B.E/B.Tech. Degree Examinations, November/December 2011.

MA2261 - Probability and Random Process. S.KANAKALAKSHMI

1. The CDF of a continuous RV is given by $F(x) = \begin{cases} 0 & , x<0 \\ 1-e^{-x/5} & , 0 \le x < \infty . \end{cases}$ Find the PDF and mean of X.

when $x<0$, $F(x) = 0$.

when $0 \le x < \infty$, $F(x) = \int_0^x (1-e^{-x/5})dx = \left[x - \frac{e^{-x/5}}{-1/5}\right]_0^x = \left[x + 5e^{-x/5} - 5\right]$, $0<x<\infty$.

$\therefore F(x) = \begin{cases} (x-5)+5e^{-x/5} & , 0<x<\infty \\ 0 & , \text{otherwise}. \end{cases}$

2. If X is a normal random variable with mean zero and variance σ^2, find the PDF of $Y=e^X$.

X is a normal random variable, $f_X(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty<x<\infty$

Given $y=e^x$. $\Rightarrow x = \log y \Rightarrow \frac{dx}{dy} = \frac{1}{y} \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{y}$.

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{y}$, $0<y<\infty$, $x = \log y$.

3. If the joint pdf of (X,Y) is $f_{XY}(x) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0 & , \text{otherwise} \end{cases}$, check whether X and Y are independent.

Given $f(x,y) = e^{-(x+y)} = e^{-x}.e^{-y}$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} e^{-x}.e^{-y}dy = e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}[-e^{-\infty}+e^0] = e^{-x}, x>0$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{\infty} e^{-x}e^{-y}dx = e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = e^{-y}[-e^{-\infty}+e^0] = e^{-y}, y>0.$

$f(x,y) = e^{-(x+y)} = e^{-x}.e^{-y} = f_X(x)f_Y(y)$. $\therefore$ X and Y are independent.

4. The regression equations are $3x+2y=26$ and $6x+y=31$. Find the correlation coefficient between X and Y.

Given $3x+2y=26$	$6x+y=31$	$\therefore r = \sqrt{b_{yx} * b_{xy}}$
$2y = 26-3x$	$6x = 31-y$	$= \sqrt{-3/2 \times -1/6}$
$y = \frac{26}{2} - \frac{3x}{2}$	$x = \frac{31}{6} - y/6$	$= \sqrt{3/12}$
$b_{yx} = -3/2$	$\therefore b_{xy} = -1/6$	$\boxed{r = 0.5}$

5. When is a random process said to be mean ergodic?

A random process $\{X(t)\}$ is mean-ergodic if $\lim_{T\to\infty} Var\,\overline{X}_T = 0$.

where $\overline{X}_T$ is time average of $\{X(t)\}$ and μ is its mean.

6. If $\{X(t)\}$ is a normal process with $\mu(t)=10$ and $C(t_1,t_2)=16e^{-|t_1-t_2|}$ find the variance of $X(10)-X(6)$.

Let $U = X(10)-X(6)$ Then U is also a random variable.

$E(U) = E[X(10)-X(6)] = E[X(10)] - E[X(6)] = 10-10=0$ (Mean of $X(t)=0$ $\therefore E[X(10)]=10$, $E[X(6)]=10$)

$E(U^2) = E[X^2(10)+X^2(6)-2X(10)X(6)]$

$= E[X^2(10)] + E[X^2(6)] - 2E[X(10)X(6)]$

$= E[X^2(10)] + E[X^2(6)] - 2\,cov(10,6)$.

$\therefore Var(U) = E(U^2)-(E(U))^2 = E[X^2(10)] + E(X^2(6)] - 2\,cov(10,6)$

$= Cov(10,10) + Cov(6,6) - 2\,Cov(10,6)$ $\left(\because Cov(10,10) = 16e^{-|10-10|} = 16\right)$

$Var(U) = 16+16-32e^{-|10-6|} = 32-32e^{-4} = \underline{31.4139}$

7. The ACF of a stationary random process is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process.

$\overline{X}^2 = \lim_{|\tau|\to\infty} R_{XX}(\tau) = \lim_{\tau\to\infty} 25 + \frac{4}{1+6\tau^2} = 25 \Rightarrow \boxed{\overline{X}=5}$ Mean

$R_{XX}(0) = 25 + \frac{4}{1+0} = 29$ $\therefore E(X^2(t)) = 29$.

$\therefore Var(X(t)) = E(X^2(t)) - \{E(X(t))\}^2 = 29-5^2 = 4$.

8. Prove that for a WSS process $\{X(t)\}$, $R_{XX}(t,t+\tau)$ is an even function of τ.

$R_{XX}(\tau) = E[X(t)X(t+\tau)]$ Replacing τ by $-\tau$ in $R_{XX}(\tau)$.

$R_{XX}(-\tau) = E[X(t)X(t-\tau)]$ put $t-\tau = t_1$; $t = t_1+\tau$

$\therefore R_{XX}(-\tau) = E[X(t_1+\tau)X(t_1)] = E[X(t_1)X(t_1+\tau)]$

$= R_{XX}(\tau)$

Hence $R_{XX}(-\tau) = R_{XX}(\tau)$. i.e $R_{XX}(t,t+\tau)$ is an even function of τ.

9. State any two properties of a linear time-invariant system.

1) If the input $x(t)$ and its output $y(t)$ are related by $y(t)=\int_{-\infty}^{\infty} h(u)\,x(t-u)\,du$ then the system is a linear time-invariant system.

2) If $\{x(t)\}$ is a WSS process and if $y(t)=\int_{-\infty}^{\infty} h(u)\,x(t-u)\,du$, then

a) $R_{xy}(\tau)=R_{xx}(\tau)*h(\tau)$

b) $R_{yy}(\tau)=R_{xy}(\tau)*h(-\tau)$ where $*$ denotes the convolution.

c) $R_{yy}(\tau)=R_{xx}(\tau)*h(\tau)*h(-\tau)$

d) $S_{xy}(\tau)=S_{xx}(\omega)H(\omega)$ and

e) $S_{yy}(\omega)=S_{xx}(\omega)\,|H(\omega)|^2$.

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10. If $\{x(t)\}$ and $\{y(t)\}$ is the system $y(t)=\int^{\infty} h(u)\,x(t-u)\,du$ are WSS process how are their auto correlation function related.

$R_{yy}(t,t+\tau)=E[y(t)\cdot y(t+\tau)]$

$$=E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\,x(t-u_1)\,h(u_2)\,x(t+\tau-u_2)\,du_1\,du_2\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)\,E[x(t-u_1)\,x(t+\tau-u_2)]\,du_1\,du_2$$

Since $\{x(t)\}$ is a WSS process, $R_{yy}(t,t+\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)\,R_{xx}(\tau)\,du_1\,du_2$

$R_{yy}(t,t+\tau)$ = a function of τ. Hence the output $\{y(t)\}$ is also a WSS process.

PART B - ($5\times16=80$ marks).

11) a) (i) The probability function of an infinite discrete distribution is given by $P(X=j)=\frac{1}{2^j}$ $(j=1,2,3,\dots)$ Find

(1) Mean of X

(2) $P(X$ is even$)$ and

(3) $P(X$ is divisible by $3)$.

(1) Mean $= E[X] = \sum_j x_j p(X=j)$

$$= 1 \cdot \frac{1}{2^1} + 2 \cdot \frac{1}{2^2} + 3\frac{1}{2^3} + 4\frac{1}{2^4} + \cdots$$

$$= \tfrac{1}{2}\left(1 + 2\frac{1}{2} + 3\frac{1}{2^2} + 4 \cdot \frac{1}{2^3} + \cdots\right)$$

$$= \frac{1}{2}\left(1 + 2(\tfrac{1}{2}) + 3(\tfrac{1}{2})^2 + 4(\tfrac{1}{2})^3 + \cdots\right)$$

$$1 + 2x + 3x^2 + 4x^3 + \cdots = (1-x)^{-2}.$$

$$= \tfrac{1}{2}(1 - \tfrac{1}{2})^{-2}$$

$$= \frac{1}{2}(\tfrac{1}{2})^{-2} = \frac{1}{2}\frac{1}{(\frac{1}{2})^2} = \frac{1}{2} \cdot \frac{1}{(\frac{1}{4})} = \frac{1}{\cancel{2}} \times \cancel{4}^{2} = 2.$$

Mean $= E[X] = 2$.

$$\text{or } \frac{1}{2}(\tfrac{1}{2})^{-2} = \tfrac{1}{2}(2)^{+2} = \frac{1}{2} \times 4 = 2.$$

(2) $p(X \text{ is even}) = p(X = 2, 4, 6, 8, \ldots)$

$$= p(X=2) + p(X=4) + p(X=6) + p(X=8) + \cdots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots$$

$$\frac{1}{2^3 \times 2^3} = 2^2 \cdot 2^4$$

$$2^6 \cdot 2^2 = 2^{6+2} = 2^8$$

$$= \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right)$$

$$= \frac{1}{4}\left(1 + \tfrac{1}{2^2} + (\tfrac{1}{2^2})^2 + (\tfrac{1}{2^2})^3 + \cdots\right)$$

$$1 + x + x^2 + \cdots = (1-x)^{-1}$$

$$= \frac{1}{4}[1 - \tfrac{1}{4}]^{-1} = \frac{1}{4}(3/4)^{-1} = \frac{1}{4}(\tfrac{4}{3})^{+1} = \frac{1}{\cancel{4}} \times \cancel{4}/3 = \tfrac{1}{3}.$$

(3) $p(X \text{ is divisible by } 3) = p(X = 3, 6, 9, 12, \ldots)$

$$= p(X=3) + p(6) + p(X=9) + \cdots$$

$$= \left(\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right) = \frac{1}{2^3} + \frac{1}{2^3 \cdot 2^3} + \frac{1}{2^3 \cdot 2^3 \cdot 2^3} + \cdots$$

$$= \frac{1}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{2^3 \cdot 2^3} + \cdots\right) = \frac{1}{2^3}\left[1 + (\tfrac{1}{2^3})^1 + (\tfrac{1}{2^3})^2 + \cdots\right]$$

$P(X \text{ is divisible by } 3) = \frac{1}{2^3}\left[1 - \frac{1}{2^3}\right]^{-1} = \frac{1}{8}\left(\frac{7}{8}\right)^{-1} = \frac{1}{8}\left(\frac{8}{7}\right)^{+1} = \frac{1}{7}.$

(ii) A continuous RV X has the pdf $f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{elsewhere} \end{cases}$. Find

(1) the value of k

(2) Distribution function of X.

(3) $P(X \geq 0)$.

Since $f(x)$ is pdf we have $\int_{-\infty}^{\infty} f(x)dx = 1$.

$k\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = 1 \Rightarrow k\left(\tan^{-1}x\right)_{-\infty}^{\infty} = 1 \Rightarrow k[\tan(\infty) - \tan^{-1}(-\infty)] = 1.$

$k\left[(\pi/2) + \pi/2\right] = 1 \Rightarrow \boxed{k = 1/\pi}$ since $\tan^{-1}(\infty) = \pi/2$ & $\tan^{-1}(-\infty) = -\pi/2$.

$\therefore f(x) = \begin{cases} \frac{1}{\pi}\cdot\frac{1}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise}. \end{cases}$

$F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{1+x^2}dx = \frac{1}{\pi}\left(\tan^{-1}x\right)_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1}x - \tan^{-1}(-\infty)\right]$

$F(x) = \frac{1}{\pi}\left[\tan^{-1}x + \pi/2\right]$

$P(X > 0) = 1 - P(X \leq 0) = 1 - F(0) = 1 - \frac{1}{\pi}[0 + \pi/2] = 1 - 1/2 = 1/2.$

11) b) Let X and Y be independent normal variates with mean 45 and 44 and standard deviation 2 and 1.5 respectively. What is the probability that randomly chosen values of X and Y differ by 1.5 or more.

12) a) i) The joint probability density function of random variable X and Y is given by $f(x,y) = \begin{cases} \frac{8xy}{9}, & 1 \le x \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$ Find the conditional density function of X and Y.

Conditional density function of X given $Y=y$ is $f(x/y) = \dfrac{f(x,y)}{f(y)}$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{1}^{y} \frac{8}{9} xy\,dx = \frac{8}{9} y \left(\frac{x^2}{2}\right)_1^y = \frac{4}{9} y (y^2-1),\ 1<y<2.$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{2} \frac{8}{9} xy\,dy = \frac{8}{9} x \left(\frac{y^2}{2}\right)_x^2 = \frac{4}{9} x (4-x^2),\ 1<x<2$$

$$\therefore f(x/y) = \frac{\frac{8}{9} xy}{\frac{4y}{9}(y^2-1)} = \frac{2x}{y^2-1},\ 1<x<y<2.$$

$$\text{and } f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{8}{9} xy}{\frac{4x}{9}(4-x^2)} = \frac{2y}{4-x^2},\ 1<x<y<2.$$

ii) The jpdf of the 2D RV (X,Y) is $f(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

Find the correlation coefficient between X and Y.

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}},\quad \mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = -\frac{1}{144}$$

$$f(x) = \frac{3}{2} - x$$

$$f(y) = \frac{3}{2} - y$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \frac{5}{12}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\left(\frac{3}{2} - y\right)dy = \frac{5}{12}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \frac{1}{4} = E(Y^2).$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144} \Rightarrow \sigma_x = \frac{\sqrt{11}}{12}\ \&\ \sigma_y = \frac{\sqrt{11}}{12}.$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy = \frac{1}{6}.$$

$$\rho(X,Y) = \frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = -\frac{1}{11}.$$

13a) Show that the random process $X(t) = A\cos(\omega t+\theta)$ is wide sense stationary, if A and ω are constants and θ is uniformly distributed in $(0, 2\pi)$.

$\theta \sim U(0,2\pi)$ $\therefore$ $f(\theta) = 1/2\pi$, $(0,2\pi)$.

$$E[X(t)] = E[A\cos(\omega t+\theta)] = A\int_0^{2\pi}\cos(\omega t+\theta)\cdot f(\theta)\,d\theta = \frac{A}{2\pi}\int_0^{2\pi}\cos(\omega t+\theta)\,d\theta$$

$$= \frac{A}{2\pi}\left[\frac{\sin(\omega t+\theta)}{1}\right]_0^{2\pi} = \frac{A}{2\pi}[\sin\omega t - \sin\omega t] = 0, \text{ a constant.}$$

$$R_{xx}(t,t+\tau) = E[X(t)X(t+\tau)] = E[A^2\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta)]$$

$\cos A\cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$= \frac{A^2}{2}E[\cos(2\omega t+\omega\tau+2\theta)+\cos(-\omega\tau)]$$

$$= \frac{A^2}{2}\cos\omega\tau + \frac{A^2}{2}E[\cos(\omega t+\omega\tau+2\theta)]$$

$$= \frac{A^2}{2}\cos\omega\tau + \frac{A^2}{2}\int_0^{2\pi}\cos(\omega t+\omega\tau+2\theta)\cdot f(\theta)\,d\theta \qquad \frac{1}{2\pi}$$

$$= \frac{A^2}{2}\cos\omega\tau + \frac{A^2}{4\pi}\left[\sin(\omega t+\omega\tau+2\theta)\right]_0^{2\pi}$$

$R_{xx}(t,t+\tau) = \frac{A^2}{2}\cos\omega\tau$, a fn of τ $\quad (\because \sin(\omega t+\omega\tau+4\pi) = \sin(\omega t+\omega\tau))$.

$\therefore X(t)$ is WSS process.

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b) State the postulates of a poisson process and derive the probability distribution. Also prove that the sum of two independent poisson process is a poisson process.

Refer:

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14) a) The ACF of a random process is given by $R(\tau) = \begin{cases}\lambda^2, & |\tau| > \epsilon, \\ \lambda^2 + \frac{\lambda}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right); & |\tau| \le \epsilon\end{cases}$

Find the power spectral density of the process.

Let $\lambda = \alpha$ & $\epsilon = \lambda$ we get

$$R(\tau) = \begin{cases}\alpha^2, & |\tau| > \lambda \\ \alpha^2 + \frac{\alpha}{\lambda}\left(1-\frac{|\tau|}{\lambda}\right), & |\tau| \le \lambda.\end{cases}$$

The spectral density function is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{-\lambda} R_{XX}(\tau) e^{-j\omega\tau} d\tau + \int_{-\lambda}^{+\lambda} R_{XX}(\tau) e^{-j\omega\tau} d\tau + \int_{\lambda}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{-\lambda} \alpha^2 e^{-j\omega\tau} d\tau + \int_{-\lambda}^{\lambda} \left(\alpha^2 + \frac{\alpha}{\lambda}\left(1 - \frac{|\tau|}{\lambda}\right)\right) e^{-j\omega\tau} d\tau + \int_{\lambda}^{\infty} \alpha^2 e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \alpha^2 e^{-j\omega\tau} d\tau + \frac{\alpha}{\lambda} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\tau|}{\lambda}\right) e^{-j\omega\tau} d\tau$$

$$= F(\alpha^2) + \frac{2\alpha}{\lambda} \int_0^{\lambda} \left(1 - \frac{\tau}{\lambda}\right) \cos\omega\tau \, d\tau$$

$$= F(\alpha^2) + \frac{2\alpha}{\lambda} \left[\left(1 - \frac{\tau}{\lambda}\right)\left(\frac{\sin\omega\tau}{\omega}\right) - \left(-\frac{1}{\lambda}\right)\left(-\frac{\cos\omega\tau}{\omega^2}\right) \right]_0^{\lambda}$$

$$= \frac{2\alpha}{\lambda^2\omega^2} 2\sin^2\left(\frac{\omega\lambda}{2}\right) + F(\alpha^2)$$

$$= 4\alpha \frac{\sin^2(\lambda\omega/2)}{\lambda^2\omega^2} + F(\alpha^2)$$

$$\left(\because F(\alpha^2) = \alpha^2 \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau \quad \delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau \quad 2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau \right)$$

$$S_{XX}(\omega) = \frac{4\alpha \sin^2(\lambda\omega/2)}{\lambda^2\omega^2} + 2\pi\alpha^2\delta(\omega)$$

(ii) Given the power spectral density of a continuous process as $S_{XX}(\omega) = \frac{\omega^2+9}{\omega^4+15\omega^2+4}$. Find the mean square value of the process.

$$R_{XX}(\tau) = F^{-1}\left[\frac{\omega^2+9}{\omega^4+15\omega^2+4}\right] \text{ consider } \frac{\omega^2+9}{\omega^4+15\omega^2+4} = \frac{u+9}{u^2+5u+4}, \; u = \omega^2.$$

$$\frac{u+9}{(u+1)(u+4)} = \frac{A}{u+1} + \frac{B}{u+4} \Rightarrow \boxed{u+9 = A(u+4) + B(u+1)}$$

put $u = -4$, $B = -5/3$

put $u = -1$, $A = 8/3$.

$\therefore \frac{u+9}{(u+1)(u+4)} = \frac{8/3}{u+1} - \frac{5/3}{u+4}$

$\therefore R_{xx}(\tau) = F^{-1}\left[\frac{8}{3} \cdot \frac{1}{\omega^2+1} - \frac{5}{3} \frac{1}{\omega^2+4} \right]$

$= \frac{8}{3} \times \frac{1}{2} F^{-1}\left[\frac{2\times 1}{\omega^2+1} \right] - \frac{5}{3} \cdot \frac{1}{2^2} \left[\frac{2\times 2^{\alpha}}{\omega^2+4} \right]$

$R_{xx}(\tau) = \frac{8}{6} e^{-|\tau|} - \frac{5}{12} e^{-2|\tau|}$

Mean squal value $R_{xx}(0) = 8/16 - 5/12 = 11/12$ //.

14) b) (i) State and prove Weiner-Khinchine theorem.

Refer: Apr/May 2011 Q.No 14 b (i)

(ii) The cross power spectrum of real random processes $\{X(t)\}$ and $\{Y(t)\}$ is given by $S_{XY}(\omega) = \begin{cases} a+bj\omega, & |\omega|<1 \\ 0, & \text{elsewhere} \end{cases}$ Find the cross correlation function

Refer: Apr/May 2011 Q.No 14 b (ii)

15) a) i) If the input to a time invariant, stable, linear system is a WSS process. Prove that the output will also be a WSS process.

The I/p and O/p are related by $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$ — ①

$\Rightarrow E[Y(t)] = \int_{-\infty}^{\infty} h(u) E[X(t-u)] du$ —— ②

Since $\{X(t)\}$ is a WSS process, Mean is a constant i.e $E[X(t-u)]$ is a const

$E[Y(t)] = E[X(t-u)] \int_{-\infty}^{\infty} h(u) du = \bar{X}_u \int_{-\infty}^{\infty} h(u) du$ = a finite constant, independent of t

$\therefore E[Y(t)]$ = a constant.

By definition $R_{YY}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)] = E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) X(t-u_1) h(u_2) X(t+\tau-u_2) du_1 du_2 \right]$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2) E[X(t-u_1) X(t+\tau-u_2)] du_1 du_2$

= a fn of τ since $\{X(t)\}$ is a WSS process

Hence the output $\{Y(t)\}$ is also a WSS process.

a) (i) Let $X(t)$ be a WSS process which is the input to a linear time invariant system with unit impulse $h(t)$ and output $Y(t)$, then prove that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$.

WKT

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) \quad \text{(Q:15 b)(i) proof.}$$

Taking Fourier Transform, we have

$$F[R_{XY}(\tau)] = F[R_{XX}(\tau) * h(\tau)]$$

$$= F[R_{XX}(\tau)]\, F[h(\tau)]$$

By convolution theorem on Fourier transform,

$$S_{XY}(\omega) = S_{XX}(\omega)\, F[h(\tau)]$$

$$S_{XY}(\omega) = S_{XX}(\omega) H(\omega) \quad \text{— (1)}$$

Similarly Applying Fourier Transform on $R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$ we get

$$S_{YY}(\omega) = S_{XY}(\omega)\, (\overline{H(\omega)}) \quad \text{— (2)}$$

From (1) and (2), $S_{YY}(\omega) = S_{XX}(\omega) H(\omega) \cdot (\overline{H(\omega)}) = \underline{S_{XX}(\omega) |H(\omega)|^2}$

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b) i) For a input-output linear system $(X(t), h(t), Y(t))$, derive $R_{XY}(\tau)$ & $R_{YY}(\tau)$.

(i) Given $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$.

WKT $R_{XY}(\tau) = E[X(t) Y(t+\tau)] = E\left[X(t) \int_{-\infty}^{\infty} h(u) X(t+\tau-u)\, du\right]$

$$= \int_{-\infty}^{\infty} E[X(t) X(t+\tau-u)]\, h(u)\, du = \int_{-\infty}^{\infty} R_{XX}(\tau-u) h(u)\, du$$

$\therefore R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$.

(ii) $R_{YY}(\tau) = E[Y(t) Y(t+\tau)] = E\left[\int_{-\infty}^{\infty} h(u) X(t-u) \cdot Y(t+\tau)\, du\right] = \int_{-\infty}^{\infty} E[X(t-u) Y(t+\tau)]\, h(u)\, du$

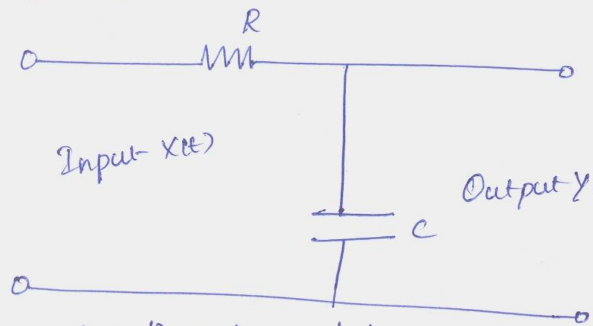
put $t-u = u_1$, $t = u+u_1$, $t+\tau = u+u_1+\tau$ | when $u = -\infty$, $u_1 = -\infty$; $u = \infty$, $u_1 = \infty$.

$$\therefore R_{YY}(\tau) = \int_{-\infty}^{\infty} E[X(u_1) Y(u_1+u+\tau)]\, h(u)\, du = \int_{-\infty}^{\infty} R_{XY}(\tau+u)\, h(u)\, du$$

$$= \int_{+\infty}^{-\infty} R_{XY}(\tau-\alpha)\, h(-\alpha)\, (-d\alpha) \quad (\text{Taking } u = -\alpha)$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau-\alpha)\, h(-\alpha)\, d\alpha \quad \therefore \boxed{R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)}$$

15b(ii) A white Gaussian noise $X(t)$ with zero mean and spectral density $\frac{N_0}{2}$ is applied to a low-pass RC filter shown in the figure.



Determine the auto correlation of the output $Y(t)$.

Answer:-

The transfer function of the filter is, $H(f) = \frac{1}{1+i2\pi fRC}$

$\therefore H(\omega) = \frac{1}{1+i\omega RC}$, $\omega = 2\pi f$.   Let Output $Y(t)$ be $N(t)$

The power spectral density of the noise $N(t)$ appearing at the low pass RC filter output is given by

$$S_{NN}(\omega) = S_{WW}(\omega)|H(\omega)|^2$$

$$= \frac{N_0}{2} \frac{1}{1+(\omega RC)^2}$$

$$= \frac{N_0}{2} \frac{\beta^2}{\beta^2+\omega^2}, \quad \beta = \frac{1}{RC}$$

$$S_{NN}(\omega) = \frac{\beta N_0}{2} \cdot \frac{\beta}{\beta^2+\omega^2}$$

Taking Inverse Fourier transform, we get the ACF

$$R_{NN}(\tau) = F^{-1}\left[\frac{\beta N_0}{2} \cdot \frac{\beta}{\beta^2+\omega^2}\right]$$

$$= \frac{\beta N_0}{4} F^{-1}\left[\frac{2\beta}{\beta^2+\omega^2}\right]$$

$$= \frac{\beta N_0}{4} e^{-\beta|\tau|} \quad \left(\text{since } F\left(e^{-\alpha|\tau|}\right) = \frac{2\alpha}{\alpha^2+\omega^2}\right)$$

$$R_{NN}(\tau) = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}} \quad \text{and } \beta = \frac{1}{RC}.$$