frac{0.13}{\sqrt{[0.198](0.00183)}} = \frac{0.13}{0.0191}$$

$$z = 6.8 \quad app$$

At 5% level of significance, the table value for $z_\propto = 1.645$.

$$|calculated\ value| \le tabulated\ value \quad \Longrightarrow \quad Accept\ H_0$$

$$|6.8| > 1.645 \quad \Longrightarrow \quad Reject\ \ H_0$$

Conclusion: We reject the null hypothesis. That is there is a significant difference in the consumption of tea before and after inverse in excise duty.

11. b. (i). Out of 8000 graduates in a town 800 are females; out of 1600 graduate employees 120 are females. Use χ^2 to determine if any distinction is made in appointment on the basis of sex. Value of χ^2 at 5% LOS with one degree of freedom is 3.84.

Solution:

The given information can be in tabular form

	Employment	Unemployment	Total
Males	1480	5720	7200
Females	120	680	800
Total	1600	6400	8000

Let us take the null hypothesis that no distinction is made in appointment on the basis of sex.

Applying χ^2 test

$$E_{11} = \frac{7200}{8000} * 1600 = 1440$$

The table of expected frequencies is given below:

1440	5760	7200
160	640	800
1600	6400	8000

Observed freq	Expected freq	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1480	1440	1600	1.111
120	160	1600	10.00
5720	5760	1600	0.278
680	640	1600	2.500
			13.889

$$\chi^2 = \mathbf{13.889}$$

Table value of $\chi^2_{0.05}$ with $(n-1)(r-1) = (2-1)(2-1) = 1$ d.o.f is 3.84.

Conclusion: Since $\chi^2 > \chi^2_{0.05}$, we reject H_o. That is some distinction is made in appointment on the basis of sex.

11. b. (ii). An automobile company gives you the following information about age groups and the liking for particular model of car which it plans to introduce. On the basis of this data can it be concluded that the model appeal is independent of the age group. $(\chi^2_{0.05} = 7.815)$

Persons who:	Below 20	20-39	40-59	60 and above
Liked the car:	140	80	40	20
Disliked the car:	60	50	30	80

Solution:

Persons who:	Below 20	20-39	40-59	60 and above	
Liked the car:	140	80	40	20	280
Disliked the car:	60	50	30	80	220
	200	130	70	100	500

Observed freq	Expected freq	$(O_i - E_i)$	$\frac{(O_i - E_i)^2}{E_i}$
140	112	28	7
80	72.8	7.2	0.712
40	39.2	0.8	0.016
20	56	-36	23.14
60	88	-28	8.909
50	57.2	-7.2	0.906
30	30.8	-0.8	0.02
80	44	36	29.45
			70.154

$$\chi^2 = \mathbf{70.154}$$

Table value of $\chi^2_{0.05}$ with $(n-1)(r-1) = (4-1)(2-1) = 3$ d.o.f is 7.815.

Conclusion:

Since $\chi^2 > \chi^2_{0.05}$, we reject null hypothesis.

12. (a). A set of data involving four tropical feed stuffs A, B, C, D tried on 20 chicks is given below. All the twenty chicks are treated alike in all respects except the feeding treatments and each of the feeding treatment is given to 5 chicks. Analyze the data.

Weight gain of baby chicks fed on different feeding materials composed of tropical feed stuffs.

						$Total\ T_i$
A	55	49	42	21	52	219
B	61	112	30	89	63	355
C	42	97	81	95	92	407
D	169	137	169	85	154	714
				Grand Total		G=1695

Solution:

$\boldsymbol{H_0}$: There is no significant difference between rows and columns.

To simplify calculation let us subtract 50 from each value, the new values are

Salesmen						Total
A	5	-1	-8	-29	2	-31
B	11	62	-20	39	13	105
C	-8	47	31	45	42	157
D	119	87	119	35	104	464
Total	127	195	122	90	161	695

$$Correction\ factor = C.F = \frac{G^2}{N} = \frac{(695)^2}{20} = 24151.25$$

$$SST = Total\ sum\ of\ squares = \{\quad\} - C.F$$

$$= \begin{bmatrix} (5)^2 + (-1)^2 + (-8)^2 + (-29)^2 + (2)^2 + (11)^2 + (62)^2 + (-20)^2 + (39)^2 \\ +(13)^2 + (-8)^2 + (47)^2 + (31)^2 + (45)^2 + (42)^2 + (119)^2 + (87)^2 + (119)^2 + (35)^2 + (104)^2 \end{bmatrix} - 24151.25$$

$$= 37793.75$$

Between Column sum of squares

$$SSC = \frac{(\sum x_1)^2}{n_1} + \frac{(\sum x_2)^2}{n_1} + \frac{(\sum x_3)^2}{n_1} + \frac{(\sum x_4)^2}{n_1} + \frac{(\sum x_5)^2}{n_1} - C.F$$

$$SSC = \frac{(127)^2}{4} + \frac{(195)^2}{4} + \frac{(122)^2}{4} + \frac{(90)^2}{4} + \frac{(161)^2}{4} - 24151.25 = 1613.5$$

Between Row sum of squares

$$SSC = \frac{(\sum y_1)^2}{m_1} + \frac{(\sum y_2)^2}{m_1} + \frac{(\sum y_3)^2}{m_1} + \frac{(\sum y_4)^2}{m_1} - C.F$$

$$R_2 = \frac{(-31)^2}{5} + \frac{(105)^2}{5} + \frac{(157)^2}{5} + \frac{(464)^2}{5} - 24151.25 = 26234.95$$

Error sum of squares

$$SSE = SST - SSR - SSC = 37793.75 - 1613.50 - 26234.95 = 9945.3$$

Degrees of freedom: $v_1 = c - 1 = 5 - 1 = 4, \quad v_2 = r - 1 = 4 - 1 = 3, \quad v_3 = v_1 * v_2 = 12$

ANOVA table for two-way classification				
Source of variation	Degrees of freedom	Sum of squares (SS)	Mean sum of squares (MS)	Variance Ration (F-Ratio)
B/W Column	$5-1=4$	SSC=1613.5	$MSC=\frac{SSC}{v_1}=403.37$	$F_1=\frac{MSE}{MSC}=2.055$
B/W Row	$4-1=3$	SSR=26234.95	$MSR=\frac{SSR}{v_2}=8744.98$	$F_2=\frac{MSR}{MSE}=10.55$
Error	$4*3=12$	SSE=9945.3	$MSE=\frac{SSE}{v_3}=828.775$	

$F_{0.05}(12,4)=5.91 \quad and \quad F_{0.05}(3,12)=8.74$

Conclusion:

1. $F_1<F_{0.05}(12,4)$. Hence we accept the null hypothesis. That is there is no difference between columns.
2. $F_2<F_{0.05}(3,12)$. Hence we reject the null hypothesis. That is there is some difference between Rows.

12. (b). An experiment was planned to study the effect of sulphate of potash and super phosphate on the yield of potatoes. All the combinations of 2 levels of super phosphate and two levels of sulphate of potash were studied an a randomized block design with 4 replications for each.

The yields (per plot) obtained are given in the following table. Analyse the data and give your exclusions.

Block	$Yields\ (l_b\ per\ plot)$			
I	(1) 23	K 25	P 22	KP 38
II	P 40	(1) 26	K 36	KP 38
III	(1) 29	K 20	KP 30	P 20
IV	KP 34	K 31	P 24	(1) 28

Analyse the data and comment on your finds.

Solution:

Treatment Combination	Blocks			
	I	II	III	IV
(1)	23	26	29	28
k	25	36	20	31
p	22	40	20	24
kp	38	38	30	34

Subtract 29 form each value, we get

Treatment Combination	Blocks				Total
	I	II	III	IV	
(1)	-6	-3	0	-1	-10
k	-4	7	-9	2	-4
p	-7	11	-9	-5	-10
kp	9	9	1	5	24
Total	-8	24	-17	1	0

H_0: the data is homogeneous w.r.t to the blocks and treatments. $N = 4 * 4 = 16$

$$Correction\ factor = C.F = \frac{G^2}{N} = \frac{(20)^2}{20} = \frac{0^2}{16} = 0$$

$$SST = Total\ sum\ of\ squares = \sum_i \sum_j y_{ij}^2 - C.F = 660 - 0 = 660$$

Between Columns sum of squares

$$R_1 = \frac{(8)^2}{4} + \frac{(24)^2}{4} + \frac{(-17)^2}{4} + \frac{(1)^2}{4} - 0 = 232.5$$

Between Row sum of squares

$$R_2 = \frac{(-10)^2}{4} + \frac{(-4)^2}{4} + \frac{(-10)^2}{4} + \frac{(24)^2}{4} - 0 = 198$$

Error sum of squares

$$R_3 = SST - SSC - SSR = 660 - 232.5 - 198 = 229.5$$

Treatment total:

$$[k] = [kp] - [p] + [k] - [1] = 24 - (-10) + (-4) + 10 = 40$$

$$[p] = [kp] + [p] - [k] - [1] = 24 + (-10) + (-4) + 10 = 28$$

$$[kp] = [kp] - [k] - [p] + [1] = 24 - (-10) - (-4) - 10 = 28$$

$$S_k = \frac{[40]^2}{4r} = \frac{1600}{4 * 4} = 100 \qquad S_p = \frac{[28]^2}{4r} = \frac{784}{4 * 4} = 49 \qquad S_{kp} = \frac{[28]^2}{4r} = \frac{784}{4 * 4} = 49$$

Source of variation	Degrees of freedom	Sum of squares (SS)	Mean sum of squares (MS)	F-Ratio	Table value
K	1	$S_k = 100$	$MS_k = 100$	$F_k = 10.56$	$F_{0.05}(4{,}12) = 3.26$
P	1	$S_p = 49$	$MS_p = 49$	$F_p = 10.56$	$F_{0.05}(4{,}12) = 3.26$
Kp	1	$S_{kp} = 49$	$MS_{kp} = 49$	$F_{kp} = 10.56$	$F_{0.05}(4{,}12) = 3.26$
Error	N-C-r+1 =16-4-4+1=9	$SSE = 229.5$	$MSE = \frac{229.5}{9} = 25.5$		

Conclusion: Calculated value is less than table value in all cases, we conclude that there is no significant difference between effects of treatments.

13. (a). (i). Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution:

We know that $[A, I] = [I, A^{-1}]$

$$\text{Now, } [A,I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Now, we need to make $[A.I]$ as a diagonal matrix.

Fix the first row, change second and third row by using first row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \Leftrightarrow R_2 - R_1 \\ R_3 \Leftrightarrow R_3 + 2R_1 \end{array}$$

Fix the first row & second row, change third row by using second row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \quad R_3 \Leftrightarrow R_3 + R_2$$

Fix the third row, change first and second row by using third row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} -4 & -4 & 0 & -7 & -3 & -3 \\ 0 & -8 & 0 & 10 & 2 & 6 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow -4R_1 - 3R_3 \\ R_2 \Leftrightarrow -4R_2 + 6R_3 \end{array}$$

Fix the second & third row, change first by using second row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 32 & 0 & 0 & 96 & 32 & 48 \\ 0 & -8 & 0 & 10 & 2 & 6 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \quad R_1 \Leftrightarrow -8R_1 + 4R_2$$

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 32 & 0 & 0 & \frac{96}{32} & \frac{32}{32} & \frac{48}{32} \\ 0 & -8 & 0 & \frac{10}{-8} & \frac{2}{-8} & \frac{6}{-8} \\ 0 & 0 & -4 & \frac{1}{-4} & \frac{1}{-4} & \frac{1}{-4} \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow R_1/32 \\ R_2 \Leftrightarrow R_2/-8 \\ R_3 \Leftrightarrow R_3/-4 \end{array}$$

$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

13. (a). (ii) Solve the system of equations by Gauss elimination method:

$2x + y + z = 10,\ \ 3x + 2y + 3z = 18,\ \ \ x + 4y + 9z = 16.$

Solution:

(i**). Gauss elimination method**:

Let the given system of equations be $\quad 2x + y + z = 10$

$$3x + 2y + 3z = 18$$

$$x + 4y + 9z = 16$$

The given system is equivalent to $AX = B$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix} \qquad \textit{Here } [A,B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

Now, we need to make A as a upper triangular matrix.

Fix the first row, change second and third row by using first row.

$$[A, B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow 2R_2 - 3\,R_1 \\ R_3 \Leftrightarrow 2R_3 - R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A, B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix} \quad R_3 \Leftrightarrow R_3 - 7\,R_2$$

This is an upper triangular matrix. From the above matrix we have

$$-4\,z = -20 \quad \Rightarrow \quad z = 5$$

$$y + 3\,z = 6 \quad \Rightarrow \quad y + 3(5) = 6$$

$$\Rightarrow \quad y = 6 - 15 \Rightarrow y = -9$$

$$\Rightarrow \quad y = -9$$

$$2x + y + z = 10$$

$$\Rightarrow 2x - 9 + 5 = 10 \Rightarrow 2x = 10 + 9 - 5 = 14$$

$$\Rightarrow x = 7$$

Hence the solution is $\boldsymbol{x = 7}$, $\boldsymbol{y = -9}$ $\boldsymbol{and}$ $\boldsymbol{z = 5}$

13. (b). (ii). Solve the following system of equations by Gauss-Siedal method of Iteration.

$5x - y + z = 10,\ 2\,x + 4y = 12,\quad x + y + 5z = -1.$

Solution:

$$5x - y + z = 10$$

$$2x + 4y + 0z = 12$$

$$x + y + 5z = -1$$

Since the diagonal elements are dominant in the coefficient matrix, we rewrite x, y, z as follows

$$x = \frac{1}{5}(10 + y - z)$$

$$y = \frac{1}{4}(12 - 2)$$

$$z = \frac{1}{5}(-1 - x - y)$$

Gauss Siedal Method:

We form the Iterations in the table

Iteration	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
1	2	2	−1
2	2.6	1.7	−1.06
3	2.552	1.724	−1.0552
4	2.5558	1.722	−1.0556
5	2.5556	1.722	−1.0556
6	2.5556	1.722	−1.0556

Hence the solution is $\boldsymbol{x = 2.556}$, $\boldsymbol{y = 1.722}$ $\boldsymbol{and}$ $\boldsymbol{z = -1.0556}$.

13. (b). (ii) Determine the largest Eigen value and the corresponding Eigen vector of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ with initial vector $X^{(0)} = [1 \quad 1 \quad 1]^T$.

Solution: Let $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0.375 \\ 0.375 \end{bmatrix} = 8\,X_2$$

$$A\,X_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.375 \\ 0.375 \end{bmatrix} = \begin{bmatrix} 3.625 \\ 1.75 \\ 1.125 \end{bmatrix} = 3.625\begin{bmatrix} 1 \\ 0.483 \\ 0.310 \end{bmatrix} = 3.625\,X_3$$

$$A\,X_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.483 \\ 0.310 \end{bmatrix} = \begin{bmatrix} 4.028 \\ 1.966 \\ 0.930 \end{bmatrix} = 4.028\begin{bmatrix} 1 \\ 0.467 \\ 0.221 \end{bmatrix} = 4.028\,X_4$$

$$A\,X_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.467 \\ 0.221 \end{bmatrix} = \begin{bmatrix} 4.023 \\ 1.934 \\ 0.663 \end{bmatrix} = 4.023\begin{bmatrix} 1 \\ 0.481 \\ 0.165 \end{bmatrix} = 4.023\,X_5$$

$$A\,X_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.481 \\ 0.165 \end{bmatrix} = \begin{bmatrix} 4.051 \\ 1.962 \\ 0.495 \end{bmatrix} = 4.051\begin{bmatrix} 1 \\ 0.484 \\ 0.122 \end{bmatrix} = 4.051\,X_6$$

$$A\,X_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.484 \\ 0.122 \end{bmatrix} = \begin{bmatrix} 4.026 \\ 1.968 \\ 0.366 \end{bmatrix} = 4.026\begin{bmatrix} 1 \\ 0.488 \\ 0.0909 \end{bmatrix} = 4.026\,X_7$$

$$A\,X_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.488 \\ 0.0909 \end{bmatrix} = \begin{bmatrix} 4.019 \\ 1.976 \\ 0.2727 \end{bmatrix} = 4.019\begin{bmatrix} 1 \\ 0.492 \\ 0.068 \end{bmatrix} = 4.02\,X_8$$

∴ The dominant **Eigen value** = 4.02.

Corresponding **Eigen vector** is $\begin{bmatrix} 1 \\ 0.492 \\ 0.068 \end{bmatrix}$.

14. (A). (I). Use Newton's divided difference formula to calculate $f(3)$, $f'(3)$ *and* $f''(3)$ by from the following table.

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{14-1}{1-0} = 13$	$\frac{1-13}{2-0} = -6$	$\frac{-2+6}{4-0} = 1$	
					0
1	14	$\frac{15-14}{2-1} = 1$	$\frac{-5-1}{4-1} = -2$	$\frac{2+2}{5-1} = 1$	
					0
2	15	$\frac{5-15}{4-2} = -5$	$\frac{1+5}{5-2} = 2$	$\frac{6-2}{6-2} = 1$	

4	5	$\frac{6-5}{5-4}=1$	$\frac{13-1}{6-4}$			
5	6	$\frac{19-6}{6-5}=13$				
6	19					

By Newton's divided difference interpolation formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)$$

Here $x_0=0,\ x_1=1,\ x_2=2,\ x_3=4,\ x_4=5\ \ and$

$f(x_0)=1,\ \ f(x_0,x_1)=13,\ \ f(x_0,x_1,x_2)=-6,\ \ f(x_0,x_1,x_2,x_3)=1,\ f(x_0,x_1,x_2,x_3,x_4)=0$

$\therefore\ \ f(x)=1+(x-0)\,(13)+(x-0)(x-1)\,(-6)+(x-0)(x-1)(x-2)\,(1)+(x-0)(x-1)(x-2)(x-4)(0)+0$

$f(x)=1+13x+(x^2-x)\,(-6)+x(x-1)(x-2)$

$f(x)=1+13x-6x^2+6x+x(x^2-3x+2)$

$f(x)=1+13x-6x^2+6x+x^3-3x^2+2x$

$f(x)=x^3-9x^2+22x+1\quad ---(1)$

$\therefore\ \ f(3)=(3)^3-9(3)^2+22(3)+1$

$f(3)=10$

$f(x)=x^3-9x^2+22x+1\quad ---(1)$

$f'(x)=3x^2-18x+22\qquad\Rightarrow\quad f'(3)=3(3)^2-18(3)+22=-5$

$f''(x)=6x-18\qquad\Rightarrow\quad f''(3)=6(3)-18=0$

14. (a). (ii) From the following table of values of x and y, obtain $\frac{dy}{dx}$ $\ and\ $ $\frac{d^2y}{dx^2}$ for y at $x=1.2$

x :	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y :	2.7183	3.3201	4.0552	4.9530	60.496	7.3891	9.025

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	2.7183	0.6018	0.1333		
1.2	**3.3201**	**0.7351**	**0.16273**	0.0294	0.0067
1.4	4.0552	0.8978	0.1988	**0.0361**	**0.008**
1.6	4.9530	1.0966	0.2429	0.0441	0.0094
1.8	6.0496	1.3395	0.2964	0.0535	

2.0 2.2	7.3891 9.0250	1.6359			

To find $y \ at \ x = 1.2, \ choose \ \ x = x_0$. Here $h = 0.2$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \ when \ x = x_0$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.2}\left[0.7351 - \frac{(0.1627)}{2} + \frac{0.0361}{3} - \frac{0.008}{4}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = 3.320$$

$$And \quad \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \cdots\right] \quad when \ x = x_0$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{0.2^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.008)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = 3.348$$

14. (b). (i) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is in the table below. Using trapezoidal rule and Simpson's $\frac{1}{3}$ rule, find the velocity of the rocket at $t = 80$ sec.

t (sec)	0	10	20	30	40	50	60	70	80
f: (cm/sec)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	40.67

Solution:

$$Accelaration \ (A) = \frac{d^2 f}{dt^2}$$

$$Velocity \ V = \int_0^{80} A \, dt$$

t (sec)	0	10	20	30	40	50	60	70	80
f: (cm/sec)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	40.67

Trapezoidal rule

$$V = \int_0^{80} A \, dt = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{10}{2}\left[(30 + 40.67) + 2(31.63 + 33.34 + 35.47 + 37.75 + 40.33 + 43.25 + 46.69)\right]$$

$V = 3037.95 \quad cm/sec.$

Simpson's 1/3 rule :

$$V = \int_0^{80} A \, dt = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 40.67) + 4(33.34 + 37.75 + 43.25) + 2(31.63 + 35.47 + 40.33 + 46.69)\right]$$

$V = 3052.766 \quad cm/sec.$

15. (a). (i). Using 4^{th} order Runge-Kutta method, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0) = 1$ for $x = 0.2$ and $x =$ 0.4with $h = 0.2$.

Solution:

$$Given \quad \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} = f(x, y) \quad , \quad y(0) = 1 \quad \Rightarrow x_0 = 0, \;\; y_0 = 1 \;, \;\; h = 0.2$$

To find $y(0.2)$: $\left[\boldsymbol{x_0 = 0.0, \;\; y_0 = 1, \;\; h = 0.2, \;\; f(x,y) = \frac{y^2-x^2}{y^2+x^2}}\right]$

$$k_1 = h\, f(x_0, y_0) = 0.2\, f(0, \;\; 1\;) = 0.2\left[\frac{(1)^2 - (0)^2}{(1)^2 + (0)^2}\right] = 0.2[1]$$

$\boldsymbol{k_1 = 0.2}$

$$k_2 = 0.2\, f\left(0 + \frac{0.2}{2}, \;\; 1 + \frac{0.2}{2}\;\right) = 0.2\, f(0.1, \;\; 1.1)$$

$\boldsymbol{k_2 = 0.1967213}$

$$k_3 = 0.2\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1967213}{2}\right) = 0.2f(0.1, \;\; 1.0983606)$$

$\boldsymbol{k_3 = 0.1967}$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.2\;\; f(0 + 0.2, \;\; 1 + 0.1967) \quad \boldsymbol{k_4 = 0.1891}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2 + 2(0.1967213) + 2(0.1967) + 0.1891)$$

$\boldsymbol{\Delta y = 0.19598}$

$$y_1(x_0 + h) = y(x_0) + \Delta y \quad \Rightarrow \quad y(0.0 + 0.2) = y(0) + 0.44321 = 1 + 0.19598$$

$\boldsymbol{y(0.2) = 0.19598} \quad [\boldsymbol{y(x_1) = y_1}]$

To find $y(0.4)$: $\left[\boldsymbol{x_1 = 0.2, \;\; y_1 = 0.19598, \;\; h = 0.2, \;\; f(x,y) = \frac{y^2-x^2}{y^2+x^2}}\right]$

$$k_1 = h\, f(x_1, y_1) = 0.2\, f(0.2, \;\; 0.19598\;)$$

$\boldsymbol{k_1 = 0.1891}$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2\, f\left(0.2 + \frac{0.2}{2}, \;\; 0.19598 + \frac{0.1891}{2}\;\right) = 0.2\, f(0.3, \;\; 1.29055)$$

$\boldsymbol{k_2 = 0.17949}$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2\, f\left(0.2 + \frac{0.2}{2}, 0.19598 + \frac{0.17949}{2}\right) = 0.2f(0.3, \;\; 1.28572)$$

$\boldsymbol{k_3 = 0.1793}$

$$k_4 = h\, f(x_1 + h, y_1 + k_3) = 0.2\;\; f(0.2 + 0.2, \;\; 0.19598 + 0.1793) = 0.2\, f(0.4, \;\; 1.37528)$$

$$k_4 = 0.1687$$

$\boldsymbol{k_4 = 0.6111}$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.1891 + 2(0.17949) + 2(0.1793) + 0.1687)$$

$\boldsymbol{\Delta y \;= 0.1792}$

$$y_2(x_1 + h) = y(x_1) + \Delta y \quad \Rightarrow \quad y(0.2 + 0.1) = y(0.2) + 0.1792 = 1.19598 + 0.44321$$

$\boldsymbol{y(0.4) = 1.3751}$

15. (a). (ii). Given $\frac{dy}{dx} = 1 + y^2$, where $x = 0$, find $y(0.2)$, $y(0.4)$ and $y(0.6)$ using Taylor's series method.

Solution:

$$Given \quad \frac{dy}{dx} = 1 + \boldsymbol{y^2} \quad \& \quad x_0 = 0, \;\; y_0 = 0$$

Taylor series formula is

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots\ldots(1)$$

$y' = 1 + y^2$	$y_0' = 1 + y_0^2 = 1 + 0$	$y_0' = 1$
$y'' = 0 + 2y\,y'$	$y_0'' = 2y_0\, y_0' = 2(0)(1)$	$y_0'' = 0$
$y''' = 2y\,y'' + 2y'y'$ $y''' = 2y\,y'' + 2(y')^2$	$y_0''' = 2y_0\, y_0'' + 2(y_0')^2$ $= 2(0)0 + 2(1)^2$	$y_0''' = 2$
$y'^{v} = 2y\,y''' + 2y'y'' + 4y'\,y''$ $y'^{v} = 2y\,y''' + 6y'y''$	$y_0'^{v} = 2y_0\, y_0''' + 6y_0'y_0''$ $= 2(0)2 + 6(1)0$	$y_0'^{v} = 0$
$y^{v} = 2y\,y'^{v} + 2y'''y' + 6y'\,y'''$ $+ 6y''y''$	$y_0^{v} = 2y_0\, y_0'^{v} + 8y_0'y_0''' + 6{y_0''}^2$ $= 0 + 8(2)(1) + 0$	$y_0^{v} = 16$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \frac{(x-x_0)^5}{5\,!}\, y_0^{v}$$

$$= 0 + \frac{(x-0)}{1}\,(1) + \frac{(x-0)^2}{2}\,(0) + \frac{(x-0)^3}{6}\,(2) + \frac{(x-0)^4}{24}\,(0) + \frac{(x-x_0)^5}{5\,!}\,16$$

$$y(x) = x + \frac{x^3}{6}\,(2) + \frac{x^5}{120}\,(16)$$

To find $\boldsymbol{y}$ $\boldsymbol{at}$ $\boldsymbol{x = 0.2}$

$$\therefore \quad y(0.2) = 0.2 + \frac{(0.2)^3}{6}\,(2) + \frac{(0.2)^5}{120}(16)$$

$$\therefore \quad y(0.2) = 0.2027$$

To find $\boldsymbol{y}$ $\boldsymbol{at}$ $\boldsymbol{x = 0.4}$

$$\therefore \quad y(0.4) = 0.4 + \frac{(0.4)^3}{6}\,(2) + \frac{(0.4)^5}{120}(16)$$

$$\boldsymbol{y(0.4) = 0.42269}$$

To find $\boldsymbol{y}$ $\boldsymbol{at}$ $\boldsymbol{x = 0.6}$

$$\therefore \quad y(0.6) = 0.6 + \frac{(0.6)^3}{6}\,(2) + \frac{(0.6)^5}{120}(16)$$

$$\boldsymbol{y(0.6) = 0.6824}$$

15. (b). (i) Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1, y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979, evaluate$ $y(1.4)$ by using Adam-Bashforth predictor corrector method Method.

Solution: Given

$$y' = f(x,y) = \frac{dy}{dx} = x^2(1+y)$$

$y(1) = 1$	$y(x_0) = y_0$	$x_0 = 1$	$y_0 = 1$
$y(1.1) = 1.233$	$y(x_1) = y_1$	$x_1 = 1.1$	$y_1 = 1.233$
$y(1.2) = 1.548$	$y(x_2) = y_2$	$x_2 = 1.2$	$y_2 = 1.548$
$y(1.3) = 1.979$	$y(x_3) = y_3$	$x_3 = 1.3$	$y_3 = 1.979$

$\boldsymbol{Here} \quad \boldsymbol{h = 0.1}$ **and** $\boldsymbol{n = 3}$

The Adam's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37\, y'_{n-2} - 9\, y'_{n-3}\right] \quad \ldots\ldots (1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37\, y'_1 - 9\, y'_0\right] \qquad \ldots\ldots (2)$$

$Given \quad y' = x^2(1 + y)$

$x_0 = 1$	$y_0 = 1$	$y'_0 = x_0^2(1 + y_0)$	$y'_0 = (1)^2(1 + 1)$	$y'_1 = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y'_1 = x_1^2(1 + y_1)$	$y'_1 = (1.1)^2(1 + 1.233)$	$y'_1 = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$y'_2 = x_2^2(1 + y_2)$	$y'_2 = (1.2)^2(1 + 1.548)$	$y'_2 = 3.66912$
$x_3 = 1.3$	$y_3 = 1.979$	$y'_3 = x_3^2(1 + y_3)$	$y'_3 = (1.3)^2(1 + 1.979)$	$y'_3 = 2.0345$

Equation (2) becomes

$$y_{4,P}(1.3 + 0.1) = 1.979 + \frac{0.1}{24}\left[55(2.0345) - 59(3.66912) + 37\,(2.70193) - 9\,(2)\right]$$

$$y_{4,P}(1.4) \;\; = 1.979 + \frac{0.1}{24}[142.33683] = 1.979 + 0.593070$$

$$\boldsymbol{y_{4,P}(1.4) \;\; = 2.5721 \quad [y(x_4) = y_4, \quad x_4 = 1.4 \;\; \& \;\; y_4 = 2.5721\,]}$$

The Adams's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_n + \frac{h}{24}\left[\,9\, y'_{n+1} + 19y'_n - 5y'_{n-1} + \, y'_{n-2}\right] \quad \ldots\ldots (3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_3 + \frac{h}{24}\left[\,9\, \boldsymbol{y'_4} + 19y'_3 - 5y'_2 + \, y'_1\right] \quad \ldots\ldots (4)$$

$x_4 = 1.4$	$y_4 = 2.5721$	$y'_4 = x_4^2(1 + y_4)$	$y'_4 = (1.4)^2(1 + 1.2751)$	$y'_4 = 7.7030716$

Equation (4) becomes

$$y_{4,C}(1.3 + 0.1) = 1.979 + \frac{0.1}{24}\left[\,9\,(7.7030716) + 19\,(5.0345) - 5\,(3.60912) + \,(2.70193)\right]$$

$$y_{4,C}(0.8) \;\; = 1.979 + \frac{0.1}{24}[143.58827] = 1.979 + 0.592844$$

$$\boldsymbol{y_{4,C}(0.8) \;\; = 2.57728 \quad [y(x_4) = y_4, \quad x_4 = 1.4 \;\; \& \;\; y_4 = 2.57728\,]}$$

Result:

$$\boldsymbol{y_{4,P}(1.4) = 2.5721} \quad \& \quad \boldsymbol{y_{4,C}(1.4) = 2.5778}$$

15. (b). (i). Using finite differences solve the boundary value problem

$y'' + 3y' - 2y = 2x + 3,\ y(0) = 2,\ y(1) = 1$ with $h = 0.2$.

Solution:

x_0	x_1	x_2	x_3	x_4	x_5
0	0.2	0.4	0.6	0.8	1

$$y(0) = 2 \quad \Longrightarrow \quad x_0 = 0,\ y_0 = 2 \quad and \quad y(1) = 1 \quad \Longrightarrow \quad x_5 = 1,\ y_5 = 1,\ h = 0.2$$

The given differential equation can be written as

$$y''(x) + 3y'(x) - 2y(x) = 2x + 3$$

Using the central difference approximation, we have

$$y'' = \frac{y_{i-1} + y_{i+1} - 2y_i}{h^2} \quad and \quad y' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\left[\frac{y_{i-1} + y_{i+1} - 2y_i}{h^2}\right] + 3\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] - 2y_i = 2x_i + 3$$

$$[y_{i-1} + y_{i+1} - 2] + 3h[y_{i+1} - y_{i-1}] - 2\,h^2\,y_i - 2\,h^2\,x_i - 3h^2 = 0$$

$$y_{i+1}(1 + 3h) + y_{i-1}(1 - 3h) + y_i(-2h^2) = 2h^2 x_i + 3h^2 + 2 \ --- (A)$$

Solving, we get

$$y_1 = 1.005,\quad y_2 = 0.6362,\ \ y_3 = 0.5937,\ \ y_4 = 0.7363$$