B.E/B.Tech DEGREE EXAMINATION, NOVEMBER/DECEMBER 2008.

MA1254 – RANDOM PROCESSES.

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Answer ALL questions

Part A – (10 × 2 = 20 marks).

1. A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Out of syllabus.

2. Define moment generating function and write the formula to find mean and variance.

The moment generating function (m.g.f) of a random variable 'x' whose probability function $f(x)$ is given by $M_X(t) = E[e^{tX}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x) \end{cases}$.

$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$.

$\therefore$ rth moment = co.eff of $\frac{t^r}{r!}$

$\mu_1' = M_X'(0) = \left(\frac{d}{dt} M_X(t)\right)_{t=0}$ & $\mu_2' = M_X''(0) = \left(\frac{d^2}{dt^2} M_X(t)\right)_{t=0}$.

3. Define exponential density function and find mean and variance of the same

A continuous random variable x is said to follow an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise}. \end{cases}$

$M_X(t) = \frac{\lambda}{\lambda - t} = (1 - t/\lambda)^{-1} = 1 + t/\lambda + (t/\lambda)^2 + \dots$

$\therefore \mu_r' = \frac{r!}{\lambda^r}, r = 1, 2, \dots$

$\therefore$ Mean $\mu_1' = \frac{1}{\lambda}$ and $\mu_2' = \frac{2}{\lambda^2}$ $\therefore$ Variance $= \frac{2}{\lambda^2} - (1/\lambda)^2 = \frac{1}{\lambda^2}$.

4. If $Y = X^2$, where X is a Normal random variable with zero mean and variance σ^2, find the pdf of the random variable Y.

Given $f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < \infty$.

Given $y = x^2 \Rightarrow x = y^{1/2}$

$$\frac{dx}{dy} = \frac{1}{2} y^{-1/2}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{2\sqrt{y}}$$

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{2\sqrt{y}}, \quad -\infty < x < \infty, \ x = \sqrt{y}.$$

5. Let x and y be any two random variables and a, b be constants. Prove that $Cov(ax, by) = ab\,Cov(x, y)$.

$Cov(X,Y) = E(XY) - E(X)E(Y)$.

$$\therefore Cov(ax, by) = E\big((ax)(by)\big) - E(ax)E(by).$$
$$= ab\,E(XY) - aE(X)bE(Y)$$
$$= ab\big(E(XY) - E(X)E(Y)\big).$$
$$Cov(ax, by) = ab\,Cov(X,Y)$$

6. The joint probability density function of the Random variable (X,Y) is given by $f(x,y) = kxy\,e^{-(x^2+y^2)}$, $x>0$, $y>0$. Find the value of k and prove that X and Y are independent.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1.$$

$$\therefore \int_0^{\infty}\int_0^{\infty} kxy\,e^{-(x^2+y^2)}\,dy\,dx = 1$$

$$k\int_0^{\infty} y e^{-y^2} dy \cdot \int_0^{\infty} x e^{-x^2} dx = 1. \quad \left(\text{Since } \int_0^{\infty} x e^{-x^2} dx = \tfrac{1}{2}\right)$$

$$k \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 1. \Rightarrow \boxed{k = 4}$$

$$f_X(x) = f(x) = \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy = 4xe^{-x^2}\int_0^{\infty} y e^{-y^2} dy = 4xe^{-x^2}\cdot\tfrac{1}{2}$$

Similarly $f_Y(y) = 4ye^{-y^2}\cdot\tfrac{1}{2} = \frac{4}{2} y e^{-y^2}$.

$$f(x,y) = 4xy\,e^{-(x^2+y^2)} = \left(\frac{4}{2}xe^{-x^2}\right)\left(\frac{4}{2}ye^{-y^2}\right) = f_X(x)\cdot f_Y(y)$$

$\therefore$ X and Y are independent.

7) Define strict sense process and wide sense stationary process.

A random process $\{X(t)\}$ is called strongly stationary process or strict sense process (SSS) if all its finite dimensional distributions are invariant under the translation of time parameter.

A random process $\{X(t)\}$ is said to be weakly stationary process if its mean is a constant and the auto correlation depends only on the time difference.

8. Define Markov Process.

A random process $\{X(t)\}$ is called a Markov process if for $(t_0 < t_1 < \ldots < t_n)$

$$P[X(t_{n+1}) \leq x_{n+1} / X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1, X(t_0) = x_0]$$
$$= P[X(t_{n+1}) \leq x_{n+1} / X(t_n) = x_n].$$

9. Define autocorrelation of the process $\{X(t)\}$.

If the process $\{X(t)\}$ is either WSS or SSS then $E[X(t)X(t+\tau)]$ is a function of τ, denoted by $R_{XX}(\tau)$. This function $R_{XX}(\tau)$ is called the Auto Correlation function of the random process $\{X(t)\}$.

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

10. Given that the autocorrelation function for a stationary ergodic process with no periodic components is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process $\{X(t)\}$.

$$\bar{X}^2 = \lim_{|\tau|\to\infty} R_{XX}(\tau) = \lim_{|\tau|\to\infty} \left[25 + \frac{4}{1+6\tau^2}\right] = 25 + 0 = 25$$

$$\therefore \bar{X} = 5 = \text{Mean} = E(X(t))$$

The variance is $V(X(x)) = E(X^2(t)) - \{E(X(t))\}^2 = R_{XX}(0) - \bar{X}^2 = 29 - 25 = \underline{\underline{4}}$

since $E(X^2(t)) = R_{XX}(0) = 25 + \frac{4}{1}$
$= 29.$

PART - B $(5 \times 16 = 80$ marks$)$

11) a) i) A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?

Out of syllabus.

a) ii) A random variable x has the following probability distribution.

x:	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find: (1) The value of k (2) $p(1.5 < x < 4.5 / x > 2)$ and (3) The smallest value of λ for which $p(x \le \lambda) > 1/2$.

(i) If $p(x)$ is the PMF, then $\sum_{x=0}^{7} p(x) = 1$.

$\therefore k+2k+2k+3k+k^2+2k^2+7k^2+k=1 \Rightarrow 10k^2+9k-1=0$

$\therefore (10k-1)(k+1)=0 \Rightarrow \boxed{k=\frac{1}{10}}$ (Since $k=-1$ is not possible)

ii) $$p(1.5<x<4.5/x>2) = \frac{p(1.5<x<4.5 \cap x>2)}{p(x>2)}$$

$$= \frac{p(x=2, x=3, x=4) \cap p(x=3, x=4, x=5, x=6, x=7)}{p(x=3, x=4, x=5, x=6, x=7)}$$

$$= \frac{p(x=3, x=4)}{p(x=3, x=4, x=5, x=6, x=7)}$$

$$= \frac{p(x=3)+p(x=4)}{p(x=3)+p(x=4)+p(x=5)+p(x=6)+p(x=7)}$$

$$= \frac{2/10 + 3/10}{2/10 + 3/10 + \frac{1}{100} + \frac{2}{100} + \left(\frac{7}{100}+\frac{1}{10}\right)} = \frac{5/10}{6/10 + \underbrace{\frac{10}{100}}_{\frac{1}{10}}} = \frac{5}{\cancel{10}} \times \frac{\cancel{10}}{7}$$

$p(1.5<x<4.5/x>2) = 5/7$

iii) $p(x \le 1) = \frac{1}{10} < 1/2$

$p(x \le 2) = \frac{1}{10} + 2/10 = 3/10 < 1/2$

$p(x \le 3) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = 5/10 = 1/2$.

$p(x \le 4) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} = 4/5 > 1/2$

$\therefore p(x \le \underset{4}{\underset{\shortparallel}{\lambda}}) > 1/2$. i.e $\boxed{\lambda=4}$

11) b) i) State and prove Baye's theorem. 3

Out of Syllabus.

b) ii) If the density function of a continuous RV X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(1) Find the value of a (2) Find the CDF of X (3) If x_1, x_2 and x_3 are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5?

1) Since $f(x)$ is a PDF of X, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1 \Rightarrow a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\Rightarrow \frac{4a}{2} = 1 \Rightarrow \boxed{a = 1/2}$$

(2) The CDF of the RV X is $F(x) = P(X \le x)$

$F(x) = 0, \quad x < 0$

For $0 \le x < 1$, $F(x) = \frac{1}{2}\int_0^x x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{4}, \quad 0 \le x < 1$

For $1 \le x < 2$, $F(x) = \frac{1}{2}\int_0^1 x\,dx + \int_1^x \frac{1}{2}\,dx = x/2 - 1/4, \quad 1 \le x < 2$

For $2 \le x < 3$, $F(x) = \frac{1}{2}\int_0^1 x\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x (3/2 - x/2)\,dx = \frac{3}{2}x - \frac{x^2}{4} - 5/4, \quad 2 \le x < 3$

For $x \ge 3$, $F(x) = 1$;

(3) $P(X > 1.5) = 1 - \{P(X \le 1.5)\} = 1 - \{P(0 \le X \le 1) + P(1 \le X \le 1.5)\}$

$$= 1 - \left\{\int_0^1 x/2\,dx + \int_1^{1.5} \frac{1}{2}\,dx\right\}$$

$$= 1 - 1/2$$

$$\boxed{\therefore P(X > 1.5) = 1/2.}$$

12) a) i) Prove that Poisson distribution is the limiting case of Binomial distribu

The binomial distribution is $P(X=x) = nC_x\, p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$ ($q = 1-p$)

$$= \frac{1.2.3.\dots(n-x)\dots n}{1.2.3.\dots(n-x)\,x!}\left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\dots(n-x+1)}{x!}\left(\frac{\lambda/n}{1-\lambda/n}\right)^x (1-\lambda/n)^n \quad \left(\because \begin{matrix} q = 1-p \ \& \\ p = \lambda/n \end{matrix}\right)$$

$$= \frac{n(n-1)(n-2)\dots(n-x+1)}{x!}\frac{\lambda^x}{n^x}\cdot\frac{1}{(1-\lambda/n)^x}(1-\lambda/n)^n.$$

$$= \frac{1(1-1/n)(1-2/n)\dots\left(1-\frac{x-1}{n}\right)}{x!}\lambda^x(1-\lambda/n)^{n-x}$$

When $n\to\infty$, $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2,\dots\infty$ since $\underset{n\to\infty}{Lt}(1-\lambda/n)^{n-x} = e^{-\lambda}$ &

$\underset{n\to\infty}{Lt}(1-1/n) = \underset{n\to\infty}{Lt}\left(1-\frac{x-1}{n}\right) = 1$

Hence the probability mass function of a random variable X which follows Poisson distribution is given by,

$$P(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0,1,2,\dots,\infty \\ 0, & \text{otherwise} \end{cases}$$

ii) Define Gamma distribution and find mean and variance of the same.

The continuous random variable X is said to follow a Gamma distribution with parameter 'λ' if its probability function is given by

$$f(x) = \begin{cases} \frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda>0, 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$$

$M_X(t) = (1-t)^{-\lambda}$. $M_X'(t) = -\lambda(1-t)^{-\lambda-1}(-1) = \lambda(1-t)^{-\lambda-1}$

$\therefore \mu_1' = M_X'(0) = \lambda$. $\therefore \boxed{\text{Mean} = \mu_1' = \lambda}$

$M_X''(t) = \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)$

μ_2' (about origin) $= M_X''(0) = \lambda(\lambda+1)$

Variance $= \mu_2 = \mu_2' - (\mu_1')^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$.

$\boxed{\text{Variance} = \lambda}$

12) b) i) In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%; between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students got distinction. Find the percentage of students who have got first class and second class. (Assuming normal distribution of marks)

10% of students get less than 45%.

From the normal table, z value corresponding to 0.4 [(40% area) = (50 − 10) = 40] = −0.1554.

i.e $z = \frac{45-\mu}{\sigma} = -0.1554$ — (1)

5% of students get distinction marks i.e 75 marks and above

∴ From the normal table, z value corresponding to 0.45 [Area 45% (50 − 5) = 45] = 0.1700

i.e $z = \frac{75-\mu}{\sigma} = 0.1700$ — (2)

From (1) and (2),

$\mu - 45 = 0.1554\sigma$ — (a)

$-\mu + 75 = 0.1700\sigma$ — (b)

Solving (a) and (b), $\sigma = 92.19$

Using σ in (a), $\mu = 59.33$. From μ and σ the area's can be found.

(ii) Define Weibull distribution and find its mean and variance.

Out of syllabus.

13) a) (i) The joint probability mass function of (X,Y) is given by $p(x,y) = k(2x+3y)$, (12 marks)

$x = 0,1,2$; $y = 1,2,3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(X+Y)$.

$p(x,y) = k(2x+3y)$

$p(0,1) = k(0+3) = 3k$ $\quad p(1,1) = 5k$ $\quad p(2,1) = 7k$

$p(0,2) = k(0+6) = 6k$ $\quad p(1,2) = 8k$ $\quad p(2,2) = 10k$

$p(0,3) = k(0+9) = 9k$ $\quad p(1,3) = 11k$ $\quad p(2,3) = 13k$.

To find k :-

The marginal distribution are given in the table:

Y \ X	0	1	2	Marginal distribution of Y $P(Y=y)$
1	$3k$ $\frac{3}{72}$	$5k$ $\frac{5}{72}$	$7k$ $\frac{7}{72}$	$15k$ $\frac{15}{72}$
2	$6k$ $\frac{6}{72}$	$8k$ $\frac{8}{72}$	$10k$ $\frac{10}{72}$	$24k$ $\frac{24}{72}$
3	$9k$ $\frac{9}{72}$	$11k$ $\frac{11}{72}$	$13k$ $\frac{13}{72}$	$33k$ $\frac{33}{72}$
$P(X=x)$ Marginal distribution of X	$18k$ $\frac{18}{72}$	$24k$ $\frac{24}{72}$	$30k$ $\frac{30}{72}$	$72k$ 1

Total Probability $=1$

$72k=1$

$\boxed{k=1/72}$

Probability distribution of $x+y$

$x+y$	Probability
1	$p(0,1)=p(1,0)=3/72$.
2	$p(1,1)+p(0,2)=\frac{5}{72}+\frac{6}{72}=11/72$
3	$p(2,1)+p(1,2)+p(3,0)=\frac{7}{72}+\frac{8}{72}+\frac{9}{72}=\frac{24}{72}$
4	$p(3,1)+p(2,2)=\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5	$P(3,2)=13/72$.

13) b) i) The life time of a certain brand of an electric bulb may be considered a RV with mean 1200h and standard deviation 250h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250 h.

mean $\mu=1200$, $S.D(\sigma)=250$, $\sigma/\sqrt{n}=\frac{250}{\sqrt{60}}$, $\overline{x}$ = mean life time of 60 lights

Lindeberg-Levy form of central limit theorem, we have

$\overline{x} \sim N(\mu, \sigma/\sqrt{n})$ i.e. $\overline{x} \sim N\left(1200, \frac{250}{\sqrt{60}}\right)$

To find $p(\overline{x}>1250) = p\left(\frac{\overline{x}-1200}{250/\sqrt{60}} > \frac{1250-1200}{250/\sqrt{60}}\right) = p(z>1.55)$

$= P(0<z<\infty) - p(0<z<1.55) = 0.5 - (\text{Area from 0 to } 1.55)$

$= 0.5 - 0.4394 = 0.0606.$

13b) (i) The random variable X and Y are statistically independent having a gamma distribution with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively. Determine/Derive the probability density function of a random variable $U = \frac{X}{X+Y}$. 5

Answer:

X has gamma distribution with parameters $(m, 1/2)$.

$$\therefore f_X(x) = \frac{1}{\Gamma(m)} \frac{1}{2^m} x^{m-1} e^{-x/2}, \quad x > 0$$

Y has gamma distribution with parameters $(n, 1/2)$.

$$\therefore f_Y(y) = \frac{1}{\Gamma(n)} \frac{1}{2^n} y^{n-1} e^{-y/2}, \quad y > 0.$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \left(\frac{1}{\Gamma(m)} \frac{1}{2^m} x^{m-1} e^{-x/2}\right)\left(\frac{1}{\Gamma(n)} \frac{1}{2^n} y^{n-1} e^{-y/2}\right)$$

$\therefore$ X and Y are independent.

Given $U = \frac{X}{X+Y}$ Let $V = X+Y$

i.e $u = \frac{x}{x+y}$ & $v = x+y$

$u = \frac{x}{v}$ | $\therefore v = uv + y$

$\Rightarrow \boxed{x = uv}$ | $\boxed{y = v - uv}$

$\therefore \frac{\partial x}{\partial u} = v$ | $\frac{\partial y}{\partial u} = -v$

$\frac{\partial x}{\partial v} = u$ | $\partial y/\partial v = 1-u$.

$$\therefore |J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$|J| = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v.$$

The joint pdf of (u,v) is, $f_{UV}(u,v) = f_{XY}(x,y)|J| = v\left[\frac{1}{\Gamma(m)2^m} x^{m-1} e^{-x/2}\right]\left[\frac{y^{n-1}}{\Gamma(n)2^n} e^{-y/2}\right]$

$$\therefore f_{UV}(u,v) = v \frac{1}{\Gamma(m)\Gamma(n) 2^{m+n}} e^{-\left(\frac{x+y}{2}\right)} \cdot x^{m-1} y^{n-1}$$

$$= \frac{1}{\Gamma(m)\Gamma(n)2^{m+n}} e^{-v/2} u^{m-1} v^{m+n-1} (1-u)^{n-1}, \quad v > 0,\ 0 \le u \le 1.$$

The pdf of $U = \frac{X}{X+Y}$ i.e $f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_0^{\infty} \frac{1}{\Gamma(m)\Gamma(n)2^{m+n}} e^{-v/2} u^{m-1} v^{m+n-1} (1-u)^{n-1}\,dv$

$$= \frac{u^{m-1}(1-u)^{n-1}}{\Gamma(m)\Gamma(n)} \int_0^{\infty} \frac{e^{-v/2} v^{m+n-1}}{2^{m+n}}\,dv$$

put $v/2 = t$

$v = 2t$

$dv = 2dt$

$$\therefore \int_0^\infty \frac{e^{-v/2}\, v^{m+n-1}}{2^{m+n}}\, dv = \frac{1}{2^{m+n}} \int_0^\infty e^{-t} (2t)^{m+n-1}\, 2dt = \frac{2^{m+n-1} \cdot 2}{2^{m+n}} \int_0^\infty e^{-t}\, t^{m+n-1}\, dt$$

$$= \int_0^\infty e^{-t}\, t^{m+n-1}\, dt = \Gamma(m+n)$$

$$\therefore f_U(u) = \frac{u^{m-1}(1-u)^{n-1}}{\Gamma(m)\Gamma(n)} \Gamma(m+n) = u^{m-1}(1-u)^{n-1} \beta(m,n).$$

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14) a) i) Define random process. Classify it with an example.

A random process is a collection of Random Variables $\{X(s,t)\}$ that are functions of t a real variable t where $s \in S$, S is the sample space and $t \in T$, T is an index set.

We can classify the random process according to the characteristics of time t and the random variable $X = X(t)$ at time 't'.

Continuous Random Process

If X is continuous and t can have any of a continum of values then $X(t)$, a Continuous Random Process.

Example: i) The maximum temperature of a particular place in $(0,t)$. The set of possible values of X is continuous in continuous time.

(ii) Thermal agitation noise in conductors.

Discrete Random Process:

If X assumes only discrete values and t is continuous, then $\{X(t)\}$ is called Discrete Random Process.

Example (i) Voltage when the switch is closed and open.

(ii) Let X denote the number of telephone calls received in the interva $(0,t)$

Then $\{X(t)\}$ is a Discrete Random Process.

Continuous Random Sequence

A random process for which X is continuous but time 't' takes only discrete values is called a Continuous Random Sequence.

Example: Let $X(t)$ denotes the outcome of the nth toss of a coin, then $X(t)$ is r $t=1,2,3,\dots$ is a random sequence since the sample space $S = \{H,T\}$ is discrete and hence $X(t)$ is also discrete.

Discrete Random Sequence

A random process in which both the random variable 'X' and time 't' are discrete is called Discrete Random Sequence.

a) (ii) If $X(t) = Y\cos\omega t + Z\sin\omega t$, where Y and Z are two independent normal RV's with $E(Y) = E(Z) = 0$, $E(Y^2) = E(Z^2) = \sigma^2$ and ω is a constant, prove that $\{X(t)\}$ is a SSS process of order 2. 6

To show that it is stationary of second order, we have to show that $E[X(t)]$ and $E(X^2(t))$ are constants.

Given $E(Y) = E(Z) = 0$ & $E(Y^2) = E(Z^2) = \sigma^2$. Since Y & Z are independent,

$$E(YZ) = E(Y) \cdot E(Z) = 0.$$

$$E[X(t)] = E[Y\cos\omega t + Z\sin\omega t] = \cos\omega t\, E(Y) + \sin\omega t\, E(Z)$$
$$= \cos\omega t\,(0) + \sin\omega t\,(0)$$
$$= 0, \text{ a constant.}$$

$$E[X^2(t)] = E[X(t) \cdot X(t)] = E[(A\sin t + B\cos t)(A\sin t + B\cos t)], \quad \left(\text{Let } Z = A,\ Y = B,\ E(AB) = E(YZ) = 0\right)$$
$$= E[A^2\sin^2 t + AB\sin t\cos t + AB\sin t\cos t + B^2\cos^2 t]$$
$$= \sin^2 t\, E(A^2) + \cos^2 t\, E(B^2)$$
$$= \sigma^2(\sin^2 t + \cos^2 t)$$
$$= \sigma^2, \text{ a constant.}$$

$\therefore$ The process is stationary of second order.

14) b) i) State the postulates of a Poisson process and derive its probability law.

Refer Q.P: April/May 2008 13 a (i)

(ii) Prove that the sum of two independent Poisson process is a poisson process.

Let $X(t) = X_1(t) + X_2(t)$.

$$P(X(t) = n) = \sum_{r=0}^{n} P(X_1(t) = r) \cdot P(X_2(t) = n-r)$$
$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$
$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{(\lambda_1 t)^r (\lambda_2 t)^{n-r}}{r!\,(n-r)!}$$
$$= e^{-(\lambda_1+\lambda_2)t} \frac{1}{n!} \sum_{r=0}^{n} nC_r\, (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$
$$= e^{-(\lambda_1+\lambda_2)t} \frac{1}{n!} \left((\lambda_1+\lambda_2)t\right)^n$$

Hence $\{X_1(t) + X_2(t)\}$ is a poisson process with parameter $(\lambda_1+\lambda_2)t$.

15 a) (i) Find the power spectral density of a WSS process with autocorrelation function. $R(\tau) = e^{-\alpha\tau^2}$.

Given the auto correlation function $R_{XX}(\tau) = e^{-\alpha\tau^2}$.

The spectral density function is given by $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$.

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-(\alpha\tau^2 + j\omega\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha(\tau^2 + \frac{j\omega\tau}{\alpha})} d\tau = \int_{-\infty}^{\infty} e^{-\alpha\left((\tau + \frac{j\omega}{2\alpha})^2 + \frac{\omega^2}{4\alpha^2}\right)} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha\omega^2}{4\alpha^2}} \cdot e^{-\alpha(\tau + j\omega/2\alpha)^2} d\tau = e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + \frac{j\omega}{2\alpha})^2} d\tau.$$

Let $u = \left(\tau + \frac{j\omega}{2\alpha}\right)\sqrt{\alpha}$ $\therefore du = \sqrt{\alpha}\, d\tau$

$$\therefore S_{XX}(\omega) = e^{-\omega^2/4\alpha^2} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{\alpha}}$$

$$= \frac{e^{-\frac{\omega^2}{4\alpha^2}}}{\sqrt{\alpha}} 2 \int_{0}^{\infty} e^{-u^2} du \quad (\because e^{-u^2} \text{ is even fn})$$

$$= \frac{e^{-\omega^2/4\alpha^2}}{\sqrt{\alpha}} \cdot 2 \cdot \sqrt{\pi}/2$$

$$S_{XX}(\omega) = \sqrt{\pi/\alpha}\; e^{-\omega^2/4\alpha^2} \quad \left(\because \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

(ii) The power spectral density function of a zero mean WSS process $\{X(t)\}$ is given by $S(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{elsewhere} \end{cases}$ Find $R(\tau)$ and show that $X(t)$ and $X(t + \frac{\tau}{\omega_0})$ are uncorrelated.

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \cdot e^{j\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-\omega_0}^{\omega_0} = \frac{1}{2\pi j\tau}\left[e^{j\omega_0\tau} - e^{-j\omega_0\tau}\right] = \frac{1}{\cancel{2}\pi \cancel{j}\tau} \cancel{2j} \sin\omega_0\tau$$

$$R_{XX}(\tau) = \frac{\sin\omega_0\tau}{\pi\tau}$$

To show that $X(t)$ and $X(t+\frac{\pi}{\omega_0})$ are uncorrelated we have to show that Auto covariance $C[X(t)X(t+\frac{\tau}{\omega_0})]=0$. 7

$$C[X(t)X(t+\tfrac{\tau}{\omega_0})] = R_{XX}(\tau/\omega_0) - E[X(t)]\,E[X(t+\tau/\omega_0)]$$

$$= R_{XX}(\tau/\omega_0) \quad (\text{Since Mean} = E[X(t)] = 0 \text{ is given})$$

$$= \frac{\sin\omega_0(\tau/\omega_0)}{\pi(\tau/\omega_0)}$$

$$= \frac{\sin\tau}{\left(\frac{\pi\tau}{\omega_0}\right)} = 0 \quad \text{if } \tau=\pi.$$

$\therefore$ For $\tau=\pi$, $X(t)$ and $X(t+\frac{\tau}{\omega_0})$ are uncorrelated.

15.) $X(t)$ is the input voltage to a circuit and $Y(t)$ is the output voltage. $\{X(t)\}$ is a stationary random process with $\mu_x=0$ and $R_{XX}(\tau)=e^{-\alpha|\tau|}$. Find μ_y, $S_{yy}(\omega)$ and $R_{yy}(\tau)$, if the power transfer function is $H(\omega)=\frac{R}{R+iLW}$

$$H(\omega) = \frac{R}{R+j\omega L}$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}\cos\omega\tau\, d\tau \quad \left(\begin{array}{c}\text{Since}\\ \sin\omega\tau \text{ is}\\ \text{odd}\end{array}\right)$$

$$= 2\int_0^{\infty} e^{-\alpha\tau}\cos\omega\tau\, d\tau = \frac{2\alpha}{\alpha^2+\omega^2}.$$

$E(Y)=0$. Since $E[X(t)]=0$.

WKT, $S_{YY}(\omega) = S_{XX}(\omega)\,|H(\omega)|^2$

$$= \frac{2\alpha}{\alpha^2+\omega^2}\cdot\left|\frac{R}{R+j\omega L}\right| = \frac{2\alpha}{\alpha^2+\omega^2}\cdot\frac{R^2}{R^2+\omega^2L^2} \quad — ①$$

$$\text{Hence } R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{j\omega\tau}d\omega \quad — ②$$

Consider $\frac{1}{(\alpha^2+\omega^2)(R^2+\omega^2L^2)} = \frac{A}{\alpha^2+\omega^2} + \frac{B}{R^2+\omega^2L^2}$

$$1 = A(R^2+\omega^2L^2) + B(\alpha^2+\omega^2)$$

Put $\omega^2 = -\alpha^2$, $A = \dfrac{1}{R^2 - \alpha^2 L^2}$

Put $\omega^2 = \dfrac{-R^2}{L^2}$, $B = \dfrac{L^2}{\alpha^2 L^2 - R^2}$.

$$\therefore \frac{1}{(\alpha^2+\omega^2)(R^2+\omega^2 L^2)} = \frac{\left(\frac{1}{R^2-\alpha^2 L^2}\right)}{\alpha^2+\omega^2} + \frac{\left(\frac{L^2}{\alpha^2 L^2 - R^2}\right)}{R^2+\omega^2 L^2}$$

$$= \frac{2\alpha R^2}{R^2-\alpha^2 L^2} \cdot \frac{1}{\alpha^2+\omega^2} + \frac{2\alpha R^2 L^2}{\alpha^2 L^2 - R^2} \cdot \frac{1}{R^2+\omega^2 L^2}$$

$$= \underbrace{\frac{2\alpha (R/L)^2}{(R/L)^2-\alpha^2}}_{\lambda} \cdot \frac{1}{\alpha^2+\omega^2} + \underbrace{\frac{2\alpha\, R^2/L^2}{\alpha^2-(R/L)^2}}_{\mu} \cdot \frac{1}{(R/L)^2+\omega^2}$$

$$\therefore R_{yy}(\tau) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega\tau}}{\alpha^2+\omega^2}\,d\omega + \frac{\mu}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega\tau}}{(R/L)^2+\omega^2}\,d\omega$$

$$\boxed{R_{yy}(\tau) = \frac{\lambda}{2\alpha} e^{-\alpha|\tau|} + \frac{\mu}{2}\,(L/R)\, e^{-R/L|\tau|}} \qquad \left(\because \int_{-\infty}^{\infty} \frac{e^{j\omega\tau}}{\omega^2+a^2}\,d\omega = \pi/a\; e^{-a|\tau|}\right)$$

where $\lambda = \dfrac{2\alpha (R/L)^2}{(R/L)^2-\alpha^2}$; $\mu = \dfrac{2\alpha (R/L)^2}{\alpha^2-(R/L)^2}$ //.