SL (1)

Au - chennai Nov/Dec 2010.

Reg 2008.

MA 2261 - Probability and Random process

Part - A

1) A continuous R.V X has pdf $f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{o.w} \end{cases}$

find k such that $P(X>k) = 0.5$

Soln:

$P(X>k) = 0.5$

$\int_k^1 f(x)\,dx = 0.5 \Rightarrow \int_k^1 3x^2\,dx = 0.05 \Rightarrow 1-k^3 = \frac{1}{20}$

$k = \left(\frac{19}{20}\right)^{1/3} = 0.9830$

2) If X is uniformly distributed in $(-\pi/2, \pi/2)$. Find the pdf of $Y = \tan X$.

Soln: $f_X(x) = \frac{1}{\pi}, \quad -\pi/2 < x < \pi/2$.

Given $y = \tan x$.

$y = \tan x \Rightarrow x = \tan^{-1}(y)$

$\left|\frac{dx}{dy}\right| = \frac{1}{1+y^2}$

$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\cdot\frac{1}{1+y^2}$ — (1)

change of domain (ie x domain into y domain)

Since $x \in (-\pi/2, \pi/2)$, $\tan x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$ — (2)

Using (1) & (2) we get

$f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$.

3) Let x and y be continuous R.V's with joint pdf

$f_{xy}(x,y) = \frac{x(x-y)}{8}$, $0<x<2$, $-x<y<x$ and $f_{xy}(x,y)=0$, elsewhere

find $f_{y/x}(y/x)$.

Soln:

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x}^{x} \frac{x}{8}(x-y)\,dy$$

$$= \frac{1}{8}\left[x^2y - \frac{xy^2}{2}\right] = x^3/4, \quad 0<x<2.$$

$$f_{y/x}(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{x(x-y)}{8}}{x^3/4} = \frac{x(x-y)}{2x^3}, \quad 0<x<2, \; -x<y<x.$$

4) State central limit theorem for iid R.V's.

Lindberg - Levy's Form.

If $X_1, X_2, \ldots X_n$ be a seq of independent identically distributed R.V with $E[X_i] = \mu$, and $var(X_i) = \sigma^2$, $i=1,2\ldots$ and if $S_n = X_1 + X_2 + \ldots X_n$ then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

5) Consider the R.P $X(t) = \cos(t+\phi)$ where ϕ is a RV with density fn $f(\phi) = \frac{1}{\pi}$, $-\pi/2 < \phi < \pi/2$. Check whether (or) not the process is WSS.

Soln:

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\phi)\,d\phi = \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\cdot\frac{1}{\pi}\,d\phi$$

$$= \frac{1}{\pi}\left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2} = \frac{2\cos t}{\pi} \neq \text{constant}$$

$X(t)$ is not a stationary process.

6) State the posulates of a poisson Process

Soln:

1) $P[1$ occurence in $(t, t+\Delta t)] = \lambda \Delta t + O(\Delta t)$

2) $P[0$ occurence in $(t, t+\Delta t)] = 1 - \lambda \Delta t + O(\Delta t)$

3) $P[2$ or more occurence in $(t, t+\Delta t)] = O(\Delta t)$

4) $x(t)$ is independent of the number of occurance of the event in any interval prior (or) after the interval $(0, t)$

5) The probability that the event occurs a specified number of times in (t_0, t_0+t) depends only on t, but not on t_0.

7) Find the Variance of the stationary process $\{x(t)\}$ whose auto correlation f^n is $R_{xx}(\tau) = 2+4e^{-2|\tau|}$.

Soln:

$$\bar{x}^2 = \lim_{|\tau| \to \infty} (2+4e^{-2|\tau|}) = 2+4\frac{1}{e^{\infty}} = 2 \Rightarrow \boxed{\bar{x} = \sqrt{2}}$$

$$E[x^2(t)] = R_{xx}(0) = 6.$$

$$Var(x(t)) = E[x^2(t)] - [E(x)]^2 = 6 - (\sqrt{2})^2 = 4.$$

8) State any two properties of cross. Correlation f^n.

1) $R_{xy}(\tau) = R_{yx}(-\tau)$

2) If $\{x(t)\}$ and $\{y(t)\}$ are two R.P then $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0) R_{yy}(0)}$.

9) If $y(t)$ is the output of a linear time invariant system with impulse response $h(t)$, then find the cross correlation of the input f^n $x(t)$ and output f^n $y(t)$.

Soln:

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

10) Define Band-limited white noise.

Noise having a non zero and constant power spectrum over a finite frequency band and zero everywhere else is called band-limited white noise.

Part - B

11) a)

(i) A R.V X has the following prob dist

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find a) The value of k

b) $P(1.5 < x < 4.5 / x > 2)$

c) The smallest value of n for which $P(x \le n) > 1/2$.

Soln:

Refer Nov/Dec 2008

Q.No 11) a) (ii)

(ii) Find the MGF of a R.V X having the pdf $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0, & \text{o.w} \end{cases}$

Also deduce that first four moments about the origin.

Soln:

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx.$$

$$= \int_0^{\infty} e^{tx} \frac{x}{4} e^{-x/2} dx.$$

$$= \frac{1}{4}\left[\frac{x e^{-(1/2-t)x}}{-(1/2-t)} - \frac{e^{-(1/2-t)x}}{-(1/2-t)^2}\right]_0^{\infty}$$

$$= \frac{1}{4}\left[\frac{4e^0}{(1-2t)^2}\right]$$

$$M_X(t) = \frac{1}{(1-2t)^2}.$$

(3)

$M_X(t) = (1-2t)^{-2}$

1st moment

$M_X'(0) = \left[(-2)(1-2t)^{-3}(-2)\right]_{t=0}$

$\boxed{M_X'(0) = 4}$

$M_X''(t) = \left[4(-3)(1-2t)^{-4}(-2)\right]_{t=0}$

2nd moment

$\boxed{M_X''(0) = 24}$

$M_X'''(t) = 24(-4)(1-2t)^{-5}(-2)$

3rd moment

$\boxed{M_X'''(0) = 192}$

4th moment

$M^{iv}(t) = 1920(1-2t)^{-6}$

$\boxed{M^{iv}(0) = 1920}$

(or)

11 b) (i) If X is uniformly distributed in $(-1,1)$ then find the pdf of $Y = \sin\frac{\pi X}{2}$.

Soln:

$$f_X(x) = \begin{cases} 1/2, & -1 < x < 1 \\ 0, & \text{o.w} \end{cases}$$

$$Y = \sin\left(\frac{\pi X}{2}\right) \Rightarrow y = \sin\left(\frac{\pi x}{2}\right)$$

$$\frac{\pi x}{2} = \sin^{-1}(y) \Rightarrow x = \frac{2}{\pi}\sin^{-1}(y)$$

$$\left|\frac{dx}{dy}\right| = \frac{2}{\pi\sqrt{1-y^2}}$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|, \quad = \frac{1}{\pi\sqrt{1-y^2}}$$

Change of domain (x-domain and y-domain)

$y = \sin\left(\frac{\pi x}{2}\right), \quad -1 < x < 1$

when $x=-1, y=-1$, $x=1, y=1$ $\Big\}$ $-1<y<1$.

$\therefore f_Y(y) = \dfrac{1}{\pi\sqrt{1-y^2}}$, $-1<y<1$.

1 b) (ii) If X and Y are independent R.V's following $N(8,2)$ and $N(12, 4\sqrt{3})$ respectively, find the value of λ such that $P[2X-Y \le 2\lambda] = P[X+2Y \ge \lambda]$.

Soln:

Let $U = 2X-Y$, $V = X+2Y$.

By additive property of normal distribution,

$U = 2X-Y$ follows $N(2\times 8-12, \sqrt{4\times 4+1\times 48})$

$N(4, 8)$.

lly $V = X+2Y$ follow $N(32, 14)$.

Consider $P(2X-Y \le 2\lambda) = P(X+2Y \ge \lambda)$

$$P(U \le 2\lambda) = P(V \ge \lambda)$$

$$P\left(\frac{U-4}{8} \le \frac{2\lambda-4}{8}\right) = P\left(\frac{V-32}{14} \ge \frac{\lambda-32}{14}\right).$$

$$P\left(Z \le \frac{2\lambda-4}{8}\right) = P\left(Z \ge \frac{\lambda-32}{14}\right).$$

where Z is a standard normal variable

$$\frac{2\lambda-4}{8} = -\left(\frac{\lambda-32}{14}\right)$$

$$\boxed{\lambda = \frac{26}{3}}$$

12)

a) Two R.v X and Y have the joint pdf given by

$$f_{xy}(x,y) = \begin{cases} k(1-x^2y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

(i) Find k (ii) obtain the marginal pdf of X and Y (iii) find the Correlation coeff b/w X and Y.

Proof:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

$$\int_0^1\int_0^1 k(1-x^2y)\,dx\,dy = 1$$

$$\boxed{k = 6/5}$$

$$\therefore f(x,y) = \begin{cases} \frac{6}{5}(1-x^2y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

M.D.F of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{6}{5}(1-x^2y)\,dy$

$$\boxed{f_X(x) = \frac{3}{5}(2-x^2),\ 0 \le x \le 1}$$

MDF of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{5}(1-x^2y)\,dx$.

$$\boxed{f_Y(y) = \frac{2}{3}(3-y),\ 0 \le y \le 1}$$

To find Correlation coeff:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\,\frac{3}{5}(2-x^2)\,dx = 9/20.$$

$$E(y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y\cdot\frac{2}{5}(3-y)\,dy = 7/15$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\cdot\frac{3}{5}(2-x^2)\,dx = 7/25$$

$$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^1 y^2 \left(\frac{2}{5}(3-y)\right) dy = \frac{3}{10}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{7}{25} - \left(\frac{9}{20}\right)^2 = \frac{31}{400}.$$

$$Var(y) = E(y^2) - [E(y)]^2 = \frac{3}{10} - \left(\frac{7}{15}\right)^2 = \frac{37}{450}$$

$$\sigma_x = \sqrt{\frac{31}{400}}, \quad \sigma_y = \sqrt{\frac{37}{450}}$$

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y) dx dy = \int_0^1\int_0^1 xy\left(\frac{6}{5}(1-x^2y)\right) dx dy.$$

$$= \frac{6}{5}\int_0^1 \left(\frac{xy^2}{2} - \frac{x^3y^3}{3}\right)_0^1 dx.$$

$$= \frac{6}{5}\left[\frac{x^2}{4} - \frac{1}{12}x^4\right]_0^1$$

$$= \frac{1}{5}.$$

$$\therefore \rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y} = \frac{\frac{1}{5} - \frac{63}{300}}{\sqrt{31/400}\sqrt{37/450}}$$

$$\boxed{\rho_{xy} = -0.125}$$

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12 b) (ii) If x and y are independent continuous R.V, show that the pdf of $U = x+y$ is given by $h(u) = \int_{-\infty}^{\infty} f_x(v) f_y(u-v) dv$

Soln:

$U = x+y.$

$$F_U(u) = P(x+y \le u) = \int_{-\infty}^{\infty}\int_{-\infty}^{u-y} f(x,y) dx dy$$

Both sides diff w.r.to u we get

$$f_U(u) = \int_{-\infty}^{\infty} f(x, u-x) dx \quad \text{—— (1)}$$

(5)

Since x and y are independent RV's

$$f(x,y) = f_X(x)\, f_Y(y)$$

$$f(x, y-x) = f_X(x)\, f_Y(u-x) \quad \text{--- (2)}$$

Sub (2) in (1) we get

$$f_U(u) = \int_{-\infty}^{\infty} f(x, u-x)\,dx = \int_{-\infty}^{\infty} f_X(x)\, f_Y(u-x)\,dx$$

Changing $f(u)$ to $h(u)$ and x to v we get

$$h(u) = \int_{-\infty}^{\infty} f_X(v)\, f_Y(u-v)\,dv$$

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b) ii) If V_i, $i = 1, 2, 3 \ldots 20$ are independent noise voltage received in an adder and V is the sum of the voltage received, find probability that the total incoming voltage V exceeds 105, Using CLT. Assume that each of the R.V V_i is uniformly distributed over $(0, 10)$

Soln:

V is uniformly distributed in $(0, 10)$ we get

Mean of $V = E(V) = \frac{1}{2}(0+10) = 5$

$$Var(V) = \frac{1}{12}(10^2) = \frac{25}{3}$$

By CLT, $n\mu = 20 \times 5 = 100$

$$n\sigma^2 = \frac{25}{3} \times 20 = \frac{500}{3}$$

Let $z = \frac{V - n\mu}{n\sigma^2}$

When $V = 105$, $z = \frac{105 - 100}{500/3} = 0.39$

$P(Y>105) = P(Z>0.39) = P(0.39<Z<\infty)$

$= 0.5 - P(0<Z<0.39)$

$= 0.5 - 0.1517$ (From table)

$= 0.3483$

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13)a)

(i) The process $\{X(t)\}$ whose probability distribution under certain condition is given by
$$P\{X(t)=n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n=1,2\ldots \\ \dfrac{at}{1+at}, & n=0 \end{cases}$$
Find the mean and variance of the process. Is the process first order stationary?

Soln:

Refer Nov/Dec 2011.

Q. No 13 a (ii)

(ii) If the WSS process $\{X(t)\}$ is given by $X(t) = 10\cos(100t+\theta)$ where θ is uniformly distributed over $(-\pi,\pi)$. Prove that $\{X(t)\}$ is correlation ergodic.

Soln:

$X(t) = 10\cos(100t+\theta)$ —— (1)

$Y(t) = X(t)\,X(t+\lambda)$ —— (2)

Sub $X(t)$ in (2),

$Y(t) = 100\cos(100t+\theta)\cos(100t+100\lambda+\theta)$ —— (3)

$Y(t) = \frac{100}{2}\left[\cos(-100\lambda) + \cos(2\times 100t + 2\theta + 100\lambda)\right]$

$= \frac{100}{2}\left[\cos(100\lambda) + \cos(2\times 100t + 2\theta + 100\lambda)\right]$ —— (4)

(6)

To prove $\{x(t)\}$ is correlation ergodic, we have to show that $\{y(t)\}$ is mean-ergodic.

$$E(y(t)) = \lim_{|T|\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)\,dt$$

$$E[y(t)] = E\left[\frac{100}{2}\cos(100\lambda) + \cos(200t+2\theta+100\lambda)\right]$$

$$= \frac{100}{2}\left\{\cos(100\lambda) + \int_{-\infty}^{\infty} f(\theta)\cos(200t+2\theta+100\lambda)\,d\theta\right\} \quad \text{—(5)}$$

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi<\theta<\pi \\ 0, & \text{o.w} \end{cases} \quad \text{—(6)}$$

sub (6) in (5) we get

$$E[y(t)] = \frac{100}{2}\left\{\cos(100\lambda) + \int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(200t+2\theta+100\lambda)\,d\theta\right\}$$

$$= \frac{100}{2}\left\{\cos 100\lambda + \frac{1}{4\pi}\left[\sin(200t+2\theta+100\lambda)\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{100}{2}\cos 100\lambda \quad \text{—(7)}$$

Now $\overline{Y_T} = \frac{1}{2T}\int_{-T}^{T} y(t)\,dt$

$$= \frac{1}{2T}\int_{-T}^{T}\frac{100}{2}\left[\cos(100\lambda) + \cos(200t+2\theta+100\lambda)\right]d$$

$$= \frac{100}{4T}\, 2T\cos(100\lambda) + \frac{1}{8T}\left[\sin(200T+2\theta+100\lambda) - \sin(-200T+2\theta+100\lambda)\right] \quad \text{—(8)}$$

$$\lim_{T\to\infty}\overline{Y_T} = \frac{100}{2}\cos(100\lambda) \quad \text{—(9)}.$$

From (7) and (9) we have

$$E[y(t)] = \lim_{T\to\infty} \bar{Y}_T = \frac{100}{2} \cos(100\lambda)$$

Hence the process $\{x(t)\}$ is Correlation ergodic.

(or)

b) (i) If the process $\{x(t); t \geq 0\}$ is a poisson process with parameter λ, obtain $P[x(t)=n]$. Is the process first order stationary.

Soln:

$$P_n(t) = P\{x(t) = n\} \quad \text{--- (1)}$$

$$P_n(t+\Delta t) = P\{x(t+\Delta t) = n\}$$

$= P\{(n-1)$ occurances in $(0,t)$ and 1 occurance in $(t, t+\Delta t)\}$

$+ P\{n$ occurances in $(0,t)$ and no occurancies in $(t, t+\Delta t)\}$.

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\{P_{n-1}(t) - P_n(t)\}$$

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t\to 0} \lambda[P_{n-1}(t) - P_n(t)]$$

$$\frac{dP_n(t)}{dt} = \lambda[P_{n-1}(t) - P_n(t)]$$

$$\frac{dP_n(t)}{dt} + \lambda P_n(t) = \lambda P_{n-1}(t) \quad \text{--- (2)}$$

$$\therefore P_n(t)e^{\lambda t} = \int_0^t \lambda P_{n-1}(t) e^{\lambda t}\, dt$$

Taking $n=1$,

$$e^{\lambda t} P_1(t) = \lambda \int_0^1 P_0(t) e^{\lambda t}\, dt \quad \text{--- (3)}$$

(7)

$$P_0(t) = e^{-\lambda t}.$$

Sub. $P_0(t)$ in (3)

$$e^{\lambda t} P_1(t) = \lambda \int_0^t e^{-\lambda t} e^{\lambda t}\, dt = \lambda t$$

$$\boxed{P_1(t) = e^{-\lambda t}(\lambda t)} \quad -(4)$$

Solving recursively, we get $P_2(t), P_3(t) \ldots$

In general $P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}, \; n = 0, 1, 2 \ldots$

Pdf of a P.P is

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \; n = 0, 1, 2 \ldots$$

Mean $E(x(t)) = \sum_{x=0}^{\infty} x\, P_x(t)$

$$= \sum_{x=0}^{\infty} x \cdot \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{0!} + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \ldots\right]$$

Mean of Poisson process.

$$\boxed{E(x(t)) = \lambda t} \neq \text{constant}.$$

$\therefore$ Poisson process is not stationary.

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13) b) (ii) Prove that a random telegraph signal process $Y(t) = \alpha X(t)$ is a WSS process when α is a R.V which is independent of $X(t)$, assume value -1 and 1 with equal probability and $R_{XX}(t_1, t_2) = e^{-2\lambda|t_1 - t_2|}$.

Soln:

$$P\{X(t) = 1\} = P\{N(t) \text{ is even}\}$$

$$= P\{N(t)=0\} + P\{N(t)=2\} + \dots$$

$$= e^{-\lambda t}\frac{(\lambda t)^0}{0!} + e^{-\lambda t}\frac{(\lambda t)^2}{2!} + \dots$$

$$P(N(t)=r) = e^{-\lambda t}\frac{(\lambda t)^r}{r!} = e^{-\lambda t}\cosh \lambda t$$

$$P\{x(t)=-1\} = P\{N(t) \text{ is odd}\}$$

$$= P\{N(t)=1\} + P\{N(t)=3\} + \dots$$

$$= e^{-\lambda t}\frac{(\lambda t)^1}{1!} + e^{-\lambda t}\frac{(\lambda t)^3}{3!} + \dots$$

$$= e^{-\lambda t}\sinh(\lambda t)$$

Mean $E\{x(t)\} = \Sigma\, x(t)\, P\{x(t)\}$

$$= e^{-\lambda t}[\cosh \lambda t - \sinh \lambda t]$$

$$= e^{-\lambda t}\left[\frac{e^{\lambda t}+e^{-\lambda t}}{2} - \frac{e^{\lambda t}-e^{-\lambda t}}{2}\right]$$

$$= e^{-2\lambda t}$$

Auto correlation of $x(t)$

$$P\{x(t_1)=1,\ x(t_2)=1\} = P\{x(t_1)=1 \mid x(t_2)=1\} \times P\{x(t_2)=1\}$$

$$= e^{-\lambda \tau}\cosh \lambda \tau \times e^{-\lambda t_2}\cosh \lambda t_2$$

III^{ly}

$$P\{x(t_1)=-1,\ x(t_2)=-1\} = e^{-\lambda \tau}\cosh \lambda \tau \times e^{-\lambda t_2}\sinh \lambda t_2$$

$$P\{x(t_1)=-1,\ x(t_2)=1\} = e^{-\lambda \tau}\sinh(\lambda \tau)\, e^{-\lambda t_2}\cosh \lambda t_2$$

$$\therefore P\{x(t_1) \times x(t_2) = 1\} = P\{x(t_1)=-1 \text{ and } x(t_2)=-1\}$$

$$= e^{-\lambda \tau}\sinh(\lambda \tau)$$

and $P\{x(t_1) \times x(t_2) = -1\} = e^{-\lambda\tau} \sinh(\lambda\tau)$.

$$R_{xx}(t_1,t_2) = E[x(t_1)\,x(t_2)] = e^{-\lambda\tau}\left[\cosh\lambda\tau - \sinh\lambda\tau\right]$$

$$= e^{-2\lambda\tau}$$

$$= e^{-2\lambda(t_1-t_2)} \neq \text{constant}$$

$R_{xx}(t_1,t_2)$ is a fn of (t_1-t_2).

$\{x(t)$ is not a stationary.

Mean and ACF of a random telegraph process:

By defn, $E(\alpha)=0$, $E(\alpha^2)=1$ ($P(\alpha=1)=1/2$, $P(\alpha=-1)=1/2$).

$$E[y(t)] = E[\alpha \times x(t)]$$

$$= E[\alpha]\;E[x(t)] = 0.$$

$$E[y(t_1) \times y(t_2)] = E[\alpha\, x(t_1) \times \alpha \times x(t_2))$$

$$= e^{-2\lambda(t_1-t_2)}$$

$$= f(\tau)$$

$\therefore$ Hence the random telegraph process is a WSS process

14)
a)
(i) If $\{x(t)\}$ and $\{y(t)\}$ are two R.P with auto correlation fn $R_{xx}(\tau)$ & $R_{yy}(\tau)$ respectively then prove that $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0)R_{yy}(0)}$. Establish any two properties of ACF $R_{xx}(\tau)$

Soln:

Consider $E\{[x(t) + k\,y(t+\tau)]^2\} \geq 0$ for any real k

$$E[x^2(t)] + k^2 E[y^2(t+\tau)] + 2k E[x(t)\, y(t+\tau)] \geq 0 \quad \text{— (1)}$$

WKT $R_{xx}(0) = E[x^2(t)]$, $R_{yy}(0) = E[y^2(t+\tau)]$

Sub in (1) we get

$$R_{xx}(0) + k^2 R_{yy}(0) + 2k R_{xy}(\tau) \geq 0$$

$$k^2 R_{yy}(0) + 2k R_{xy}(\tau) + R_{xx}(0) \geq 0 \quad \text{— (2)}$$

$$(2R_{xy}(\tau))^2 - 4 R_{xx}(0) R_{yy}(0) \leq 0 \qquad [b^2 \leq 4ac]$$

$$R_{xy}(\tau)^2 \leq R_{xx}(0) R_{yy}(0)$$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}.$$

Hence the property is proved.

Property of ACF.

1) The mean square value of the R.P may be obtained from the ACF $R_{xx}(\tau)$ by putting $\tau = 0$.

$$R_{xx}(0) = E[x^2(t)] = \bar{x}^2$$

$$\boxed{\bar{x}^2 = R_{xx}(0)}$$

2) $R_{xx}(\tau)$ is an even fn of τ (ie) $R_{xx}(\tau) = R_{xx}(-\tau)$.

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)] \quad \text{— (1)}$$

Replace $\tau = -\tau$.

$$R_{xx}(-\tau) = E[x(t)\, x(t-\tau)] \qquad \text{put } t - \tau = t_1,\ t = t_1 + \tau.$$

$$= E[x(t_1+\tau)\, x(t_1)]$$

$$= E[x(t_1)\, x(t_1+\tau)]$$

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

(1)

14) a) (ii) Find the power density of the R.P whose ACF is

$$R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & \text{for } |\tau| \le T \\ 0 & \text{for } |\tau| > T. \end{cases}$$

Soln:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau.$$

$$= \int_{-\infty}^{-T} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau + \int_{-T}^{T} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$+ \int_{-T}^{\infty} R_{XX}(\tau)\, e^{i\omega\tau}\, d\tau.$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) e^{-i\omega\tau}\, d\tau.$$

$$= 2\int_{0}^{T} \left(1 - \frac{\tau}{T}\right) \cos\omega\tau\, d\tau.$$

$$= 2\left[\left(1 - \frac{\tau}{T}\right) \frac{\sin\omega\tau}{\omega} + \frac{1}{T}\left(-\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^T$$

$$\boxed{S_{XX}(\omega) = \frac{2}{T\omega^2}(1 - \cos\omega T)}$$

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14) b (i) State and prove Weiner – Khintchine theorem.

Refer Apr/May 2011.

Q.No 14 b (i).

(ii) The cross – power spectrum of a real R.P $\{x(t)\}$ and $\{y(t)\}$ is given by $S_{xy}(\omega) = \begin{cases} a + bj\omega, & \text{for } |\omega| < 1 \\ 0, & \text{o.w} \end{cases}$ Find the cross correlation fn.

Refer Apr/May 2011

Q.No 14 b(ii).

15
a)
(i) Show that if the input $\{x(t)\}$ is a wss process for a linear system the output $\{y(t)\}$ is a wss process. Also find $R_{xy}(\tau)$.

Soln: Input & output are related by

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$$

$$E[y(t)] = \int_{-\infty}^{\infty} h(u) E(x(t-u))\, du$$

$$= E[x(t-u)] \int_{-\infty}^{\infty} h(u)\, du$$ $\left\{ \because \{x(t)\} \text{ is wss } E\{x(t)\} \text{ is constant} \right\}$

$$= E\, \bar{X}_u \int_{-\infty}^{\infty} h(u)\, du$$

$=$ a finite constant.

$E[y(t)] =$ constant.

$$R_{yy}(t, t+\tau) = E[y(t)\, y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)\, x(t-u_1)\, h(u_2)\, x(t+\tau-u_2)\, du_1\, du_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2)\, E[x(t-u_1)\, x(t+\tau-u_2)]\, du_1\, du_2$$

$$= g(\tau) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2)\, du_1\, du_2$$

$=$ a fn of τ.

Hence $\{y(t)\}$ is also a wss process.

W.K.T $R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$$= E\left[X(t) \int_{-\infty}^{\infty} h(u)\, X(t+\tau-u)\,du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)\, X(t+\tau-u)]\, h(u)\,du.$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau-u)\, h(u)\,du$$

$$\boxed{R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)}$$

15) a) (ii) If $X(t)$ is the input voltage to a circuit and $Y(t)$ is the output voltage $\{X(t)\}$ is a stationary process with $\mu_X = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean μ_Y and power spectrum $S_{YY}(\omega)$ of the output if the system transfer fn is given by $H(\omega) = \frac{1}{\omega + 2i}$

Refer Nov/Dec 2008

Q.No 15. b).

15) b (i) If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$. where A is a constant, θ is a R.V with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is a band limited Gaussian white noise with PSD

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega - \omega_0| < \omega_B \\ 0, & \text{o.w} \end{cases}$$

find power spectral density $\{Y(t)\}$. Assume that $\{N(t)\}$ and θ are independent.

Refer Apr/May 2011, Q.No 15 b (ii).

b (ii) A system has an impulse response $h(t) = e^{-\beta t} U(t)$, find the psd of the output $Y(t)$ corresponding to the input $X(t)$

Ref Apr/May 2011. Q.No 15 a (ii)