BE/B.Tech Degree Examinations, Nov/DEC 2011

MA 2161 - Mathematics - II

Part - A.

1. Transform the equation $x^2y'' + xy' = x$ into a Linear DE with Constant Coefficients.

Soln: put $x = e^z$ (or) $z = \log x$ $\therefore \theta = d/dz$

$$xDy = \theta y,\ x^2D^2y = \theta(\theta-1)y$$

$$\therefore (\theta^2 - \theta + \theta)y = e^z$$

$$\theta^2 y = e^z.$$

2. Find the particular integral of $(D^2+4)y = \sin 2x$.

Soln:- $P.I = \frac{1}{D^2+4}\sin 2x = \frac{1}{-4+4}\sin 2x = x\frac{1}{2D}\sin 2x$

$$= -\frac{x\cos 2x}{4}.$$

3. P.T $\bar{F} = yzi + zxj + xyk$ is irrotational.

Soln:- $\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = 0$. $\therefore$ It is irrotational.

4. Prove by Green's theorem that the area bdd by a Simple closed curve C is $\frac{1}{2}\int_c x\,dy - y\,dx$.

Soln:- Green's thm $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$.

$M = -y/2$, $N = x/2$ $\Rightarrow M_y = -1/2$, $N_x = 1/2$

$\therefore \frac{1}{2}\int_c x\,dy - y\,dx = \iint_R dx\,dy =$ Area of the region R enclosed by C.

5. S.T an analytic fun. with Constant imaginary part is Constant.

Soln:- $f(z) = u + iv$ is analytic, Gn $v =$ Constant $\therefore v_x = v_y = 0$.

By CR eqn $u_x = v_y$ & $u_y = -v_x$.

$\therefore u_x = u_y = v_x = v_y = 0$. $\therefore$ u is indep. of x & y $\therefore$ u is constant.

$\therefore f(z) = u + iv =$ Constant.

6. Find the invariant points of the transformation $w = \frac{1+z}{1-z}$

Soln: Invariant pt means $w = z = \frac{1+z}{1-z}$

$(\text{i.e.})\ z^2+1=0,\ z=\pm i.$

7. Evaluate $\int_c \frac{z\,dz}{(z-1)(z-2)}$ where c is $\bigcirc^{le}$ $|z| = \frac{1}{2}$.

Soln: $z=1$ & $z=2$ lies outside $\therefore$ Gn $=0$.

8. Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its poles.

Soln: pole $z=-1$ is of order 2.

$\therefore \left. \text{Res}\, f(z) \right]_{z=-1} = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{e^{2z}}{(z+1)^2}\right] = 2e^{-2}.$

9. Find $L[e^{-3t}\sin t \cos t]$

Soln: $L\left[e^{-3t}\frac{\sin 2t}{2}\right] = \frac{1}{2} L[\sin 2t]_{s\to s+3}$

$= \frac{1}{(s+3)^2+4}.$

10. Find $L^{-1}\left[\frac{e^{-as}}{s}\right]$

Soln: $= L^{-1}[1/s]_{t\to t-1} = 1.$

Part - B.

11. a) i) Solve $(D^2-3z+2)y = 2\cos(2x+3)+2e^x$.

Soln: refer AU Q.P Nov/Dec 2009 — Q. No. 11 a(i)

a) ii) Apply the mtd of Variation of parameters to Solve $(D^2+4)y = \cot 2x$.

Soln: refer AU Q.P Nov/Dec 2009. — Q. No. 11 a(ii)

(or).

b i) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos(\log(1+x))$.

Soln: $(1+x) = e^z$, $z = \log(1+x)$.

$(1+x)\frac{dy}{dx} = D'y \qquad (1+x)^2\frac{d^2y}{dx^2} = D'(D'-1)y.$

Gn $\Rightarrow (D'^2+1)y = 4\cos z$.

C.F $m^2+1=0$ $\therefore$ C.F $= A\cos z + B\sin z$ — (1).

$L(t\cos t) = -\frac{d}{ds}L(\cos t) = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\left[\frac{1-s^2}{(s^2+1)^2}\right].$

When $s=2$, $L[t\cos t] = -\left[\frac{1-4}{(4+1)^2}\right] = 3/25.$

15. b) ii) Solve $y''+4y'+4y = \sin t$, $\frac{dy}{dt}=0$ & $y=2$ when $t=0$ Using Laplace transform.

Soln: $L[y''+4y'+4y] = L(\sin t)$

$s^2 L(y) - s\,y(0) - y'(0) + 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s^2+1}$

$s^2 L(y) - 2s + 4sL(y) + 4L(y) = \frac{1}{s^2+1} + 8$

$\therefore L(y) = \frac{2s^3+8s^2+2s+9}{(s^2+1)(s+2)^2}.$

Using P.F

$L[y(t)] = \frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{C}{s+2} + \frac{D}{(s+2)^2}.$

$y(t) = -45/25\, L^{-1}\left[\frac{s}{s^2+1}\right] + 3/25\, L^{-1}\left[\frac{1}{s^2+1}\right] + 54/25\, L^{-1}\left(\frac{1}{s+2}\right)$
$+ 21/5\, L^{-1}\left(\frac{1}{(s+2)^2}\right).$

$y(t) = -45/25 \cos t + 3/25 \sin t + 54/25\, e^{-2t} + 21/5\, t e^{-2t}.$

$P.I = \frac{1}{D^2+1} 4\cos x = \frac{1}{-1+1} 4\cos x = 4x \frac{1}{2D} \cos x = 2x \sin x.$

$\therefore y = A\cos x + B\sin x + 2x\sin x$ put $z = \log(1+x)$.

ii) Solve $\frac{dx}{dt} - y = t$ & $\frac{dy}{dt} + x = t^2$ given $x(0) = 2,\ y(0) = 2$.

Soln:

$$Dx - y = t$$
$$Dx + D^2 y = 2t$$
$$(D^2+1)y = t.$$

$\therefore$ C.F $m^2+1=0$ $\therefore$ $C.F = A\cos t + B\sin t$.

$P.I = \frac{1}{D^2+1} t = (1+D^2)^{-1} t = (1 - D^2 + \cdots) t = t.$

$\therefore y = A\cos t + B\sin t + t.$

$\frac{dy}{dt} = -A\sin t + B\cos t + 1$

$\therefore x = t^2 + A\sin t - B\cos t - 1.$

$t=0,\ x=2$ $\qquad t=0\ \ y=2$

$B = -3$ $\qquad A = 2$

$\therefore$ $x = t^2 + 2\sin t + 3\cos t - 1$

$y = 2\cos t - 3\sin t + t.$

12 a) i) Evaluate $\int_C (x^2+xy)dx + (x^2+y^2)dy$, where C is the square bdd by the lines $x=0,\ x=1,\ y=0,\ y=1$

Soln: Refer AU QP Nov/Dec 2009 Q.No 12 a ii)

(or)

b) Verify GDT for $\bar{F} = (x^3-yz)\bar{i} - 2x^2y\bar{j} + 2\bar{k}$ over the cube $x=y=z=0$ & $x=y=z=a$.

Soln:- Refer AUQP May/June 2010 Q.No. 12(b).

13. a) i) Find the analytic fun $w = u+iv$ when $v = e^{-2y}(y\cos 2x + x\sin 2x)$ & find u.

Soln:- By Milne's thomson Mtd.

$$f(z) = \int v_{y(z,0)}\,dx + i\int v_{x(z,0)}\,dx + C.$$

$$= \int (\cos 2z - 2z\sin 2z)\,dz + i\int (2z\cos 2z + \sin 2z)\,dz + C$$

$$= z(\cos 2z + i\sin 2z) + C = z e^{i2z} + C.$$

$\therefore du = u_x dx + u_y dy = v_y dx - v_x dy$

$= \int e^{-2y}(\cos 2x - 2(y\cos 2x + x\sin 2x))\,dx - \int e^{-2y}(-2y\sin 2x + 2x\cos 2x + \sin 2x)\,dy$

$u = e^{-2y}(x\cos 2x + y\sin 2x) + C.$

13 a) ii) Show that the map $w = 1/z$ maps the totality of cycles & straight lines as circles or st. lines.

Soln: $w = 1/z \Rightarrow z = 1/w$

$\therefore \quad x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$\therefore \quad x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \quad \therefore \quad x^2+y^2 = \frac{1}{u^2+v^2}$.

Consider the eqn. $a(x^2+y^2)+bx+cy+d=0$ — ①, represents a circle if $a \neq 0$. it is a st. line sub the value of x & y in ①

we get $d(u^2+v^2)+bu-cv+a=0$ ②. If $d \neq 0$ ② becomes circle.

If $d=0$ it a st. line.

Case (i) when $a \neq 0, d \neq 0$. Eqn ① & ② represent circle in z-plane & w-plane not passing through the origin. Under $w = 1/z$.

Case (ii) $a \neq 0, d = 0$, eqn ① is circle in z-plane & ② is st. line, in w-plane

Case (iii) $a = 0, d \neq 0$, ① a st. line in z-plane & ② is a circle in w-plane

Case (iv) $a = 0, d = 0$, ① & ② is a st. line. Thus the lines through the origin z-plane map into same in w-plane.

13 b) If $u(x,y)$ & $v(x,y)$ are harmonic fun in a R, P.T the fun $(u_y - v_x) + i(u_x + v_y)$ is an analytic fun of $z = x+iy$.

(or)

Soln:- As u & v are harmonic, in R (i) i) $u_{xx}+u_{yy}=0$, ii) $v_{xx}+v_{yy}=0$ & iii) 2nd order partial derivatives of u & v are continuous

Let $U = u_y - v_x$ & $V = u_x + v_y$

$U_x = u_{xy} - v_{xx}$, $\quad V_x = u_{xx} + v_{xy}$

$U_y = u_{yy} - v_{yx}$, $\quad V_y = u_{yx} + v_{yy}$.

$U_x = V_y$ if $v_{xx}+v_{yy}=0$ & $U_y = -V_x$ if $u_{xx}+u_{yy}=0$.

Hence the Proof.

14 a) i) Using CSF $\int_c \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1-i| = 2$.

Soln:- $Dr = 0 \quad z = -1 \pm 2i$.

$z = -1-2i$ lies outside the $C: |z+1-i| = 2$.

$\therefore \left[\text{Res } f(z)\right]_{z=-1-2i} = 0$.

$\therefore$ $z=-1+2i$ lies inside $C: |z+1-i|=2$. Find Res $f(z)$ for this (or)

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{\frac{z+4}{z-(-1-2i)}}{z-(-1+2i)}\,dz$$

$$= 2\pi i \times f(-1+2i) \quad \because f(z) = \frac{z+4}{z+1+2i}$$

$$= \frac{\pi}{2}[3+2i]. \qquad f(z-(-1+2i)) = \frac{(3+2i)}{4i}$$

14 a) ii) Find Laurent's series of $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ valid in $2<|z|<3$.

Soln:- Refer UQ May / June 2009 – Q.No. 14 b(i)

14. b) i) Evaluate using CRT (or) $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$, $C: |z|=3$.

Soln:- $Dr=0 \Rightarrow z=1$ lies inside $|z|=3$
$z=2$ " " " $|z|=3$.

$$\therefore \text{Res } f(z)\Big]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \frac{\sin\pi z^2 + \cos\pi z^2}{\cancel{(z-1)}(z-2)} = \frac{\overset{-1}{\cos\pi} + \overset{0}{\sin\pi}}{-1} = 1$$

$$\text{\& Res } f(z)\Big]_{z=2} = \lim_{z\to 2} \cancel{(z-2)} \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)\cancel{(z-2)}} = 1.$$

By CRT, $\int_C f(z)dz = 2\pi i \times$ sum of the residues.
$= 2\pi i[1+1] = 4\pi i$.

14 b) ii) Evaluate using Contour integration $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$.

Soln:- $\int_C \phi(z)dz = \int_{-R}^{R} \phi(z)dz + \int_\Gamma \phi(z)dz$.

as $R\to\infty$ $\int_\Gamma \phi(z)dz = 0$

$$\therefore \int_{-R}^{R} \phi(z)\,dz = \int_{-\infty}^{\infty} \phi(x)\,dx.$$

The poles of $\phi(z)$ are $(z^2+1)^2=0$, $z=\pm i$ (twice).

$z = \pm i$ is a poles of order 2, $z = i$ lies in the upper ½ of plane

$$\therefore \operatorname{Res} f(z)\Big]_{z=i} = \lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-i)^2}\frac{z^2}{(z+i)^2\cancel{(z-i)^2}}\right]$$

$$= \frac{-i}{4}.$$

$$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}\,dx = 2\pi i\left(-\tfrac{i}{4}\right) = \tfrac{\pi}{2}.$$

15 a) i) Find $L^{-1}\left[\frac{1}{S(s^2+4)}\right]$ using Convolution Theorem.

<u>Soln:-</u> $Gn = L^{-1}(1/s) * L^{-1}\left[\frac{1}{s^2+4}\right) = 1 * \frac{\sin 2t}{2}$

$$= \tfrac{1}{2}\int_0^t \sin 2(t-u)\,du = \tfrac{1}{2}\left[\frac{-\cos 2(t-u)}{2(-1)}\right]_0^t$$

$$= \tfrac{1}{4}\left[\cos 2(t-t) - \cos 2t\right] = \tfrac{1}{4}(1-\cos 2t).$$

15 a) ii) Find the Laplace transform of a square wave fun. is

$$f(t) = \begin{cases} E, & 0 \le t \le a/2 \\ -E, & a/2 \le t \le a \end{cases} \quad \& \quad f(t+a) = f(t).$$

<u>Soln:-</u> $L[f(t)] = \frac{1}{1-e^{-as}}\left[\int_0^{a/2} E\,e^{-st}dt + \int_{a/2}^{a} -E\,e^{-st}dt\right]$

$$= \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} + \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E\,(1+e^{-sa}-2e^{-sa/2})}{s(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{E}{s}\frac{(1-e^{-as/2})^{\cancel{2}}}{\cancel{(1-e^{-as/2})}(1+e^{-as/2})} = E/s \tanh\left(\frac{as}{4}\right).$$

(or).

15 b) i) Evaluate $\int_0^\infty t\,e^{-2t}\cos t\,dt$ using Laplace transforms.

<u>Soln:-</u> $L[t\cos t] = \int_0^\infty e^{-st}\,t\cos t\,dt$

Hence the gn integral is the value of $L(t\cos t)$ when $s=2$.