AU- chennai - Nov / Dec 2007

MA 1254 - Random Processes

(Regulation 2004)

Part A

1) In a coin tossing exp, if the coin shows head, 1 dice is thrown and the result is recorded. But if the coin shows tail, 2 dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2?

Out of syllabus.

2) Find the MGF of binomial distribution.

Soln:

$$M_X(t) = E[e^{tx}] = \sum e^{tx} p(x) = \sum_{x=0}^{n} e^{tx}\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= (q+pe^t)^n.$$

3) Find the mean variance of Geometric distribution.

Soln:

Mean $E(X) = 1/p$

Variance $= q/p^2$.

4) If X is a ~~Gaussian~~ R. Gaussian R.V with mean zero and variance σ^2, find the pdf of $Y = |X|$.

Soln:

$$F_Y(y) = P(Y \le y) = P\{|X| \le y\} = P(-y < X < y)$$

$$= F_X(y) - F_X(-y)$$

$$\therefore f_Y(y) = f_X(y) - f_X(-y),\ y > 0$$

$X \sim N(0, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad \therefore f_Y(y) = \frac{2}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2},\ y > 0.$$

5) The joint pdf of the R.V (X, Y) is given by

$f(x,y) = kxy\,e^{-(x^2+y^2)}$, $x>0,\ y>0$. Find the value of k.

Soln:

$$\int_0^\infty \int_0^\infty kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1.$$

$$k\int_0^\infty x e^{-x^2}\,dx \int_0^\infty y e^{-y^2}\,dy = 1$$

$$k\,(1/2)\,(1/2) = 1$$

$$\boxed{k = 4}$$

6) If the probability density fn of X is $f_X(x) = 2x,\ 0<x<1$, find the pdf of $Y = 3X+1$.

Ans:

$$\frac{dx}{dy} = \frac{1}{2}$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{2}{9}(y-1),\quad 1<y<4.$$

7) Prove that sum of two independent poisson process is also poisson

Ans:

$$X(t) = X_1(t) + X_2(t)$$

$$P(X(t)=n) = \sum_{r=0}^{n} P(X_1(t)=r)\,P(X_1(t)=n-r)$$

$$= e^{-(\lambda_1+\lambda_2)t}\,(\lambda_1+\lambda_2)^n.$$

$= X(t)$ is a poisson R.V

8) Define wide-sense stationary process.

A R.P $\{X(t)\}$ with finity first and second order moments is called a weakly stationary process (or) wss process if its mean is a constant and the auto correlation depends only on the time differente.

9 Find the PSD of wss Process with ACF $R(\tau) = e^{-a\tau^2}$.

Soln:

$$S(\omega) = \int_{-\infty}^{\infty} e^{-a\tau^2}\,e^{-i\omega\tau}\,d\tau$$

$$= e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + i\omega/2\alpha)^2} d\tau$$

$$= \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

10) If the PSD of a WSS process is given by $S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|), & |\omega| < a \\ 0, & |\omega| > a \end{cases}$ Find the ACF of the process.

Ans: $R(\tau) = F^{-1}[S(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega = \frac{2b}{2\pi}\left(\frac{\sin \alpha\tau/2}{\alpha\tau/2}\right)^2$

Part B

11) a) (i) A R.V X has the following Probability distribution

x :	-2	-1	0	1	2	3
$p(x)$ =	0.1	k	0.2	$2k$	0.3	$3k$

(i) Find k

(ii) Evaluate $p(-2 < x < 3)$

(iii) Find the CDF of X.

Soln: Refer AU - chennai May/June 2009 Q.No 11 b (i).

(ii) out of syllabus.

11) b) (i) If the density fn of a continuous R.V X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{o.w} \end{cases}$$

Refer Nov/Dec 2008 Q.No 11 b (i).

Find 1) the value of a

2) CDF of X

3) If X_1, X_2 and X_3 are 3 independent observations of X. what is probability that exactly one of these 3 is greater than 1.5?

11) b) (ii) Find the MGF of poisson distribution and hence find mean and variance of the same.

Ans: Refer Apr/May 2011

Q-No: 11 b (i).

12) a) (i) Find first four central moments of normal distribution.

Soln:

$$E[X-\mu]^n = \int_{-\infty}^{\infty} (x-\mu)^n f(x)\,dx = \int_{-\infty}^{\infty} (x-\mu)^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$$

$\cancel{\mu_1' = E(x) = 0}$ (or)

$\cancel{\mu_x} =$

$$E[e^{t(x-\mu)}] = e^{-\mu t} E[e^{tx}] = e^{-\mu t} M_x(t)$$

$$\therefore E[e^{t(x-\mu)}] = e^{\frac{t^2\sigma^2}{2}}$$

$$= 1 + \frac{t^2\sigma^2}{2} + \frac{1}{2!}\frac{t^4\sigma^2}{4} + \frac{t^6\sigma^6}{8(3!)} + \dots$$

Since there is no term with odd powers

$$\mu_1 = \mu_3 = \mu_5 = \dots = 0.$$

and $\mu_{2n} =$ coeff of $\frac{t^{2n}}{(2n)!} = \frac{\sigma^{2n} \times (2n)!}{(2^n n)!}$

$$= \frac{\sigma^{2n}}{2^n n!} 1.3.5 \dots (2n-1) \dots 2^n n!$$

$$= 1.3.5 \dots (2n-1)\sigma^{2n}.$$

The first four moments are given by

$\mu_1 = 0$

$\mu_2 = \sigma^2$

$\mu_3 = 0$

$\mu_4 = 3\sigma^4$.

12) a)

(ii) VLSI chips, essential to the running of a computer system, fail in accordance with a poisson distribution with the rate of one chip is about 5 weeks. If there are two spare chips on hand, and if a new supply will arrive in 8 weeks. what is the probability that during the next 8 weeks the system will be down for a week or more, owing to a lack of chips.

Soln: $\lambda = 1$ chips per 5 weeks

$= \frac{1}{5}$ chips per week.

$= 0.2$ chips per week.

$= 1.4$ for 7 weeks

P(system down for atleast one week before new supply in 8 weeks $= P($ 3 or more failure within 7 weeks$)$

$= 1 - P($0, 1, 2 failure in 7 weeks$)$

$= 1 - [p(0) + p(1) + p(2)]$

$= 1 - e^{-1.4}\left[1 + \frac{1.4}{1!} + \frac{(1.4)^2}{2!} + \cdots\right]$

$= 0.1665$.

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12 b)

(i) weibull distribution out of syllabus.

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12) b)

(ii) Find mean and variance of Gamma distribution and hence find mean and variance of exponential distribution.

Soln:

W.K.T $M_X(t) = (1-t)^{-\lambda}$.

$M_X'(t) = -\lambda(1-t)^{-\lambda-1}(-1)$

$$= \lambda (1-t)^{-\lambda-1}$$

μ_1' (about origin) $= M_x'(0) = \lambda$

$$M_x''(t) = \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)$$

μ_2' (about origin) $= M_x''(0) = \lambda(\lambda+1)$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2$$

$$= \lambda(\lambda+1) - \lambda^2$$

$\boxed{\text{Variance} = \lambda}$, $\boxed{\text{Mean} = \lambda}$

Exponential distribution:

W.K.T $M_x(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r$ (or) $\frac{\lambda}{\lambda-t}$

$$M_x(t) = \frac{\lambda}{\lambda-t}$$

$$M_x'(0) = \frac{-\lambda}{(\lambda-t)^2}(-1)\Big|_{t=0}$$

$$\boxed{\mu_1' = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}}$$

$$\mu_2' = M_x''(0) = \frac{\lambda(-2)}{(\lambda-t)^3}(-1)\Big|_{t=0}$$

$$\boxed{\mu_2' = \frac{2}{\lambda^2}}$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

13) a)

(i) The R.v X and Y are statistically independent having a gamma distribution with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively. Derive the probability density fn of a R.v $V = \frac{X}{X+Y}$.

Soln:

Refer AU - chenna Nov/Dec 2008

Q.No 13 b(ii).

13) a)

(ii) State and prove central limit theorem.

Liapounoff's Form:

If $x_i (i = 1, 2, \ldots n)$ be independent R.v's such that $E(x_i) = \mu_i$ and $Var(x_i) = \sigma_i^2$ then under certain general conditions, the R.v $S_n = x_1 + x_2 + \ldots + x_n$ is asympototically normal with mean μ and SD σ where

$$\mu = \sum_{i=1}^{n} \mu_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

Proof: $M_{x_i}(t) = E[e^{tx_i}] = e^{t1} p + e^{t.0} q = q + pe^t$

$$M_{S_n}(t) = M_{x_1 + x_2 + \ldots x_n}(t)$$

$$= M_{x_1}(t) \, M_{x_2}(t) \ldots M_{x_n}(t)$$

$$= [M_{x_i}(t)]^n$$

$$= (q + pe^t) \ldots n \text{ times}$$

$$= (q + pe^t)^n.$$

which is MGF of a binomial variate with parameters n & p

Hence by uniqueness theorem of mgf

$S_n \sim B(n, p)$, $B(n, p)$ is the Binomial distribution

$E(S_n) = np = \mu$

$Var(S_n) = npq = \sigma^2$

Let $z = \dfrac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = \dfrac{S_n - \mu}{\sigma}$

Then $M_z(t) = e^{-\mu t/\sigma} M_{S_n}(t/\sigma) = e^{\frac{-npt}{\sqrt{npq}}} \left[q + pe^{\frac{t}{\sqrt{npq}}}\right]^n$

$= \left[1 + \dfrac{t^2}{2n} + o(n^{-3/2})\right]^n$

where $o(n^{-3/2})$ represent terms involving $n^{-3/2}$ and higher powers of n in the denominator.

$\lim_{n\to\infty} M_z(t) = \lim_{n\to\infty} \left[1 + \dfrac{t^2}{2n} + o(n^{-3/2})\right]^n = e^{t^2/2}$

which is the MGF of the standard normal variable

Hence $S_n = x_1 + x_2 + \dots + x_n$ is asymptotically equivalent to $N(\mu, \sigma^2)$ as $n\to\infty$.

13) b)

(i) Find the Correlation coefficient for the following data.

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Ans:

Take origin of x is 22 and $h = 4$

Take origin of y is 24 and $h = 6$

then $\bar{u} = -0.5$

$\bar{v} = -0.5$

$Cov(u,v) = 7/4$, $\sigma_x = 1.708$, $\sigma_y = 1.708$

$\rho_{xy} = \dfrac{cov(x,y)}{\sigma_x \sigma_y} = \dfrac{7/4}{1.708 \times 1.708}$

$\boxed{\rho_{xy} = 0.6}$

13)
b)(i) The joint PMF of (X, Y) is given by $P(x, y) = k(2x+3y)$ $x = 0, 1, 2$, $y = 1, 2, 3$. Find all the marginal and conditional probability distributions.

Soln: Refer AU Chennai Nov/Dec 2008

Q. No: 13) a).

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14)
a)(i) Define random process. classify it with an example.

Soln:

Refer AU chenna May/June 2009

Q.No 14 a) (i).

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14)
a)
(ii) The process $\{X(t)\}$ whose probability distribution over certain conditions is given by

$$P\{X(t) = n\} = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\ \frac{at}{1+at}, & n = 0. \end{cases}$$

Show that it is not stationary.

Soln:

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Q.No 13 a(ii).

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14) b)
(i) If $X(t) = Y\cos\omega t + Z\sin\omega t$, where Y and Z are two independent norm random variables with $E(Y) = E(Z) = 0$, $E(Y^2) = E(Z^2) = \sigma^2$ and ω is a constant. prove that $\{X(t)\}$ is a SSS process of order 2.

Soln:

Refer AU Chennai Nov/Dec 2008.

14) b)

(ii) Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

Soln.

AU - chennai May | June 2009.

Q.No 14. b(i)

15) a)

(i) The auto correlation fn of the random telegraph signal process is given by $R(\tau) = a^2 e^{-\gamma|\tau|}$. Determine the power density spectrum of the random telegraph signal.

Soln:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{i\omega\tau} d\tau.$$

$$= a^2 \int_{-\infty}^{\infty} e^{-\gamma|\tau|} (\cos\omega\tau - i\sin\omega\tau)\, d\tau.$$

$$= 2a^2 \int_0^{\infty} e^{-\gamma\tau} \cos\omega\tau \, d\tau$$

$$= 2a^2 \left[\frac{e^{-\gamma\tau}}{\gamma^2+\omega^2} (-\gamma\cos\omega\tau + \omega\sin\omega\tau) \right]_0^{\infty}.$$

$$= \frac{2a^2\gamma}{\gamma^2+\omega^2}.$$

15) b)

(i) State and prove Wiener- Khintchine theorem.

Soln: Refer Apr | May 2011

Q.NO. 14 b(i)

15) b

(ii) A R.P $\{x(t)\}$ is given by $x(t) = A\cos pt + B\sin pt$, where A and B are independent RVs such that $E(A)=E(B)=0$ and $E(A^2)=E(B^2)=\sigma^2$. Find the PSD of the process.

Soln:

$$R(\tau) = E[x(t)\,x(t+\tau)]$$

$$= E[(A\cos pt + B\sin pt)(A\cos p(t+\tau) + B\sin p(t+\tau))]$$

$$= \cos pt\cos p(t+\tau)E(A^2) + \cos pt\sin p(t+\tau)E(AB)$$
$$+ \sin pt\cos p(t+\tau)E(AB) + \sin pt\sin p(t+\tau)E(B^2)$$

A & B are independent $E(AB) = E(A)\,E(B) = 0$.

$$R(\tau) = [\cos pt\cos p(t+\tau) + \sin pt\sin p(t+\tau)]\sigma^2$$

$$= \cos[p(t+\tau) - pt]\,\sigma^2$$
$$= \sigma^2\cos p\tau.$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\,e^{-i\omega\tau}\,d\tau = \int_{-\infty}^{\infty}(\sigma^2\cos p\tau)\,e^{-i\omega\tau}\,d\tau$$

$$= \sigma^2\int_{-\infty}^{\infty}\cos p\tau\, e^{-i\omega\tau}\,d\tau = \sigma^2 F(\cos p\tau)$$

$$\boxed{S(\omega) = \sigma^2\pi[\delta(\omega+p) + \delta(\omega-p)]}$$