ANNA UNIVERSITY CHENNAI

MATHEMATICS III

NOVEMBER / DECEMBER 2011

Part-A

1. State the dirichlet's conditions for the existence of the Fourier expansion of $f(x)$, in the intreval $(0,\ 2\pi)$.

Ans:

A function $f(x)$ is defined in $0 \le x \le 2\pi$, it can be expanded as a Fourier series of the form

$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, the following conditions should be satisfied

i. $f(x)$ is a single valued and finite in $(0,\ 2\pi)$.

ii. $f(x)$ is continuous or piece wise continuous with finite number of finite discontinuities in $(0,\ 2\pi)$.

iii. $f(x)$ has a finite number of maxima or minima in $(0,\ 2\pi)$.

2. Find the root mean square value of the function $f(x) = x$ in $(0,\ l)$.

Ans:

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}} = \sqrt{\frac{\int_0^l [x]^2\,dx}{l-0}} = \sqrt{\frac{\left(\frac{x^3}{3}\right)_0^l}{l}}$$

$$\boldsymbol{R.M.S} = \sqrt{\frac{\frac{l^3}{3}}{l}} = \sqrt{\frac{l^3}{3l}} = \sqrt{\frac{l^2}{3}} = \frac{\boldsymbol{l}}{\sqrt{\mathbf{3}}}$$

3. Write the Fourier Transform Pair

Ans: The Fourier Transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{isx}\ d$$

The Inverse Fourier Transform of $f(x)$ is

$$f(x) = [F(s)]\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\ e^{-isx}\,ds$$

4. State Parseval's identity on Fourier transform.

Ans:

$$\int_0^{\infty} |f(x)|^2\,dx = \int_0^{\infty} |F(s)|^2\,ds \qquad or \qquad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

5. Find the P.D.E of the family of spheres having their centres on the z –axis.

M. Vijaya Kumar

Ans:

The equation of the sphere is

$$(x-a)^2+(y-a)^2+(z-a)^2=r^2$$

partially differentiating with respect to $'x'$ and $'y'$, we get

$$2(x-a)+2(z-a)p=0$$
$$(x-a)+(z-a)p=0$$
$$x-a+zp-ap=0$$
$$x+zp-a(1+p)=0$$
$$\boldsymbol{a=\frac{x+zp}{1+p}} \quad ---(\mathbf{1})$$

And

$$2(y-a)+2(z-a)q=0$$
$$(y-a)+(z-a)q=0$$
$$y-a+zq-aq=0$$
$$y+zq-a(1+q)=0$$
$$\boldsymbol{a=\frac{x+zp}{1+p}} \quad ---(\mathbf{2})$$

From (1) and (2), we have

$$\boldsymbol{\frac{x+zp}{1+p}=\frac{x+zp}{1+p}}$$
$$(x+zp)(1+q)=(1+p)(y+zq)$$
$$x+qx+zp+zpq=y+zq+py+pqz$$
$$\boldsymbol{p(z-y)+q(x-z)=y-x}$$

which is the required P.D.E.

6. Solve the equation $(\boldsymbol{D-D'})^3\boldsymbol{z}=\mathbf{0}$.

Ans:

The auxiliary equation is $\quad (m-1)^3=0 \quad \Rightarrow \quad m=1,1,1$

$$\therefore \quad z=f_1(y+x)+xf_2(y+x)+x^2f_3(y+x)$$

7. In the wave equation $\boldsymbol{\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}}$, what does stands for?

Ans:

$$c^2=\frac{T}{m}=\frac{Tension}{mass\ per\ unit\ length\ of\ the\ string}$$

8. A plate is bounded by the lines $x=0,\ y=0,\ x=l\ and\ y=l$. Its faces are insulated. The edge coinciding with x –axis is kept at 100^0C. The edge coinciding with y –axis is kept at 50^0c. The other

M. Vijaya Kumar

two edges are kept at 0^0C . Write the boundary conditions that are needed for solving two dimensional heat flow equation.

Ans:

i. $u(o,\ y) = 50$ $\qquad$ $ii.$ $u(l,\ 0) = 100$

ii. $u(l,\ y) = 0^0C$ $\qquad$ $iii.$ $u(x,\ y) = 0^0C$

9. Find z –transform of $\frac{1}{n!}$.

Ans:

$$\text{W.k.t} \quad Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) * \left(\frac{1}{z}\right)^n$$

$$= \frac{1}{0!}\left(\frac{1}{z}\right)^0 + \frac{1}{1!}\left(\frac{1}{z}\right)^1 + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right)^1 + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$$Z\left\{\frac{1}{n!}\right\} = e^{\left(\frac{1}{z}\right)} \qquad \left[\because\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

10. Form a difference equation by eliminating arbitrary constants from $U_n = A\,2^{n+1}$.

Ans:

$$U_n = A\,2^{n+1}$$

$$U_{n+1} = A\,2^{n+2} = A\,2^{n+1} + 2 = 2\,U_n$$

$$i.e.,\quad U_{n+1} - 2\,U_n = 0$$

Part B

11. (a). (i). Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2 \quad in\ \ 0 < x < 3$.

Ans:

$$\text{Given} \quad 2l = 3 \quad \Longrightarrow \quad l = \frac{3}{2}$$

The Fourier series for the function $f(x)$ in $(0,\ 2l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

M. Vijaya Kumar

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\ dx\ = \frac{1}{l}\int_0^{2l} (2x - x^2)\ dx$$

$$= \frac{1}{l}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^{2l} = \frac{1}{l}\left[4l^2 - \frac{8l^3}{3}\right]$$

Put $\boldsymbol{l = \frac{3}{2}}$***, we get***

$$= \frac{2}{3}\left[4\left(\frac{9}{4}\right) - \frac{8\left(\frac{27}{8}\right)}{3}\right]$$

$$\boldsymbol{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\ \cos\left(\frac{n\pi x}{l}\right)\ dx$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\ \cos\left(\frac{n\pi x}{l}\right)\ dx$$

$$= \frac{1}{l}\left[(2x - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2 - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[(2 - 2x)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{2l} = \frac{1}{l}\left[(2 - 4l)(\cos 2n\pi)\left(\frac{l^2}{n^2\pi^2}\right) - (2 - 0)(\cos 0)\left(\frac{l^2}{n^2\pi^2}\right)\right]$$

$$= \frac{1}{l}\left[(2 - 4l)\left(\frac{l^2}{n^2\pi^2}\right) - 2\left(\frac{l^2}{n^2\pi^2}\right)\right] = \frac{1}{l}\left(\frac{l^2}{n^2\pi^2}\right)[2 - 4l - 2]$$

$$\boldsymbol{a_n = \frac{-4l^2}{n^2\pi^2}}$$

Put $\boldsymbol{l = \frac{3}{2}}$***, we get*** $\quad \boldsymbol{a_n = \frac{-4\left(\frac{9}{4}\right)}{n^2\pi^2} = \frac{-9}{n^2\pi^2}}$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\ \sin\left(\frac{n\pi x}{l}\right)\ dx$$

$$b_n = \frac{2}{l}\int_0^{2l} (2x - x^2)\sin\left(\frac{n\pi x}{l}\right)\ dx$$

M Vijaya Kumar

$$= \frac{1}{l}\left[(2x - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{-1}{l}\left[(2x - x^2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{-1}{l}\left[\left((4l - 4l^2)(\cos 2n\pi)\left(\frac{l}{n\pi}\right) + 2\ \cos 2n\pi\left(\frac{l^3}{n^3\pi^3}\right)\right) - \left(0 + 2\left(\frac{l^3}{n^3\pi^3}\right)\right)\right]$$

$$= \frac{-1}{l}\left[(4l - 4l^2)\left(\frac{l}{n\pi}\right) + \frac{2l^3}{n^3\pi^3} - \frac{2l^3}{n^3\pi^3}\right]$$

$$= (l - l^2)\left(\frac{-4}{n\pi}\right)$$

Put $\boldsymbol{l = \frac{3}{2}}$***, we get*** $\quad b_n = \left(\frac{3}{2} - \frac{9}{4}\right)\left(\frac{-4}{n\pi}\right)$

$$\boldsymbol{b_n = \frac{3}{n\pi}}$$

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty}\left(\left(\frac{-9}{n^2\pi^2}\right)\cos\left(\frac{2n\pi x}{3}\right) + \left(\frac{3}{n\pi}\right)\sin\left(\frac{2n\pi x}{3}\right)\right)}$$

M. Vijaya Kumar

11. (a). (ii). Obtain the Fourier series of $f(x) = x\sin x$ in $(-\pi,\ \pi)$.

Ans:

$$f(x) = x\ sin\ x\ ,\quad f(-x) = -x\ sin\ (-x) = x\ sin\ x$$

$$\therefore\quad f(x) = f(-x),\quad f(x)\ \ is\ \ even\ \ function$$

The Fourier series of $f(x)\ \ in\ \ (-\pi,\ \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ----(1)$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{+\pi} x\sin x\,dx = \frac{2}{\pi}[x\,(-\cos x) - (-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}[\pi\cos\pi - 0] = \frac{2}{\pi}[\pi] = 2$$

$$\boldsymbol{a_0 = 2}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{+\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{+\pi} x\sin x\cos nx\,dx = \frac{2}{\pi}\int_0^{+\pi} x\ \cos nx\ \sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{+\pi} x\left(\frac{\sin(n+1)x - \sin(n-1)x}{2}\right)dx = \frac{1}{\pi}\int_0^{+\pi} x\,[\sin(n+1)x - \sin(n-1)\,x]\,dx$$

$$= \frac{1}{\pi}\left[\left(x\left(\frac{-\cos(n+1)\,x}{n+1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right) - \left(x\left(\frac{-\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)\,x}{n+1}\right) - x\left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-\cos(n+1)\pi}{n+1}\right) + \pi\left(\frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= \frac{\pi}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right] \qquad \langle \because \cos(n+1)\pi = \cos(n-1)\pi = (-1)^n \rangle$$

$$= (-1)^{n+1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right] = (-1)^{n+1}\left[\frac{-n+1+n+1}{n^2-1^2}\right] = (-1)^{n+1}\left[\frac{2}{n^2-1}\right]$$

$$\boldsymbol{a_n = (-1)^{n+1}\left[\frac{2}{n^2-1}\right] \quad if\ \ n \neq 1}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{+\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{+\pi} x\,\frac{sin\,2x}{2}\,dx = \frac{1}{\pi}\int_0^{+\pi} x\,sin\,2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-\cos 2\pi}{2}\right) - 0\right] = -\frac{1}{2}$$

$$\boldsymbol{a_1 = -\frac{1}{2}}$$

$$\boldsymbol{\therefore\ \ x\,sin\,x = 1 - \frac{1}{2}cos\,x + \sum_{n=2}^{\infty}\left((-1)^{n+1}\left[\frac{2}{n^2-1}\right]cos\,nx\right)}$$

11. (b). (i). Obtain the Fourier cosine series expansion of $x\sin x$ in $(0,\ \pi)$ and hence find the value of $1 + \frac{2}{1\,.\,3} - \frac{2}{3.\ 5} + \frac{2}{5.\ 7} + \cdots$

Ans: The half range Fourier cosine series of $f(x)\ in\ (0,\ \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \quad ----(1)$$

To find a_0:

M. Vijaya Kumar

$$a_0 = \frac{2}{\pi}\int_0^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{+\pi} x\sin x\,dx = \frac{2}{\pi}[x\,(-\cos x) - (-\sin x)]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}[\pi\cos\pi - 0] = \frac{2}{\pi}[\pi] = \mathbf{2}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{+\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{+\pi} x\sin x\cos nx\,dx = \frac{2}{\pi}\int_0^{+\pi} x\,\cos nx\,\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{+\pi} x\left(\frac{\sin(n+1)x - \sin(n-1)x}{2}\right)dx = \frac{1}{\pi}\int_0^{+\pi} x\,[\sin(n+1)x - \sin(n-1)\,x]\,dx$$

$$= \frac{1}{\pi}\left[\left(x\left(\frac{-\cos(n+1)\,x}{n+1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right) - \left(x\left(\frac{-\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)\,x}{n+1}\right) - x\left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-\cos(n+1)\pi}{n+1}\right) + \pi\left(\frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= \frac{\pi}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right] \qquad \langle \because \cos(n+1)\pi = \cos(n-1)\pi = (-1)^n\rangle$$

$$= (-1)^{n+1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right] = (-1)^{n+1}\left[\frac{-n+1+n+1}{n^2-1^2}\right] = (-1)^{n+1}\left[\frac{2}{n^2-1}\right]$$

$$\boldsymbol{a_n = (-1)^{n+1}\left[\frac{2}{n^2-1}\right] \quad if\ \ n \neq 1}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{+\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{+\pi} x\,\frac{\sin 2x}{2}\,dx = \frac{1}{\pi}\int_0^{+\pi} x\sin 2x\,dx$$

$$\boldsymbol{a_1} = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-\cos 2\pi}{2}\right) - 0\right] = -\frac{\mathbf{1}}{\mathbf{2}}$$

$$\therefore\ \ \boldsymbol{x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left((-1)^{n+1}\left[\frac{2}{n^2-1}\right]\cos nx\right)}$$

Deduction: **Put $x = \frac{\pi}{2}$, a point of continuity, we get**

$$\therefore\ \ \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + \sum_{n=2}^{\infty}\left((-1)^n(-1)\left[\frac{2}{n^2-1}\right]\cos\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = 1 - \sum_{n=2}^{\infty}\left((-1)^n\left[\frac{2}{(n+1)(n-1)}\right]\cos\frac{n\pi}{2}\right)$$

M. Vijaya Kumar

$$\frac{\pi}{2} = 1 - \left[\frac{\mathbf{1}}{\mathbf{1\,.\,3}} - \frac{\mathbf{1}}{\mathbf{3.\ 5}} + \frac{\mathbf{1}}{\mathbf{5.\ 7}} - \cdots\right]$$

$$\left[\frac{\mathbf{1}}{\mathbf{1\,.\,3}} - \frac{\mathbf{1}}{\mathbf{3.\ 5}} + \frac{\mathbf{1}}{\mathbf{5.\ 7}} - \cdots\right] = \mathbf{1} - \frac{\boldsymbol{\pi}}{\mathbf{2}} = \frac{\mathbf{2} - \boldsymbol{\pi}}{\mathbf{2}}$$

11. (b). (ii). The following table gives the variations of a periodic function over a period T

x:	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$:	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

Find the fundamental and first harmonics of $f(x)$ to express $f(x)$ in a Fourier series in the form

$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$, where $\theta = \frac{2\pi x}{T}$.

Solution: Here $n = 6$.

$$Given \quad \theta = \frac{2\pi x}{T} \quad ---(1)$$

$By\ using\ (1), \theta\ takes\ the\ values\ 0, \quad \frac{\pi}{3}, \quad \frac{2\pi}{3}, \ \pi, \frac{4\pi}{3}, \ \frac{5\pi}{3}.$

θ	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\ \sin\theta$
0^0	1.98	1	0	1.98	0
$\frac{\pi}{3}$	1.3	0.5	0.866	0.65	1.1258
$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.525	0.9093
π	1.3	-1	0	-1.3	0
$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5\pi}{3}$	-0.25	0.5	-0.866	-0.125	0.2165
	4.5			1.12	3.013

The Fourier series takes the form

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$

Where

$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{4.5}{6}\right) = 1.5, \qquad a_1 = 2\left(\frac{\sum y\cos\theta}{n}\right) = 2\left(\frac{1.12}{6}\right) = 0.37$$

$$b_1 = 2\left(\frac{\sum y\sin\theta}{n}\right) = 2\left(\frac{3.013}{6}\right) = 1.00456$$

M. Vijaya Kumar

$$\therefore \quad f(x) = \frac{1.5}{2} + 0.37\cos\theta + 1.0045\ \sin\theta$$

12. (a). (i). Show that $e^{-\frac{x^2}{2}}$ is a self reciprocal with respect to Fourier transform.

Ans: The Fourier transform of $f(x)$ is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{isx}\, dx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{-a^2x^2}\right]\ e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\, dx$$

$$a^2x^2 - isx = A^2 - 2AB \quad \Rightarrow \quad A = ax\ \ \&\ \ 2AB = isx \Rightarrow\ 2(ax)B = isx \Rightarrow B = \frac{is}{2a}$$

$$(A-B)^2 = \left(ax - \frac{is}{2a}\right)^2 = a^2x^2 - isx + \left(\frac{is}{2a}\right)^2 \Rightarrow a^2x^2 - isx = \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\langle -\left[\left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]\right\rangle\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}\ e^{\left(\frac{is}{2a}\right)^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\ e^{\left(\frac{-s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}\, dx$$

$$Put \quad t = ax - \frac{is}{2a} \qquad dt = a\, dx \ \Rightarrow \quad dx = \frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}}\ e^{\left(\frac{-s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-t^2}\ \frac{dt}{a} = \frac{1}{\sqrt{2\pi}}\ \frac{e^{\left(\frac{-s^2}{4a^2}\right)}1}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{1}{a\sqrt{2\pi}}\ e^{\left(\frac{-s^2}{4a^2}\right)}\ \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}\right]$$

M. Vijaya Kumar

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}}\, e^{\left(\frac{-s^2}{4a^2}\right)}$$

Deduction:

We have to find $F\left[e^{-\frac{x^2}{2}}\right]$, use $a^2 = \frac{1}{2}$, $a = \frac{1}{\sqrt{2}}$, we get

$$F\left[e^{-\frac{x^2}{2}}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}}\, e^{\left(\frac{-s^2}{4\left(\frac{1}{2}\right)}\right)} = e^{-\frac{s^2}{2}}$$

Hence the proof.

12. (a). (ii). Find the Fourier Transform of the function

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$ **and hence find the value of** $\int_0^{\infty}\left(\frac{\sin^4 t}{t^4}\right) dt = \frac{\pi}{3}$.

Solution:

Take $f(x) = \begin{cases} a-|x|, & -a<x<+a \\ 0, & |x|>a \end{cases}$

The Fourier transform of $f(x)$ is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a} (a-|x|)\,[\cos sx + i\sin sx]\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a} (a-|x|)\cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-a}^{+a} (a-|x|)\sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{+a} (a-x)\cos sx\, dx + 0 \quad [odd\ fn]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right) - \left(\frac{\cos sx}{s^2}\right)\right]_0^a$$

M. Vijaya Kumar

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\cos sa}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$\boldsymbol{F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos sa}{s^2}\right]}$$

Deduction:

By definition of Parseval's identity, we have

$$\boldsymbol{\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F(s)|^2\, ds}$$

$$\int_0^a (a-x)^2\, dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left[\frac{1-\cos sa}{s^2}\right]\right)^2 ds$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{2\sin^2 \frac{sa}{2}}{s^2}\right)^2 ds = \left[\frac{(a-x)^3}{3(-1)}\right]_0^a = \left[\frac{0-a^3}{-3}\right]$$

$$\int_0^\infty 4\left(\frac{\sin^2 \frac{sa}{2}}{s^2}\right)^2 ds = \frac{a^3}{3}\, \frac{\pi}{2}$$

$$Put \quad a = 1, \quad \frac{s}{2} = t \Rightarrow \quad s = 2t\,, \quad ds = 2\, dt\,, \quad we\ get$$

$$\int_0^\infty 4\left(\frac{\sin^2 t}{(2t)^2}\right)^2 2\, dt = \frac{1}{3}\, \frac{\pi}{2}$$

$$\int_0^\infty 4\left(\frac{\sin^2 t}{t^2}\right)^2 \left(\frac{1}{16}\right) 2\, dt = \frac{1}{3}\, \frac{\pi}{2}$$

$$\int_0^\infty \left(\frac{\sin^2 t}{t^2}\right)^2 \left(\frac{8}{16}\right) dt = \frac{1}{3}\, \frac{\pi}{2}$$

$$\int_0^\infty \left(\frac{\sin^2 t}{t^2}\right)^2 dt = \frac{\pi}{3}$$

$$\boldsymbol{\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}}$$

12. (b). (i). Find the Fourier sine transform of e^{-ax} and hence evaluate Fourier cosine transform of xe^{ax} *and* $xe^{ax} \sin ax$.

M. Vijaya Kumar

Ans:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx \; dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin sx \; dx = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2}\right]$$

Deduction (i):

$$F_c[x\, f(x)] = \frac{d}{ds} F_s[f(x)] \qquad [By\ Property]$$

$$F_c[x\, e^{-ax}] = \frac{d}{ds} F_s[e^{-ax}] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{(a^2+s^2)-s(2s)}{(a^2+s^2)^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{(a^2+s^2)-2s^2}{(a^2+s^2)^2}\right]$$

$$\boldsymbol{F_c[x\, e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(a^2+s^2)^2}\right]}$$

Deduction (ii):

$$\boldsymbol{F_c[\, e^{-ax} \sin ax]} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin ax \cos sx \; dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\frac{[\sin(s+a)x - \sin(s-a)x]}{2} \; dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin(s+a)x \; dx - \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin(s-a)x \; dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s-a)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{a^2+s^2+a^2+2sa} - \frac{s-a}{a^2+s^2+a^2-2sa}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{2a^2+s^2+2sa} - \frac{s-a}{2a^2+s^2-2sa}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(2a^2+s^2-2sa)(s+a)-(2a^2+s^2+2sa)(s-a)}{(2a^2+s^2+2sa)(2a^2+s^2-2sa)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(2a^2s+s^3-2s^2a+2a^3+as^2-2sa^2)-(2a^2s+s^3+2s^2a-2a^3-as^2-2sa^2)}{(2a^2+s^2)^2-(2sa)^2}\right]$$

M. Vijaya Kumar

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{4a^3 - as^2}{4a^4 + s^4 + 2(2a^2s^2) - 4s^2a^2} \right]$$

$$\boldsymbol{F_c[\, e^{-ax} \sin ax]} = \frac{1}{\sqrt{2\pi}} \left[\frac{4a^3 - as^2}{4a^4 + s^4} \right]$$

12. (b). (ii) State and prove convolution theorem of Fourier transforms.

Ans:

Statement: If (s) and $G(s)$ are the functions of $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform

$$F\,[(f * g)\, x] = F(s)\,.G(s)$$

Proof:

$$\boldsymbol{F[(f * g)x]} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)x \; e^{isx}\, d$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).g(x-t)\, dt \right] e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).g(x-t)\, dt \right] e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} .g(x-t)\, e^{isx}\, dx \right] dt \quad \left[\begin{matrix} on\ interchanging\ the \\ order\ of\ \ integration \end{matrix} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(F[g(x-t)])\, dt \quad [Defn\ of\ F.T]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} G(s)\, dt \quad [\because\ \ f(x-a) = e^{ias}F(s)]$$

$$= G(s)\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its}\, dt$$

$$\boldsymbol{F[(f * g)x] = G(s) \, * F(s)}$$

Hence the proof.

13. (a).(i). Find the singular integral of $\boldsymbol{z = px + qy + \sqrt{1 + p^2 + q^2}}$**.**

M. Vijaya Kumar

Ans:

$$Given\ z = px + qy + \sqrt{1 + p^2 + q^2} \quad ---(1)$$

The complete solution is

$$z = ax + by + \sqrt{1 + a^2 + b^2} \quad ---(2)$$

Diff (2) partially w. r. t $'a'\ and\ 'b'$, we get

$$0 = x + \frac{1}{2\sqrt{1 + a^2 + b^2}}(2a) \Rightarrow x = \frac{-a}{\sqrt{1 + a^2 + b^2}} \Rightarrow \boldsymbol{a = -x\sqrt{1 + a^2 + b^2}} \quad ---(3)$$

$$0 = y + \frac{1}{2\sqrt{1 + a^2 + b^2}}(2b) \Rightarrow y = \frac{-b}{\sqrt{1 + a^2 + b^2}} \Rightarrow \boldsymbol{b = -y\sqrt{1 + a^2 + b^2}} \quad ---(4)$$

$$\text{From (3),} \quad \boldsymbol{x^2 = \frac{a^2}{1 + a^2 + b^2}} \quad \text{and From (4),} \quad \boldsymbol{y^2 = \frac{b^2}{1 + a^2 + b^2}} \quad ---(\mathbf{5})$$

From the above equations, we have

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - (x^2 + y^2) = 1 - \left(\frac{a^2 + b^2}{1 + a^2 + b^2}\right)$$

$$1 - (x^2 + y^2) = \left(\frac{1 + a^2 + b^2 - a^2 - b^2}{1 + a^2 + b^2}\right)$$

$$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2} \Rightarrow 1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

$$\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}} \quad ---(6)$$

Using the value (6) in equation $(3)\ and\ (4)$, we get

$$\boldsymbol{a = \frac{-x}{\sqrt{1 - x^2 - y^2}} \quad and \quad b = \frac{-y}{\sqrt{1 - x^2 - y^2}}} \quad ---(\mathbf{7})$$

Using the value (7) in equation (2), we get

$$z = \frac{-x^2}{\sqrt{1 - x^2 - y^2}} + \frac{-y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}} = \sqrt{1 - x^2 - y^2}$$

M. Vijaya Kumar

$$z^2 = 1 - x^2 - y^2$$

$$\boldsymbol{x^2 + y^2 + z^2 = 1}$$

This is the required singular integral.

13. (a). (ii). Solve the partial differential equation $\boldsymbol{x(y-z)p + y(z-x)q = z(x-y)}$.

Ans:

The subsidiary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Using the multipliers $1,\ 1,\ 1$ we get

$$\frac{dx + dy + dz}{xy - xz + yz - yx + zx - zy} = \frac{dx + dy + dz}{0}$$

$$i.e.,\ \ dx + dy + dz = 0 \qquad [\because\ \ Nr = 0]$$

Integrating, we get

$$x + y + z = c_1$$

$$\boldsymbol{i.e., \qquad u = x + y + z = c_1}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$i.e.,\ \ \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \qquad [\because\ \ Nr = 0]$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$\boldsymbol{i.e., \qquad v = xyz = c_2}$$

The solution of given equation is $\Phi(u,\ v) = 0$.

$$\boldsymbol{i.e., \qquad \Phi(x + y + z,\ xyz) = 0}$$

13. (b). (i). Solve $\mathbf{(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y}$.

M. Vijaya Kumar

Ans:

The auxiliary equation is $m^3 - 2m^2 = 0$

$$m^2(m-2) = 0,$$

The roots are $m = 0,\ 0,\ 2.$

The complementary function is

$$\boldsymbol{C.F = f_1(y) + xf_2(y) + f_3(y+2x)}$$

To find Particular Integral:

$$P.I = \frac{1}{D^3 - 2D^2D'}\ [2e^{2x} + x^2y]$$

$$= \left[\frac{1}{D^3-2D^2D'}\ 2(e^{2x})\right] + \left[\frac{1}{D^3-2D^2D'}\ (3x^2y)\right]$$

$$= 2\left[e^{2x}\ \frac{1}{2^3 - 2(2^2)(0)}\right] + 3\left[\frac{1}{D^3\left(1-\frac{2D'}{D}\right)}(x^2y)\right]$$

$$= 2\left[e^{2x}\ \frac{1}{8}\right] + 3\left[\frac{1}{D^3}\left(1-\frac{2D'}{D}\right)^{-1}\ (x^2y)\right]$$

$$= \frac{e^{2x}}{4} + 3\frac{1}{D^3}\left[1 - \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 - \left(\frac{2D'}{D}\right)^3 + \cdots\right]x^2y$$

$$= \frac{e^{2x}}{4} + 3\frac{1}{D^3}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \frac{8D'^3}{D^3} + \cdots\right]x^2y$$

$$= \frac{e^{2x}}{4} + 3\left[\frac{1}{D^3}(x^2y) + \frac{1}{D^3}\frac{2D'}{D}(x^2y) + \frac{1}{D^3}\frac{4D'^2}{D^2}(x^2y)\right]$$

$$= \frac{e^{2x}}{4} + 3\left[\frac{x^5y}{60} + 2\frac{x^6}{360}\right]$$

The complete solution is $\boldsymbol{z = C.F + P.I}$

$$\boldsymbol{z = f_1(y) + +xf_2(y) + f_3(y+2x) + \frac{e^{2x}}{4} + 3\left[\frac{x^5y}{60} + 2\frac{x^6}{360}\right]}$$

13. (b). (ii). Solve $\boldsymbol{(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = e^{2x-y}}$.

Ans:

The given equation can be written as

M. Vijaya Kumar

$$\{(D-D'-1)(D-D'-2)\}\boldsymbol{z}=\boldsymbol{e^{2x-y}}.$$

We know that the complementary function of $\{(\boldsymbol{D}-\boldsymbol{m_1}\boldsymbol{D'}-\propto_{\boldsymbol{1}})(\boldsymbol{D}-\boldsymbol{m_2}\boldsymbol{D'}-\propto_{\boldsymbol{1}})\}Z=0$ IS

$$z=e^{\propto_1 x}f_1(y+m_1x)+e^{\propto_2 x}f_2(y+m_2x)$$

$$Here\ \ m_1=1,\ \ m_2=1,\ \ \propto_1=1\ \ and\ \ \propto_2=2$$

$$\therefore\ \ z=e^{x}f_1(y+x)+e^{2x}f_2(y+x)$$

To find Particular Integral:

$$P.I=\frac{1}{\{(D-D'-1)(D-D'-2)\}}\left[\boldsymbol{e^{2x-y}}\right]$$

$$=\left[\boldsymbol{e^{2x-y}}\right]\frac{1}{\{(2-(-1)-1)(2-(-1)-2)\}}$$

$$=\boldsymbol{e^{2x-y}}\,\frac{1}{2}$$

The complete solution is $\boldsymbol{z}=\boldsymbol{C.F}+\boldsymbol{P.I}$

$$\boldsymbol{z}=\boldsymbol{e^{x}f_1(y+x)+e^{2x}f_2(y+x)+\frac{e^{2x-y}}{2}}$$

14. (a). (i). A tightly stretched string of length $'l'$ is initially at rest in its equiliburium position and each of its points is given the velocity $\boldsymbol{V_0\sin^3\left(\frac{\pi x}{l}\right)}$. Find the displacement $y(\boldsymbol{x},\ \boldsymbol{t})$.

Ans:

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$$

The suitable solution of one dimensional wave equation is

$$y(x,\ t)=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos\lambda at+c_4\sin\lambda at)\qquad\ldots(1)$$

The initial and boundary conditions of $y(x,\ t)$ are

i. $y(0,\ t)=0,\ \text{for all } t>0$
ii. $y(l,\ t)=0,\ \text{for all } t>0$
iii. $\frac{\partial y}{\partial t}(x,\ 0)=0,\ 0<x<l$
iv. $y(x,\ 0)=V_0\sin^3\left(\frac{\pi x}{l}\right),\ 0<x<l$

Applying the boundary condition (1), we get

M. Vijaya Kumar

$$y(0,\ t) = (c_1 + 0)(c_3 \cos \lambda at + c_4 \sin \lambda at)$$

$$0 = c_1(c_3 \cos \lambda at + c_4 \sin \lambda at)$$

$$\boldsymbol{c_1 = 0}, \qquad \because (c_3 \cos \lambda at + c_4 \sin \lambda at) \neq 0$$

Equation (1), becomes

$$y(x,\ t) = c_2 \sin \lambda x\,(c_3 \cos \lambda at + c_4 \sin \lambda at) \quad \ldots(2)$$

Applying the boundary condition (2), we get

$$y(l,\ t) = c_2 \sin \lambda l\,(c_3 \cos \lambda at + c_4 \sin \lambda at)$$

$$0 = c_2 \sin \lambda l\,(c_3 \cos \lambda at + c_4 \sin \lambda at)$$

$$c_2 \sin \lambda l = \mathbf{0}, \qquad \because (c_3 \cos \lambda at + c_4 \sin \lambda at) \neq 0$$

If $c_2 = 0$, we get a trivial solution, therefore

$$\sin \lambda l = 0 = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{l}$$

Equation (2), becomes

$$y(x,\ t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(c_3 \cos\left(\frac{n\pi at}{l}\right) + c_4 \sin\left(\frac{n\pi at}{l}\right)\right) \quad \ldots(3)$$

Differentiating partially with respect to $'t'$ on both sides, we get

$$\frac{\partial y}{\partial t} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-c_3 \sin\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right) + c_4 \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)\right]$$

Applying the boundary condition (3), we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,\ 0)} = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_4\left(\frac{n\pi a}{l}\right)\right]$$

$$0 = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_4\left(\frac{n\pi a}{l}\right)\right]$$

$$\boldsymbol{c_4 = 0}$$

Equation (3), becomes

$$y(x,\ t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left(c_3 \cos\left(\frac{n\pi at}{l}\right)\right)$$

$$y(x,\ t) = c_2\, c_3\, \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

$$y(x,\ t) = b\, \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right) \quad \ldots.(4) \quad where \quad b = c_2 c_3$$

The most general solution is

M. Vijaya Kumar

$$y(x,\ t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right) \quad \dots(4)$$

Applying the initial condition (4), we get

$$y(x, 0\ ) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\cos 0$$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

14. (b). A Square plate is bounded by the lines $x = 0,\ y = 0,\ x = 20\ \&\ y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,\ 20) = x(20 - x), 0 < x, 20$ while the other edges are kept at 0^0C. Find the steady state temperature distribution in the plane.

Ans:

Let us take the sides of the plate be $l = 20$.

Let $u(x, y)$ be the temperature at any point $(x,\ y)$.

Then $u(x,\ y)$ satisfies the Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

M. Vijaya Kumar

From the given problems we have the following boundary conditions.

i. $u(0,\ y) = 0,\ \text{for } 0 < x < l$
ii. $y(l,\ y) = 0,\ \text{for}\ \ 0 < y < lt > 0$
iii. $u(x,\ 0) = 0,\ for\ \ 0 < x < l$
iv. $u(x,\ l) = x(l - x),\ for\ \ 0 < x < l$

The possible solution of this equation is

$$u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \dots.(1)$$

Applying the boundary condition (i), we get

$$u(0, y) = c_5(c_7 e^{py} + c_8 e^{-py}) = 0$$

$$\boldsymbol{C_5 = 0}$$

Equation (1), becomes

$$u(x, y) = c_6 \sin px\,(c_7 e^{py} + c_8 e^{-py}) \quad \dots(2)$$

Applying the boundary condition (2), we get

$$u(l,y) = c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$$

$$0 = c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py})$$

$$c_6 \sin pl = \mathbf{0}, \qquad \because (c_7 e^{py} + c_8 e^{-py}) \neq 0$$

If $c_6 = 0$, we get a trivial solution, therefore

$$\sin pl = 0 = n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}$$

Equation (2), becomes

$$u(x,y) = c_6 \sin\left(\frac{nx\pi}{l}\right)\left(c_7 e^{\left(\frac{n\pi}{l}\right)y} + c_8 e^{-\left(\frac{n\pi}{l}\right)y}\right) \quad \dots(3)$$

Applying the boundary condition (3), we get

$$u(x,o) = c_6 \sin\left(\frac{nx\pi}{l}\right)(c_7 + c_8)$$

$$c_7 + c_8 = \quad or \quad \boldsymbol{c_8 = -c_7}$$

Equation (3), becomes

$$u(x,y) = c_6 \sin\left(\frac{nx\pi}{l}\right)\left(c_7 e^{\left(\frac{n\pi}{l}\right)y} - c_7 e^{-\left(\frac{n\pi}{l}\right)y}\right)$$

$$u(x,y) = c_6 c_7 \sin\left(\frac{nx\pi}{l}\right)\left(e^{\left(\frac{n\pi}{l}\right)y} - e^{-\left(\frac{n\pi}{l}\right)y}\right)$$

$$u(x,y) = c_6 c_7 \sin\left(\frac{nx\pi}{l}\right)\, 2\sin h\left(\frac{n\pi}{l}\right)y$$

$$u(x,y) = c_n \sin\left(\frac{nx\pi}{l}\right)\, \sin h\left(\frac{n\pi}{l}\right)y \quad \dots.(4) \quad where \quad c_n = 2c_6 c_7$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx\pi}{l}\right)\, \sin h\left(\frac{n\pi}{l}\right)y \quad \dots(4)$$

Applying the initial condition (4), we get

$$u(x,\ l\,) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx\pi}{l}\right)\, \sin hn\pi = x(l-x) \quad \dots.(5)$$

To find c_n, expand $x(l-x)$ in a half range Fourier sine series in the interval $0 < x < l$.

M Vijaya Kumar

$$x(l-x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{nx\pi}{l}\right) \quad .... \ (6)$$

From (9) and (10), we have

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{nx\pi}{l}\right) \sin h n\pi = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx\pi}{l}\right) \quad ....(7)$$

Equating like co-efficient, we have

$$c_n \sin h n\pi = b_n \quad or \quad c_n = \frac{b_n}{\sin h n\pi}$$

$$Now, \quad b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l}\int_0^l (lx-x^2)\sin\left(\frac{n\pi x}{l}\right) dx$$

$$=\frac{2}{l}\left[(lx-x^2)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right)-(l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{-2}{l}\left[(lx-x^2)\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)+2\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l^3}{n^3\pi^3}\right)\right]_0^l$$

$$=\frac{-2}{l}\left[\left(0+2\cos n\pi\left(\frac{l^3}{n^3\pi^3}\right)\right)-\left(0+2\cos 0\left(\frac{l^3}{n^3\pi^3}\right)\right)\right]$$

$$=\frac{-2}{l}\left(\frac{2l^3}{n^3\pi^3}\right)[\cos n\pi-1]=\frac{-4k\,l^2}{n^3\pi^3}[(-1)^n-1]$$

$$\boldsymbol{b_n}=\begin{cases}\boldsymbol{0}, & \boldsymbol{if \ \ n \ is \ even}\\ \dfrac{\boldsymbol{8\,l^2}}{\boldsymbol{n^3\pi^3}}, & \boldsymbol{if \ \ n \ is \ odd}\end{cases}$$

$$\therefore \quad c_n = \frac{\boldsymbol{8\,l^2}}{\boldsymbol{n^3\pi^3}} * \frac{\boldsymbol{1}}{\sin h n\pi}$$

Equation (4), becomes

$$u(x,y)=\sum_{n=1}^{\infty}\left[\frac{\boldsymbol{8\,l^2}}{\boldsymbol{n^3\pi^3}} * \frac{\boldsymbol{1}}{\sin h n\pi}\right]\sin\left(\frac{nx\pi}{l}\right) \sin h\left(\frac{n\pi}{l}\right)y$$

15. (a). (i). If $Z[f(n)]=F(z)$, find $Z[f(n-k)]$ and $Z[f(n+k)]$.

M. Vijaya Kumar

Ans:

$$Z[f_{n-k}] = \sum_{n=0}^{\infty} f_{n-k}\, z^{-n} = \sum_{n=0}^{\infty} f_{n-k}\, z^{-n}\, z^{k} z^{-k}$$

$$= z^{-k} \sum_{n=0}^{\infty} f_{n-k}\, z^{-(n-k)}$$

$$= z^{-k}\left[f_{0-k} z^{-(0-k)} + f_{1-k} z^{-(1-k)} + f_{2-k} z^{-(2-k)} + \cdots\right]$$

Put $k = -n$, we get

$$= z^{-k}\left[f_n z^{-n} + f_{1+n} z^{-(1+n)} + f_{2+n} z^{-(2+n)} + \cdots\right]$$

Put $n = 0$, we get

$$= z^{-k}[f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots]$$

$$Z[f_{n-k}] = z^{-k} \sum_{n=0}^{\infty} f_n\, z^{-n}$$

$$Z[f_{n+k}] = \sum_{n=0}^{\infty} f_{n+k}\, z^{-n} = \sum_{n=0}^{\infty} f_{n+k}\, z^{-n}\, z^{k} z^{-k}$$

$$= z^{k} \sum_{n=0}^{\infty} f_{n+k}\, z^{-(n+k)}$$

$$= z^{k}\left[f_k z^{-k} + f_{k+1} z^{-(k+1)} + f_{k+2} z^{-(k+2)} + \cdots\right]$$

$$= z^{k}\left[\sum_{n=0}^{\infty} f_n\, z^{-n} - \sum_{n=0}^{k-1} f_n\, z^{-n}\right]$$

$$= z^{k}[F(z) - f_0 - f_1 z^{-1} - f_2 z^{-2} - \cdots]$$

15. (a). (ii). Evaluate $Z^{-1}\left[(z-5)^{-3}\right]$ for $|z| > 5$.

Ans:

$$\text{Let } \; X(z) = \frac{1}{(z-5)^3}$$

$$X(z)\, z^{n-1} = \frac{1}{(z-5)^3}\, z^{n-1}$$

Residue of $F[z]$

$$(z-5)^3 = 0$$

M. Vijaya Kumar

$z = 5$ is a simple pole of order 3

$$\text{Res of } F[z] = \lim_{z\to 5} \frac{1}{2!} \frac{d^2}{dz^2} (z-5)^3 \frac{z^{n-1}}{(z-5)^3}$$

$$= \lim_{z\to 5} \frac{1}{2} \frac{d^2}{dz^2} [z^{n-1}]$$

$$= \lim_{z\to 5} \frac{1}{2} \frac{d}{dz} [(n-1)z^{n-2}]$$

$$= \lim_{z\to 5} \frac{1}{2} [(n-1)(n-2)\, z^{n-3}]$$

$$\text{Res of } F[z] = \frac{1}{2} [(n-1)(n-2)\,(5)^{n-3}]$$

$$\therefore \quad \boldsymbol{x(n) = \frac{(n-1)(n-2)\,(5)^{n-3}}{2}}$$

15. (b). (i). Solve: $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$, given that $y_0 = 0$ and $y_1 = 1$.

Ans:

Given $\boldsymbol{y_{n+2} + 4y_{n+1} + 3y_n = 3^n}$ also $\boldsymbol{y_0 = 0 \;\; and \;\; y_1 = 1}$

Taking z-transform on both sides, we get

$$z[y_{n+2}] + 4\, z[y_{n+1}] + 3\, z[y_n] = z[3^n]$$

$$\left(z^2 y(z) - z^2 y(0) - zy(1)\right) + 4\left(zy(n) - zy(0)\right) + 3y(z) = \frac{z}{z-3}$$

$$(z^2 y(z) - z) + 4\left(zy(z)\right) + 3y(z) = \frac{z}{z-3}$$

$$y(z)[z^2 + 4z + 3] - z = \frac{z}{z-3}$$

$$y(z)[(z+3)(z+1)] = \frac{z}{z-3} + z$$

$$y(z) = \frac{z}{(z-3)(z+3)(z+1)} + \frac{z}{(z+3)(z+1)}$$

$$y(n) = z^{-1}\left[\frac{z}{(z+1)(z+3)(z-3)}\right] + z^{-1}\left[\frac{z}{(z+3)(z+1)}\right]$$

Method of Partial Fraction:

$$\frac{X(z)}{z} = \left[\frac{z}{(z+1)(z+3)(z-3)}\right] + \left[\frac{z}{(z+3)(z+1)}\right] \quad ---(1)$$

M. Vijaya Kumar

$$\frac{X(z)}{z}=\left\{\frac{A}{(z+1)}+\frac{B}{(z+3)}+\frac{C}{(z-3)}\right\}+\left\{\frac{A}{(z+1)}+\frac{B}{(z+3)}\right\}$$

$$\frac{X(z)}{z}=\frac{z}{(z+1)(z+3)(z-3)}$$

$$\frac{X(z)}{z}=\frac{A}{(z+1)}+\frac{B}{(z+3)}+\frac{C}{(z-3)}$$

$$z=A(z+3)(z-3)+B(z+1)(z-3)+C(z+1)(z+3)$$

If $z=3,$ then $3=24C \quad\Rightarrow\quad C=\frac{1}{8}$

If $z=-3,$ then $-3=12B \quad\Rightarrow\quad B=-\frac{1}{4}$

Equating $\text{Co.eff of } z^2$, we get $0=A+B+C$

$$0=A-\frac{1}{4}+\frac{1}{8}\quad\Rightarrow\quad A=\frac{1}{8}$$

$$\frac{X(z)}{z}=\frac{z}{(z+1)(z+3)}$$

$$\frac{X(z)}{z}=\frac{A}{(z+1)}+\frac{B}{(z+3)}$$

$$z=A(z+3)+B(z+1)$$

If $z=-3,$ then $-3=-2B \Rightarrow B=\frac{3}{2}$

If $z=-1,$ then $-1=2A \Rightarrow A=-\frac{1}{2}$

$$\frac{X(z)}{z}=\left\{\frac{\frac{1}{8}}{(z+1)}+\frac{-\frac{1}{4}}{(z+3)}+\frac{\frac{1}{8}}{(z-3)}\right\}+\left\{\frac{-\frac{1}{2}}{(z+1)}+\frac{\frac{3}{2}}{(z+3)}\right\}$$

$$=\left\{\frac{1}{8}z^{-1}\left(\frac{1}{(z+1)}\right)-\frac{1}{4}z^{-1}\left(\frac{1}{(z+3)}\right)+\frac{1}{8}z^{-1}\left(\frac{1}{(z-3)}\right)\right\}+\left\{-\frac{1}{2}z^{-1}\left(\frac{1}{(z+1)}\right)+\frac{3}{2}z^{-1}\left(\frac{1}{(z+3)}\right)\right\}$$

$$\boldsymbol{y(n)=\left\{\frac{1}{8}[(-1)^{n-1}]-\frac{1}{4}[(-3)^{n-1}]+\frac{1}{8}(3)^{n-1}\right\}+\left\{-\frac{1}{2}[(-1)^{n-1}]+\frac{3}{2}(3)^{n-1}\right\}}$$

15. (b). (ii). Form the difference equation of second order by eliminating the constants A and B from

$$\boldsymbol{y_n=A\,(-2)^n+Bn.}$$

Ans:

$$y_n=A\,(-2)^n+Bn$$

$$y_{+1n}=A\,(-2)^{n+1}+B(n+1)$$

$$=A\,(-2)^n(-2)+B(n+1)$$

$$=-2\,A\,(-2)^n+B(n+1)$$

$$y_{n+2}=A\,(-2)^{n+2}+B(n+2)$$

$$=A\,(-2)^n(-2)^2+B(n+2)$$

$$=4A\,(-2)^n+B(n+2)$$

$$\begin{vmatrix} y_n & 1 & n\\ y_{n+1} & -2 & n+1\\ y_{n+2} & 4 & n+2\end{vmatrix}=0$$

M. Vijaya Kumar

$$y_n[-2(n+2)-4(n+1)]-1\big(y_{n+1}(n+2)-y_{n+2}(n+1)\big)+n(4y_{n+1}+2y_{n+2})=0$$

$$y_n[-2n-4-4n-4]-1(ny_{n+1}+2y_{n+1}-ny_{n+2}-y_{n+2})+4ny_{n+1}+2ny_{n+2}=0$$

$$y_n[-6n-8]-ny_{n+1}-2y_{n+1}+ny_{n+2}+y_{n+2}+4ny_{n+1}+2ny_{n+2}=0$$

$$-y_n(6n+8)+y_{n+1}(-n-2+4n)+y_{n+2}(n+1+2n)=0$$

$$\boldsymbol{-y_n(6n+8)+y_{n+1}(-3n+2)+y_{n+2}(3n+1)=0}$$

This is the required difference equation.

ANNA UNIVERSITY CHENNAI

MATHEMATICS III

NOVEMBER / DECEMBER 2010

M. Vijaya Kumar

1. **Find the constant term in the expansion for $\cos^2 x$ as a Fourier series in the interval $(-\pi,\ \pi)$.**

   Ans:

$$\boldsymbol{a_0}=\frac{1}{\pi}\int_{-\pi}^{+\pi}\cos^2 \boldsymbol{x}\ \boldsymbol{dx}=\frac{1}{\pi}\int_{-\pi}^{+\pi}\left(\frac{1+\cos 2x}{2}\right)\boldsymbol{dx}=\frac{1}{2\pi}\left[x+\frac{\sin 2x}{2}\right]_{-\boldsymbol{\pi}}^{\boldsymbol{\pi}}$$

$$\boldsymbol{a_0}=\frac{1}{2\pi}[x]_{-\boldsymbol{\pi}}^{\boldsymbol{\pi}}=\frac{\boldsymbol{1}}{\boldsymbol{2}\pi}[\pi-(-\pi)]=\boldsymbol{1}$$

2. **Find the root mean square value of the function $f(x)=x^2$ in $(0,\ l)$.**

Ans:

$$R.M.S\ =\sqrt{\frac{\int_a^b[f(x)]^2\,dx}{b-a}}=\sqrt{\frac{\int_0^l[x^2]^2\,dx}{l-0}}=\sqrt{\frac{\left(\frac{x^5}{5}\right)_0^l}{l}}$$

$$\boldsymbol{R.M.S}=\sqrt{\frac{\frac{l^5}{5}}{l}}=\sqrt{\frac{l^5}{5l}}=\sqrt{\frac{l^4}{5}}=\frac{\boldsymbol{l^2}}{\sqrt{\boldsymbol{5}}}$$

3. Write the Fourier Transform Pair

Ans: The Fourier Transform of $f(x)$ is

$$F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\ e^{isx}\ d$$

The Inverse Fourier Transform of $f(x)$ is

$$f(x)=[F(s)]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)\ e^{-isx}\,ds$$

4. **Find the Fourier sine transform of e^{-ax}, a >0.**

Ans:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \; dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \; dx = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2}\right]$$

5. Form the P.D.E by eliminating the arbitrary constants from $z^2 - xy = f\left(\frac{x}{z}\right)$.

Solution:

This is a P.D.E of first order

$$Let \quad z^2 - xy = f\left(\frac{x}{z}\right) \quad ---(1)$$

Diff partially w. r. t 'x' & 'y', we get

$$2zp - y = f'\left(\frac{x}{z}\right) * \left(\frac{z - x.p}{z^2}\right) \quad ---(2)$$

$$2zq - x = f'\left(\frac{x}{z}\right) * \left(\frac{0 - x.q}{z^2}\right) \quad ---(3)$$

$$(2) \;\; becomes \qquad f'\left(\frac{x}{z}\right) = z^2\left(\frac{2zp - y}{z - xp}\right) \; ---(4)$$

Using (4) in (3), we have

$$2zq - x = z^2\left(\frac{2zp - y}{z - xp}\right)\left(-\frac{xq}{z^2}\right)$$

$$(2zq - x)(z - xp) = xq(2zp - y)$$

$$2z^2q - 2xzpq - zx + x^2p = -2xzpq + xyq$$

$$\boldsymbol{x^2p + q(2z^2 - xy) = zx}$$

6. **Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{x-y}$.**

Ans:

$$P.I = \frac{1}{(D^2 - 2DD' + D'^2)} \; e^{x-y}$$

$$P.I = \frac{1}{(1^2 + 1(-1) - 6(-1)^2)} \; e^{x-y}$$

$$P.I = \frac{e^{x-y}}{4}$$

7. Write down the possible solution of one dimensional heat equation.

M. Vijaya Kumar

Ans:

i. $u(x,\ t) = (c_1 x + c_2)$

ii. $u(x,\ t) = e^{\alpha^2 p^2 t}(c_3 e^{px} + c_4 e^{-px})$

iii. $u(x,\ t) = e^{-\alpha^2 p^2 t}(c_5 \cos px + c_6 \sin px)$

8. **Write down possible solutions of the Laplace equation.**

Ans:

i. $\boldsymbol{u(x,y)} = (\boldsymbol{c_1} e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

ii. $\boldsymbol{u(x,y)} = (\boldsymbol{c_5 \cos px} + \boldsymbol{c_6} \sin px)(c_7 e^{py} + c_8 e^{-py})$

iii. $\boldsymbol{u(x,y)} = (\boldsymbol{c_9} x + c_{10})(c_{11} y + c_{12})$

9. Define unit step sequence. Write its $\boldsymbol{Z}$ −ttransform.

Ans: A discrete unit step function is defined as

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$Hence\ \ Z[u(k)] = \sum_{k=0}^{\infty} u(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$Z[u(k)] = \left(1 - \frac{1}{z}\right)^{-1} = \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

10. Form a difference equation by eliminating arbitrary constants A from $\boldsymbol{y_n = A\ 3^n}$.

Ans:

$$y_n = A\ 3^n$$

$$y_{n+1} = A\ 3^{n+1} = A\ 3^n 3 = 3A3^n = 3\ y_n$$

$$i.e.,\ \ y_{n+1} - 3\ y_n = 0$$

Part – B

11. (a). (i). Find the Fourier series expansion $\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x} & \boldsymbol{for}\ \ \boldsymbol{0} < x < \pi \\ \boldsymbol{2\pi - x} & \boldsymbol{for}\ \ \boldsymbol{\pi} < x < 2\boldsymbol{\pi} \end{cases}$.

Hence deduce that $\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}}$.

Ans:

The Fourier series of $f(x)\ in\ (0,\ 2\pi)$ **is**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\ \ -----(1)$$

To find $\boldsymbol{a_0}$:

M. Vijaya Kumar

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right] = \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(\frac{(2\pi - x)^2}{-2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left(0 - \left(\frac{\pi^2}{-2}\right)\right)\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$\boldsymbol{a_0 = \pi}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\;\;dx + \int_{\pi}^{2\pi}(2\pi - x)\;\;\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos nx}{n^2}\right]_0^{\pi} - \left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\frac{1}{n^2}\{(\cos n\pi - \cos 0) - (\cos 2n\pi - \cos n\pi)\}$$

$$= \frac{1}{\pi}\frac{1}{n^2}\{((-1)^n - 1) - (1 - (-1)^n)\}$$

$$\boldsymbol{a_n = \frac{2}{\pi\, n^2}[1 - (-1)^n]}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\;\;dx + \int_{\pi}^{2\pi}(2\pi - x)\;\;\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right)\right]_{\pi}^{2\pi}\right\} = \frac{1}{n\pi}[-(\pi\;\cos n\pi - 0) - (0 - \pi\cos n\pi)]$$

$$\boldsymbol{b_n = 0}$$

The required Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$$\boldsymbol{f(x) = \frac{\pi}{2} + \sum_{n=1,3,5}^{\infty}\left(\frac{2}{\pi\, n^2}[1 - (-1)^n]\right)\cos nx}$$

Deduction:

M. Vijaya Kumar

$$f(x) = \begin{cases} x & for \;\; 0 < x < \pi \\ 2\pi - x & for \;\; \pi < x < 2\pi \end{cases}$$

$$\text{Put x} = 0, \text{a point of discontinuity} \quad \therefore \; \frac{f(0) + f(2\pi)}{2} = 0$$

$$\therefore \;\; 0 = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \left(\frac{-2}{n^2}\right) \cos n0$$

$$\therefore \;\; 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5}^{\infty} \left(\frac{1}{n^2}\right)$$

$$\therefore \; \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ..\right]$$

11. (a). (ii). Find the Fourier series expansion of $f(x) = 1 - x^2 \;\; in \;\; (-1, \; 1)$.

Ans:

Given $f(x) = 1 - x^2 \;\; in \;\; (-1, \; 1)$

The given function is even function.

$Here \;\; l = 1,$ The Fourier series for the function $f(x)$ in $(-l, \; l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l} \int_0^l f(x) \, dx = \frac{2}{1} \int_0^1 (1 - x^2) \; dx = 2\left[1 - \frac{x^3}{3}\right]_0^1 = 2\left[\left(1 - \frac{1}{3}\right) - (1 - 0)\right]$$

Put $l = 1$

$$\boldsymbol{a_0 = -[-1]}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \; \cos\left(\frac{n\pi x}{l}\right) \; dx$$

$$a_n = \frac{2}{1} \int_0^1 (1 - x^2) \; \cos n\pi x \; dx$$

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[2x\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 = -4\left[(\cos n\pi)\left(\frac{1}{n^2\pi^2}\right) - 0\right]$$

M. Vijaya Kumar

$$\boldsymbol{a_n = \frac{-4(-1)^n}{n^2\pi^2}}$$

The required Fourier series is

$$\therefore \quad \boldsymbol{1 - x^2 = \frac{2}{3} - \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x\right)}$$

11. (b). (i). Obtain the half range cosine series for $\boldsymbol{f(x) = x \quad in \ (0,\ \pi)}$.

Ans:

The half range cosine series for the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx\,,$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\pi} x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boldsymbol{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx = \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{-2}{\pi}\left[2x\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{-2}{\pi}\left[2\pi\left(\frac{\cos n\pi}{n^2}\right) - 0\right]$$

$$\boldsymbol{a_n = -\frac{4(-1)^n}{n^2}}$$

$$\therefore \quad \boldsymbol{x^2 = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(-\frac{4(-1)^n}{n^2}\right)\cos nx}$$

11. (b). (ii). Find the Fourier series as for as the second harmonic to represent the function $\boldsymbol{f(x)}$ with period 6, given the following table.

x:	0	1	2	3	4	5
$f(x)$:	9	18	24	28	26	20

Ans:

Here $n = 5, \quad 2l = 6, \quad l = 3$

M. Vijaya Kumar

The Fourier series takes the form

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3} \quad --(1)$$

x	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	y	$y\ \cos\frac{\pi x}{3}$	$y\ \sin\frac{\pi x}{3}$	$y\ \cos\frac{2\pi x}{3}$	$y\ \sin\frac{2\pi x}{3}$
0	0	0	0	9	0	9	0
1	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	18	9	15.7	-9	15.6
2	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	24	-12	20.9	-24	0
3	π	2π	28	-28	0	28	0
4	$\frac{4\pi}{3}$	$\frac{8\pi}{3}$	26	-13	-22.6	-13	22.6
5	$\frac{5\pi}{3}$	$\frac{10\pi}{3}$	10	10	-17.4	-10	-17.4
-19			125	-25	-3.4	-19	20.8

M. Vijaya Kumar

$$a_0 = 2\left(\frac{\sum y}{n}\right) = 2\left(\frac{125}{6}\right) = 41.66, \qquad a_1 = 2\left(\frac{\sum y \cos\frac{\pi x}{3}}{6}\right) = 2\left(\frac{-25}{6}\right) = -8.33$$

$$b_1 = 2\left(\frac{\sum y \sin\frac{\pi x}{3}}{6}\right) = 2\left(\frac{-3.4}{6}\right) = -1.336 \qquad a_2 = 2\left(\frac{\sum y \cos\frac{2\pi x}{3}}{6}\right) = -6.33$$

$$b_2 = 2\left(\frac{\sum y \sin\frac{2\pi x}{3}}{6}\right) = 6.9$$

$$\therefore \quad y = \frac{41.66}{2} - 8.33 \cos\frac{\pi x}{3} - 6.33\ \cos\frac{2\pi x}{3} - 1.336 \sin\frac{\pi x}{3} 6.9 \sin\frac{2\pi x}{3}$$

12. (a). (i). Derive Parseval's identity for the Fourier Transforms.

Statement:

If $f(x)$ is a given function defined in $(-\infty, +\infty)$ then it satisfy the identity

$$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F(s)|^2\, ds \qquad or \qquad \int_{-\infty}^\infty |f(x)|^2\, dx = \int_{-\infty}^\infty |F(s)|^2\, ds$$

Proof:

We know that

$$F[f*g] = F(s)*G(s)$$

$$F^{-1}[F(s)*G(s)] = f*g$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)*G(s)\ e^{-isx}\,ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)*g(x-t)\,dt$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} F(s)*G(s)\ \ ds = \int_{-\infty}^{\infty} f(t)*g(-t)\,dt \quad ...(1)$$

$$Let\ \ g(-t) = \overline{f(t)} \qquad ...(2)$$

$$g(t) = \overline{\overline{f(-t)}} \qquad ...(3)$$

$$\therefore \qquad G(s) = \overline{F[f(-x)]} = \overline{F(s)}$$

$$i.e., \quad G(s) = \overline{F(s)} \qquad ...(4)$$

Using the value $(2)\ \ in\ \ (4)$, we get

$$\int_{-\infty}^{\infty} F(s)*\overline{F(s)}\ \ ds = \int_{-\infty}^{\infty} f(t)*\overline{f(t)}\,dt$$

$$\int_{-\infty}^{\infty} |\boldsymbol{F(s)}|^2\,\boldsymbol{ds} = \int_{-\infty}^{\infty} |\boldsymbol{f(t)}|^2\,\boldsymbol{dt}$$

$$\boldsymbol{or} \quad \int_{0}^{\infty} |\boldsymbol{F(s)}|^2\,\boldsymbol{ds} = \int_{0}^{\infty} |\boldsymbol{f(x)}|^2\,\boldsymbol{dx}$$

12. (a). (ii). Find the Fourier integral representation of $f(x)$ defined as $f(x) = \begin{cases} \boldsymbol{0}, & \boldsymbol{for}\ \ \boldsymbol{x} < 0 \\ \frac{1}{2}, & \boldsymbol{for}\ \ \boldsymbol{x} = \boldsymbol{0} \\ \boldsymbol{e^{-x}} & \boldsymbol{for}\ \ \boldsymbol{x} > 0 \end{cases}$

Ans:

The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\ \cos\lambda(t-x)dt\ d\lambda \quad ....(1)$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{0} f(t)\ \cos\lambda(t-x)dt\ d\lambda + \int_0^{\infty} f(t)\ \cos\lambda(t-x)dt\ d\lambda\right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{0} 0 \ \cos\lambda(t-x)dt \ d\lambda + \int_0^{\infty} e^{-t} \ \cos\lambda(t-x)dt \ d\lambda \right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[0 + \int_0^{\infty} e^{-t} \ \cos\lambda(t-x)dt \ d\lambda \right]$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{e^{-t}}{\lambda^2+1} (-1 * \cos\lambda(t-x) + \lambda \sin\lambda(t-x)) \right]_0^{\infty} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[0 - \frac{1}{\lambda^2+1} (-1 * \cos\lambda x + \lambda \sin\lambda(-x)) \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left[-\frac{1}{\lambda^2+1} (-\cos\lambda x - \lambda \sin\lambda x) \right] d\lambda$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{1}{\lambda^2+1} (\cos\lambda x + \lambda \sin\lambda x) \right] d\lambda$$

12 (b). (i). Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

M. Vijaya Kumar

Ans:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \ \sin sx \ dx + \int_1^2 (2-x) \ \cos sx \ dx + \int_2^{\infty} 0 \ \sin sx \ dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[x \left(-\frac{\cos sx}{s} \right) - \left(-\frac{\sin sx}{s^2} \right) \right]_0^1 + \left[(2-x) \left(-\frac{\cos sx}{s} \right) - (-1) \left(-\frac{\sin sx}{s^2} \right) \right]_1^2 + 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[-\frac{\cos s}{s} + \frac{\sin s}{s^2} \right] + \left[0 - \frac{\sin 2s}{s^2} + \left(\frac{\cos s}{s} \right) + \left(\frac{\sin s}{s^2} \right) \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2 \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s^2} [2 \ \sin s - \sin 2s] \quad = \sqrt{\frac{2}{\pi}} \frac{1}{s^2} [2 \ \sin s - 2 \sin s \cos s]$$

$$F_s[f(x)] = 2\sin s\sqrt{\frac{2}{\pi}}\,\frac{1}{s^2}[1-\cos s]$$

12. (b). (ii). Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}.

Ans:

Let us take $f(x) = e^{-ax}$ *and* $g(x) = e^{-bx}$.

The Fourier cosine transforms of $f(x) = e^{-ax}$ *and* $g(x) = e^{-bx}$ is given by

$$F_c\,[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right) \quad and \quad F_c\,[g(x)] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)$$

$$We\ know\ that\ \int_0^{\infty} F_c[f(x)] * F_c[g(x)]\ ds = \int_0^{\infty} f(x) * g(x)\ dx$$

$$i.e.,\ \int_0^{\infty}\left(\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\right)\left(\sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)\right) ds = \int_0^{\infty} e^{-ax} * e^{-bx}\ dx$$

$$i.e.,\ \frac{2}{\pi}\int_0^{\infty}\frac{ab}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^{\infty} e^{-(a+b)x}\,dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty}$$

$$= \left[\frac{1}{-(a+b)}(e^{-\infty} - e^0)\right] = \left[\frac{1}{-(a+b)}(0-1)\right]$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{ab}{(s^2+a^2)(s^2+b^2)}\,ds = \left(\frac{1}{a+b}\right)$$

$$\int_0^{\infty}\frac{1}{(s^2+a^2)(s^2+b^2)}\,dx = \frac{\pi}{2ab}\left(\frac{1}{a+b}\right)$$

13. (a). (i). Form the P.D.E by eliminating the function from $\Phi(x^2+y^2+z^2,\ ax+by+cz) = 0$.

Ans:

$$\Phi(x^2+y^2+z^2,\ ax+by+cz) = 0 \quad ---(1)$$

This equation is of the form $\boldsymbol{\Phi(u,\ v) = 0}$.

Here $u = x^2+y^2+z^2, \quad v = ax+by+cz$

$$\frac{\partial u}{\partial x} = 2x + 2z\,\frac{\partial z}{\partial x} = 2x + 2zp \qquad \frac{\partial v}{\partial x} = a + c\,\frac{\partial z}{\partial x} = a + cp$$

M. Vijaya Kumar

$$\frac{\partial u}{\partial y} = 2y + 2z\,\frac{\partial z}{\partial y} = 2y + 2zq \qquad \frac{\partial v}{\partial y} = b + c\,\frac{\partial z}{\partial y} = b + cq$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 2x + 2zp & a + cp \\ 2y + 2zq & b + cq \end{vmatrix} = 0$$

$$(2x + 2zp)(b + cq) - (2y + 2zq)(a + cp) = 0$$

which is the required P.D.E.

13. (a). (ii). Solve the partial differential equation $\boldsymbol{x^2(y - z)p + y^2(z - x)q = z^2(x - y)}$.

Ans:

The subsidiary equations are

$$\frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ we get

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y - z) + (z - x) + (x - y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$i.e., \quad \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0 \qquad [\because \ Nr = 0]$$

$$i.e., \quad x^{-2}dx + y^{-2}dy + z^{-2}dz = 0$$

Integrating, we get

$$\left(\frac{x^{-2+1}}{-2+1}\right) + \left(\frac{y^{-2+1}}{-2+1}\right) + \left(\frac{z^{-2+1}}{-2+1}\right) = c_1$$

$$\left(\frac{x^{-1}}{-1}\right) + \left(\frac{y^{-1}}{-1}\right) + \left(\frac{z^{-1}}{-1}\right) = c_1$$

$$-\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = c_1$$

$$\boldsymbol{i.e., \qquad u = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get

M. Vijaya Kumar

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z) + y(z-x) + z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$i.e., \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \qquad [\because \ Nr = 0]$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$\boldsymbol{i.e., \qquad v = xyz = c_2}$$

The solution of given equation is $\Phi(u, \ v) = 0$.

$$\boldsymbol{i.e., \qquad \Phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \ xyz\right) = 0}$$

13. (b). (i). Solve the eqaution $\mathbf{(D^3 + D^2D' + 4DD'^2 + 4D'^3)z = \cos(2x + y)}$.

Ans:

The auxiliary equation is $m^3 + m^2 + 4m + 4 = 0$

The roots are $m = -1, +2i, -2i$.

$$\therefore \quad \boldsymbol{C.F = f_1(y - x) + f_2(y + 2ix) + f_3(y - 2ix)}$$

To find Particular Integral:

$$P.I = \frac{1}{(\mathrm{D^3 + D^2D' + 4DD'^2 + 4D'^3})} [\cos(2\mathrm{x} + \mathrm{y})]$$

$$= \frac{1}{(\mathrm{D^2D + D^2D' + 4DD'^2 + 4D'^2D'})} \cos(2\mathrm{x} + \mathrm{y})$$

Replace $\mathbf{D^2} \rightarrow \ -(\mathbf{2})^2$, $\mathbf{D'^2} \rightarrow -(\mathbf{1})^2$ & *and* $\mathbf{DD'} \rightarrow -(\mathbf{2} * \mathbf{1}) = -\mathbf{2}$

$$= \frac{1}{(-4\mathrm{D} + (-4)\mathrm{D'} + 4\mathrm{D}(-1) + 4(-1)\mathrm{D'})} \cos(2\mathrm{x} + \mathrm{y})$$

$$= \frac{1}{(-4\mathrm{D} - 4\mathrm{D'} - 4\mathrm{D} - 4\mathrm{D'})} \cos(2\mathrm{x} + \mathrm{y})$$

$$= \frac{1}{-8(\mathrm{D} + \mathrm{D'})} \cos(2\mathrm{x} + \mathrm{y})$$

$$= -\frac{1}{8}\left[\frac{1}{(\mathrm{D} + \mathrm{D'})} * \frac{D - D'}{D - D'}\right] \cos(2\mathrm{x} + \mathrm{y})$$

M. Vijaya Kumar

$$= -\frac{1}{8}\left[\frac{D-D'}{D^2-D'^2}\right] \cos(2x+y)$$

$$= -\frac{1}{8}\left[\frac{D-D'}{-4-(-1)}\right] \cos(2x+y)$$

$$= -\frac{1}{8}\left[\frac{1}{-3}\right] (D-D')\cos(2x+y)$$

$$= -\frac{1}{8}\left[\frac{1}{-3}\right] [-\sin(2x+y)\ 2-(-\sin(2x+y)]$$

$$\boldsymbol{P.I} = \frac{\mathbf{1}}{\mathbf{24}}\ [-\mathbf{sin(2x+y)}]$$

The complete solution is $\boldsymbol{z = C.F + P.I}$

$$\boldsymbol{z = f_1(y-x) + f_2(y+2ix) + f_3(y-2ix) - \frac{\sin(2x+y)}{24}}$$

14. (a). A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance $'b'$ transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Ans: The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

The equation of a string $\boldsymbol{OA}$ is $(0,0)\ and\ \left(\frac{l}{2}, b\right)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \quad \Rightarrow \quad \frac{y-0}{b-0} = \frac{x-0}{\frac{l}{2}-0}$$

$$\Rightarrow \quad \frac{y}{b} = \frac{2x}{l} \qquad \Rightarrow \quad \boldsymbol{y = \frac{2bx}{l}}$$

The equation of a string $\boldsymbol{AB}$ is $\left(\frac{l}{2}, b\right)\ and\ (l, 0)$

$$\frac{y-b}{0-b} = \frac{x-l/2}{l-l/2} \quad \Rightarrow \quad \frac{y-b}{-b} = \frac{x-l/2}{l/2}$$

$$y-b = \frac{bl-2bx}{l} \qquad \Rightarrow \quad y = \frac{bl-2bx}{l} + b \qquad \Rightarrow \quad \boldsymbol{y = \frac{2b}{l}(l-x)}$$

The initial displacement of the string is in the form

M. Vijaya Kumar

$$y(x,\ 0)=\begin{cases}\dfrac{2bx}{l}, & 0<x<\dfrac{l}{2}\\ \dfrac{2b}{l}(l-x), & \dfrac{l}{2}<x<l\end{cases}$$

The suitable solution of one dimensional wave equation is

$$y(x,\ t)=(c_1\cos\lambda x+c_2\sin\lambda x)(c_3\cos\lambda at+c_4\sin\lambda at)\quad ...(1)$$

The initial and boundary conditions of $y(x,\ t)$ are

i. $y(0,\ t)=0,\ \text{for all } t>0$

ii. $y(l,\ t)=0,\ \text{for all } t>0$

iii. $\frac{\partial y}{\partial t}(x,\ 0)=0,\ 0<x<l$

iv. $y(x,\ 0)=\begin{cases}\frac{2bx}{l}, & 0<x<\frac{l}{2}\\ \frac{2b}{l}(l-x), & \frac{l}{2}<x<l\end{cases}$

Applying the boundary condition (i) , (ii), (iii) we get

$$y(x,\ t)=b_n\ \sin\left(\frac{n\pi x}{l}\right)\ \cos\left(\frac{n\pi at}{l}\right)\ ....(4)\quad where\quad b_n=c_2c_3$$

The most general solution is

$$y(x,\ t)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi at}{l}\right)\quad ...(4)$$

Applying the initial condition (4), we get

$$y(x,0\ )=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)=f(x)$$

$$where\quad f(x)=\begin{cases}\dfrac{2bx}{l}, & 0<x<\dfrac{l}{2}\\ \dfrac{2b}{l}(l-x), & \dfrac{l}{2}<x<l\end{cases}$$

$$b_n=\frac{2}{l}\int_0^{\frac{l}{2}}f(x)\ \sin\left(\frac{n\pi x}{l}\right)\ dx=\frac{2}{l}\int_0^{\frac{l}{2}}\left(\frac{2bx}{l}\right)\ \sin\left(\frac{n\pi x}{l}\right)\ dx+\frac{2}{l}\int_{\frac{l}{2}}^{l}\ \frac{2b}{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)\ dx$$

$$=\frac{2}{l}\left(\frac{2b}{l}\right)\int_0^{\frac{l}{2}}x\ \sin\left(\frac{n\pi x}{l}\right)\ dx+\frac{2}{l}\left(\frac{2b}{l}\right)\int_{\frac{l}{2}}^{l}\ (l-x)\sin\left(\frac{n\pi x}{l}\right)\ dx$$

M. Vijaya Kumar

$$= \left(\frac{4b}{l^2}\right)\left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{2}} + \left(\frac{4b}{l^2}\right)\left[(l-x)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{2}}^{l}$$

$$= \left(\frac{4b}{l^2}\right)\left[\frac{l}{2}\left(-\cos\frac{n\pi}{2}\right)\left(\frac{l}{n\pi}\right) - \left(-\sin\frac{n\pi}{2}\right)\left(\frac{l^2}{n^2\pi^2}\right)\right]$$

$$+ \left(\frac{4b}{l^2}\right)\left[0 - \left(l - \frac{l}{2}\right)\left(-\cos\frac{n\pi}{2}\right)\left(\frac{l}{n\pi}\right) - \left(-\sin\frac{n\pi}{2}\right)\left(\frac{n^2\pi^2}{l^2}\right)\right]$$

$$= \left(\frac{4b}{l^2}\right)\left[\frac{-l^2}{2n\pi}\left(\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] + \left(\frac{4b}{l^2}\right)\left[\left(\cos\frac{n\pi}{2}\right)\left(\frac{l^2}{n\pi}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]$$

$$= \left(\frac{4b}{l^2}\right)\left[\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]$$

$$= \left(\frac{8b}{l^2}\right)\left[\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]$$

$$\boldsymbol{b_n = \frac{8b}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)}$$

Equation (4), becomes

$$\boldsymbol{y(x,\ t) = \sum_{n=odd}^{\infty} \frac{8b}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi a t}{l}\right)}$$

M. Vijaya Kumar

14. (b) A Square plate is bounded by the lines $x = 0,\ y = 0,\ x = 20\ \&\ y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,\ 20) = x(20 - x), 0 < x, 20$ while the other edges are kept at 0^0C. Find the steady state temperature distribution in the plane.

Ans: Refer Previous Question Paper

15. (a). (i). Find the z-transforms of $\cos n\theta$ and $\sin n\theta$. Hence deduce that the z-transforms of $\cos(n+1)\theta$ and $a^n \sin n\theta$.

Ans:

$$\cos n\theta + i \sin n\theta = e^{in\theta}$$

$$z\left[e^{in\theta}\right] = \sum_{n=0}^{\infty} \left(e^{in\theta}\right) z^{-n}$$

$$z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z - e^{i\theta}}$$

$$z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z - [\cos n\theta + i\sin n\theta]}$$

$$= \frac{z}{(z - \cos n\theta) - i\sin n\theta}$$

$$= \frac{z}{(z - \cos n\theta) - i\sin n\theta} * \frac{(z - \cos n\theta) + i\sin n\theta}{(z - \cos n\theta) + i\sin n\theta}$$

$$= \frac{z\ [(z - \cos n\theta) + i\sin n\theta]}{(z - \cos n\theta)^2 - i^2\sin^2 n\theta}$$

$$= \frac{z\ [(z - \cos n\theta) + i\sin n\theta]}{z^2 + \cos^2 n\theta - 2z\cos n\theta + \sin^2 n\theta}$$

$$z\left[e^{in\theta}\right] = \frac{z\ [(z - \cos n\theta) + i\sin n\theta]}{z^2 - 2z\cos n\theta + 1}$$

$$z[\cos n\theta + i\sin n\theta] = \left(\frac{z\ (z - \cos n\theta)}{z^2 - 2z\cos n\theta + 1}\right) + i\left(\frac{z\ \sin n\theta}{z^2 - 2z\cos n\theta + 1}\right)$$

Equating real and imaginary parts, we have

$$\boldsymbol{z[\cos n\theta] = \frac{z^2 - z\cos n\theta}{z^2 - 2z\cos n\theta + 1}}$$

$$\boldsymbol{z[\sin n\theta] = \frac{z\ \sin n\theta}{z^2 - 2z\cos n\theta + 1}}$$

By shifting theorem, we have

$$z[\cos(n+1)\theta] = z\left(\frac{z(z - \cos\theta)}{z^2 - 2z\cos n\theta + 1} - \cos 0\right)$$

$$z[\cos(n+1)\theta] = z\left(\frac{z(z - \cos\theta)}{z^2 - 2z\cos n\theta + 1} - 1\right)$$

$$z[a^n \sin n\theta] = z[\sin n\theta]_{z \to \frac{z}{a}}$$

$$= \left(\frac{\boldsymbol{z\ \sin n\theta}}{\boldsymbol{z^2 - 2z\cos n\theta + 1}}\right)_{z \to \frac{z}{a}} = \left(\frac{\boldsymbol{\frac{z}{a}\ \sin n\theta}}{\boldsymbol{\frac{z^2}{a^2} - \frac{2z}{a}\ \cos n\theta + 1}}\right)$$

$$= \left(\frac{\boldsymbol{z\ \sin n\theta}}{\boldsymbol{z^2 - 2za\ \cos n\theta + a^2}}\right)$$

15. (a). (ii). Find the inverse z-transform of $\frac{z(z+1)}{(z-1)^3}$ by residue method.

Ans:

M. Vijaya Kumar

$$\text{Let } X(z) = \frac{z(z+1)}{(z-1)^3}$$

$$X(z)\, z^{n-1} = \frac{z(z+1)}{(z-1)^3}\, z^{n-1}$$

$$X(z)\, z^{n-1} = \frac{z^n\,(z+1)}{(z-1)^3}$$

Residue of $F[z]$

$$(z-1)^3 = 0$$

$z = 1$ is a simple pole of order 3

$$\text{Res of } F[z] = \lim_{z\to 1} \frac{1}{2!}\frac{d^2}{dz^2}(z-1)^3\ \frac{z^n\,(z+1)}{(z-1)^3}$$

$$= \lim_{z\to 1} \frac{1}{2}\frac{d^2}{dz^2}\ [z^n\,(z+1)]$$

$$= \lim_{z\to 1} \frac{1}{2}\frac{d}{dz}\ [z^n + (z+1)nz^{n-1}]$$

$$= \lim_{z\to 1} \frac{1}{2}\ [n\,z^{n-1} + n(z+1)(n-1)z^{n-2} + nz^{n-1}]$$

$$= \frac{1}{2}\ [n\ + n(2)(n-1) + n] = \frac{1}{2}\ [2n + 2n^2 - 2n]$$

$$\text{Res of } F[z] = n^2$$

$$\therefore \quad \boldsymbol{x(n) = n^2}$$

M. Vijaya Kumar

15. (b). (i). Form the difference equation form the relation $\boldsymbol{y_n = a + b\,3^n}$.

Ans:

$$y_n = a + b\,3^n$$

$$y_{n+1} = a + b\,3^{n+1}$$

$$y_{n+2} = a + b\,3^{n+2}$$

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(9-3) - 1(9y_{n+1} - 3y_{n+2}) + 1(y_{n+1} - y_{n+2}) = 0$$

$$6y_n - 9y_{n+1} + 3y_{n+2} + y_{n+1} - y_{n+2} = 0$$

$$\boldsymbol{6y_n - 8y_{n+1} + 2y_{n+2} = 0}$$

15. (b). (ii). Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$, with $y_0 = 0$ and $y_1 = 1$, using $z -$ transform.

Ans:

$$\text{Given} \quad y_{n+2} + 4y_{n+1} + 3y_n = 2^n \quad \text{also} \quad y_0 = 0 \;\; \& \;\; y_1 = 1$$

Taking z-transform on both sides, we get

$$z[y_{n+2}] + 4\, z[y_{n+1}] + 3\, z[y_n] = z[2^n]$$

$$\left(z^2 y(z) - z^2 y(0) - zy(1)\right) + 4\left(zy(z) - zy(0)\right) + 3\, y(z) = \frac{z}{z-2}$$

$$(z^2 y(z) - z) + 4\left(zy(z)\right) + 3\, y(z) = \frac{z}{z-2}$$

$$y(z)[z^2 + 4z + 3] - z = \frac{z}{z-2}$$

$$y(z)[z^2 + 4z + 3] = \frac{z}{z-2} + z$$

$$y(z) = \frac{z}{(z^2 + 6z + 9)(z-2)} = \frac{z}{(z+3)(z+3)(z-2)}$$

$$\boldsymbol{y(z) = \frac{z}{(z+3)^2(z-2)}}$$

Method of Cauchy Residue theorem:

$$x(z) = \frac{z}{(z+3)^2(z-2)}$$

$$X(z)\, z^{n-1} = \frac{z}{(z+3)^2(z-2)}\, z^{n-1}$$

$$X(z)\, z^{n-1} = \frac{z^n}{(z+3)^2(z-2)}$$

Residue of $\boldsymbol{F[z]}$

$$\boldsymbol{(z+3)^2 = 0}$$

$z = -3$ is a simple pole of order 2

$$\text{Res of } F[z] = \lim_{z \to -3} \frac{1}{1!} \frac{d^1}{dz^1} (z+1)^2 \; \frac{z^n}{(z+3)^2(z-2)}$$

$$= \lim_{z \to -3} \frac{d}{dz} \left[\frac{z^n}{(z-2)}\right]$$

$$= \lim_{z \to -3} \left[\frac{(z-2)(nz^{n-1}) - z^n}{(z-2)^2}\right]$$

M. Vijaya Kumar

$$= \left[\frac{(-3-2)(n(-3)^{n-1}) - (-3)^n}{(-3-2)^2}\right]$$

$$R_1 = \left[\frac{-5n(-3)^{n-1} - (-3)^n}{25}\right]$$

$z = 2$ is a simple pole

$$\text{Res of } F[z] = \lim_{z \to 2} (z-2)\frac{z^n}{(z+3)^2(z-2)}$$

$$= \lim_{z \to 2} \left[\frac{z^n}{(z+3)^2}\right] = \left[\frac{2^n}{5^2}\right] = \frac{2^n}{25}$$

$$R_2 = \frac{2^n}{25}$$

$$\therefore \quad \boldsymbol{x(n) = \frac{2^n}{25} - \frac{5n(-3)^{n-1} + (-3)^n}{25}}$$

M. Vijaya Kumar