B.E./B.Tech. DEGREE EXAMINATION, November/December 2010
Fifth Semester
Computer Science and Engineering
MA2265 – DISCRETE MATHEMATICS
(Regulation 2008)

Part - A

1. When do you say that two compound propositions are equivalent?

Answer:

Let A and B are the two compound propositions. $A \Leftrightarrow B$ if $A \leftrightarrow B$ is a tautology.

2. Prove that $\boldsymbol{p, p \to q, q \to r \Rightarrow r}$.

Solution:

1.	p	Rule P
2.	$p \to q$	Rule P
3.	$q \to r$	Rule P
4.	$p \to r$	Rule T,2,3, chain rule
5.	r	Rule T,1,4, Modus phones

3. State pigeonhole principle.

Solution:

If k pigeons are assigned to n pigeonholes and $n < k$ then there is at least one pigeonhole containing more than one pigeons.

4. Find the recurrence relation satisfying the equation $\boldsymbol{y_n = A(3)^n + B(-4)^n}$.

Solution: $y_n = A(3)^n + B(-4)^n$.

$$y_{n+1} = A(3)^{n+1} + B(-4)^{n+1} = 3A3^n - 4B(-4)^n$$
$$y_{n+2} = A(3)^{n+2} + B(-4)^{n+2} = 9A3^n + 16B(-4)^n$$
$$y_{n+2} + y_{n+1} - 12y_n = 0$$

5. Define strongly connected graph.

Answer:

A digraph is said to be strongly connected graph, if there is a path between every pair of vertices in the digraph.

6. State the necessary and sufficient conditions for the existence of an Eulerian path in a connected graph.

Answer:

A connected graph has an Euler path but not Euler circuit if and only if it has exactly two vertices of odd degree.

7. State any two properties of a group.

Answer:

Identity element of a group is unique.

Inverse element of a group is unique.

8. Define a commutative ring.

Answer:

A ring $(R,+,\times)$ is said to be a commutative ring if it satisfies the following condition

$$\forall a,b \in R, a\times b = b\times a$$

9. Define Boolean algebra.
Answer:
A complemented distributive lattice is called Boolean algebra.

10. Define sub-lattice.
Answer:
A lattice $(S,\leq)$ is called a sub-lattice of a lattice $(L,\leq)$ if $S\subseteq L$ and S is a lattice.

Part - B

11. a)i) Prove that the premises $\boldsymbol{a \to (b\to c), d\to (b\wedge\sim c)}$ and $\boldsymbol{(a\wedge d)}$ are inconsistent.
Solution:

1.	$a \to (b\to c)$	Rule P
2.	$d\to (b\wedge\sim c)$	Rule P
3.	$(a\wedge d)$	Rule P
4.	a	Rule T,3, $p\wedge q\Rightarrow p$
5.	d	Rule T,3, $p\wedge q\Rightarrow q$
6.	$(b\to c)$	Rule T,1,4, Modus phones
7.	$(b\wedge\sim c)$	Rule T,2,5, Modus phones
8.	$\sim(\sim b\vee c)$	Rule T,7, Demorgan's law
9.	$\sim(b\to c)$	Rule T,8, $\sim a\vee b\Rightarrow a\to b$
10.	$(b\to c)\wedge\sim(b\to c)$	Rule T,9, $a,b\Rightarrow a\wedge b$
11.	F	Rule T,10, $a\wedge\sim a\Rightarrow F$

∴ The premises $a \to (b\to c), d\to (b\wedge\sim c)$ and $(a\wedge d)$ are inconsistent.

ii) Obtain the principal disjunctive normal form and principal conjunction form of the statement

$$\boldsymbol{P\vee\left(\sim P\to\left(Q\vee(\sim Q\to R)\right)\right)}$$

Solution:
Let $S\Leftrightarrow P\vee\left(\sim P\to\left(Q\vee(\sim Q\to R)\right)\right)$
$A:\sim P\to\left(Q\vee(\sim Q\to R)\right)$

P	Q	R	$\sim P$	$\sim Q$	$\sim Q\to R$	$Q\vee(\sim Q\to R)$	A	S	$Minterm$	$Maxterm$
T	T	T	F	F	T	T	T	T	$P\wedge Q\wedge R$	
T	F	T	F	T	T	T	T	T	$P\wedge\sim Q\wedge R$	
F	T	T	T	F	T	T	T	T	$\sim P\wedge Q\wedge R$	
F	F	T	T	T	T	T	T	T	$\sim P\wedge\sim Q\wedge R$	
T	T	F	F	F	T	T	T	T	$P\wedge Q\wedge\sim R$	
T	F	F	F	T	F	F	T	T	$P\wedge\sim Q\wedge\sim R$	
F	T	F	T	F	T	T	T	T	$\sim P\wedge Q\wedge\sim R$	
F	F	F	T	T	F	F	F	F		P ∨ Q ∨ R

$S\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge\sim Q\wedge R)\vee(\sim P\wedge Q\wedge R)\vee(\sim P\wedge\sim Q\wedge R)\vee(P\wedge Q\wedge\sim R)$
$\vee(P\wedge\sim Q\wedge\sim R)\vee(\sim P\wedge Q\wedge\sim R)$ is a PDNF
$S\Leftrightarrow$ P ∨ Q ∨ R is a PCNF

b) i) Prove that $\forall x(P(x) \to Q(x)), \forall x(R(x) \to \sim Q(x)) \Rightarrow \forall x(R(x) \to \sim P(x))$

Solution:

1.	$\forall x(P(x) \to Q(x))$	Rule P
2.	$\forall x(R(x) \to \sim Q(x))$	Rule P
3.	$P(a) \to Q(a)$	Rule T,1,US
4.	$R(a) \to \sim Q(a)$	Rule T,2,US
5.	$\sim Q(a) \to \sim p(a)$	Rule T,3, $p \to q \Rightarrow \sim q \to \sim p$
6.	$R(a) \to \sim P(a)$	Rule T,4,5, chain rule
7.	$\forall x(R(x) \to \sim P(x))$	Rule T,6,UG

ii) Without using the truth table, prove that $\sim P \to (Q \to R) \equiv Q \to (P \vee R)$.

Proof:

$\sim P \to (Q \to R)$

$\Leftrightarrow \sim\sim P \vee (Q \to R)$ $[Implication\ law]$

$\Leftrightarrow P \vee (Q \to R)$ $[negation\ law]$

$\Leftrightarrow P \vee (\sim Q \vee R)$ $[Implication\ law]$

$\Leftrightarrow (P \vee \sim Q) \vee R$ $[Associate\ law]$

$\Leftrightarrow (\sim Q \vee P) \vee R$ $[Commutative\ law]$

$\Leftrightarrow \sim Q \vee (P \vee R)$ $[Associate\ law]$

$\Leftrightarrow Q \to (P \vee R)$ $[Implication\ law]$

12.a) i) Prove, by mathematical induction, that for all $\boldsymbol{n \geq 1, n^3 + 2n}$ **is a multiple of 3.**

Solution:

$Let\ P(n): n \geq 1, n^3 + 2n$ is a multiple of 3. ... (1)

$P(1): 1^3 + 2(1) = 1 + 2 = 3$ is a multiple of 3.

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): (n+1)^3 + 2(n+1) \text{ is a multiple of 3}$$

$$(n+1)^3 + 2(n+1) = n^3 + 3n + 3n^2 + 1 + 2n + 2$$
$$= n^3 + 2n + 3n + 3n^2 + 3$$
$$= n^3 + 2n + 3(n^2 + n + 1)$$

From (1) $n^3 + 2n$ is a multiple of 3

$$\therefore (n+1)^3 + 2(n+1)\ is\ a\ multiple\ of\ 3$$

$\therefore P(n+1)$ is true.

$\therefore$ By induction method,

$P(n): n \geq 1, n^3 + 2n$ is a multiple of 3, is true for all positive integer n.

ii) Using the generating function, solve the difference equation

$$\mathbf{y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2}$$

Solution:

$Let\ G(x) = \sum_{n=0}^{\infty} y_n x^n$ ... (1) where $G(x)$ is the generating function for the sequence $\{y_n\}$.

Given $y_{n+2} - y_{n+1} - 6y_n = 0$

Multiplying by x_n and summing from 0 to ∞, we have

$$\sum_{n=0}^{\infty} y_{n+2}x^n - \sum_{n=0}^{\infty} y_{n+1}x^n - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} y_{n+2}x^{n+2} - \frac{1}{x}\sum_{n=0}^{\infty} y_{n+1}x^{n+1} - 6\sum_{n=0}^{\infty} y_n x^n = 0$$

$$\frac{1}{x^2}(G(x) - y_1x - y_0) - \frac{1}{x}(G(x) - y_0) - 6G(x) = 0 \qquad [from\ (1)]$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{y_1}{x} - \frac{y_0}{x^2} + \frac{y_0}{x} = 0$$

$$G(x)\left(\frac{1}{x^2} - \frac{1}{x} - 6\right) - \frac{1}{x} - \frac{2}{x^2} + \frac{2}{x} = 0 \Rightarrow G(x)\left(\frac{6x^2 - x + 1}{x^2}\right) = \frac{2}{x^2} - \frac{1}{x}$$

$$G(x)\left(\frac{1 - x - 6x^2}{x^2}\right) = \frac{2 - x}{x^2}$$

$$G(x) = \frac{2 - x}{1 - x - 6x^2} = \frac{2 - x}{(1 - 3x)(1 + 2x)}$$

$$\frac{2 - x}{(1 - 3x)(1 + 2x)} = \frac{A}{(1 - 3x)} + \frac{B}{(1 + 2x)}$$

$$2 - x = A(2x + 1) + B(1 - 3x) \ldots (2)$$

Put $x = -\frac{1}{2}$ in (2)

$$2 - \left(-\frac{1}{2}\right) = B\left(1 + \frac{3}{2}\right) \Rightarrow \frac{5}{2}B = \frac{5}{2} \Rightarrow B = 1$$

Put $x = \frac{1}{3}$ in (2)

$$2 - \left(\frac{1}{3}\right) = A\left(\frac{2}{3} + 1\right) \Rightarrow \frac{5}{3}A = \frac{5}{3} \Rightarrow A = 1$$

$$G(x) = \frac{1}{(1 - 3x)} + \frac{1}{(1 + 2x)} = \frac{1}{(1 - 3x)} + \frac{1}{\big(1 - (-2x)\big)}$$

$$\sum_{n=0}^{\infty} y_n x^n = \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (-2)^n x^n \qquad \left[\because \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n\right]$$

y_n = Coefficient of x^n in $G(x)$

$y_n = 3^n + (-2)^n$

b) i) How many positive integers n can be formed using the digits $3, 4, 4, 5, 5, 6, 7$ if n has to exceed 5000000?

Solution:

The positive integer n exceeds 5000000 if the first digit is either 5 or 6 or 7.

If the first digit is 5 then the remaining six digits are 3,4,4,5,6,7.

Then the number of positive integers formed by six digits is

$$\frac{6!}{2!} = 360 \qquad [Since\ 4\ appears\ twice]$$

If the first digit is 6 then the remaining six digits are 3,4,4,5,5,7.

Then the number of positive integers formed by six digits is

$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$

If the first digit is 7 then the remaining six digits are 3,4,4,5,6,5.
Then the number of positive integers formed by six digits is

$$\frac{6!}{2!\,2!} = 180 \qquad [Since\ 4\ \&\ 5\ appears\ twice]$$

∴ The number of positive integers n can be formed using the digits 3,4,4,5,5,6,7 if n has to exceed 5000000 is $360 + 180 + 180 = 720$.

ii) Find the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7.

Solution:

Let A, B, C and D represents the integer from 1 to 250 that are divisible by 2,3,5 and 7 respectively.

$$|A| = \left\lfloor\frac{250}{2}\right\rfloor = 125, |B| = \left\lfloor\frac{250}{3}\right\rfloor = 83, |C| = \left\lfloor\frac{250}{5}\right\rfloor = 50, |D| = \left\lfloor\frac{250}{7}\right\rfloor = 35$$

$$|A \cap B| = \left\lfloor\frac{250}{2 \times 3}\right\rfloor = 41, |A \cap C| = \left\lfloor\frac{250}{2 \times 5}\right\rfloor = 25, |A \cap D| = \left\lfloor\frac{250}{2 \times 7}\right\rfloor = 17, |B \cap C| = \left\lfloor\frac{250}{3 \times 5}\right\rfloor = 16$$

$$|B \cap D| = \left\lfloor\frac{250}{3 \times 7}\right\rfloor = 11, |C \cap D| = \left\lfloor\frac{250}{5 \times 7}\right\rfloor = 7, |A \cap B \cap C| = \left\lfloor\frac{250}{2 \times 3 \times 5}\right\rfloor = 8$$

$$|A \cap B \cap D| = \left\lfloor\frac{250}{2 \times 3 \times 7}\right\rfloor = 5, |A \cap C \cap D| = \left\lfloor\frac{250}{2 \times 5 \times 7}\right\rfloor = 3, |B \cap C \cap D| = \left\lfloor\frac{250}{3 \times 5 \times 7}\right\rfloor = 2$$

$$|A \cap B \cap C \cap D| = \left\lfloor\frac{250}{2 \times 3 \times 5 \times 7}\right\rfloor = 1$$

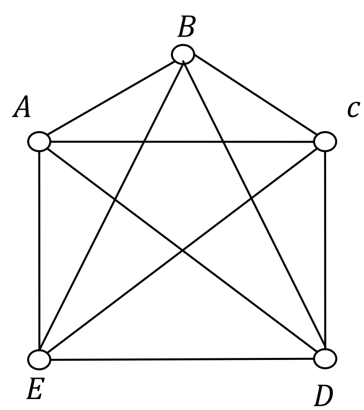
∴The number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3,5,7 is

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D|$$
$$-|C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$
$$|A \cup B \cup C \cup D| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$
$$|A \cup B \cup C \cup D| = 193$$

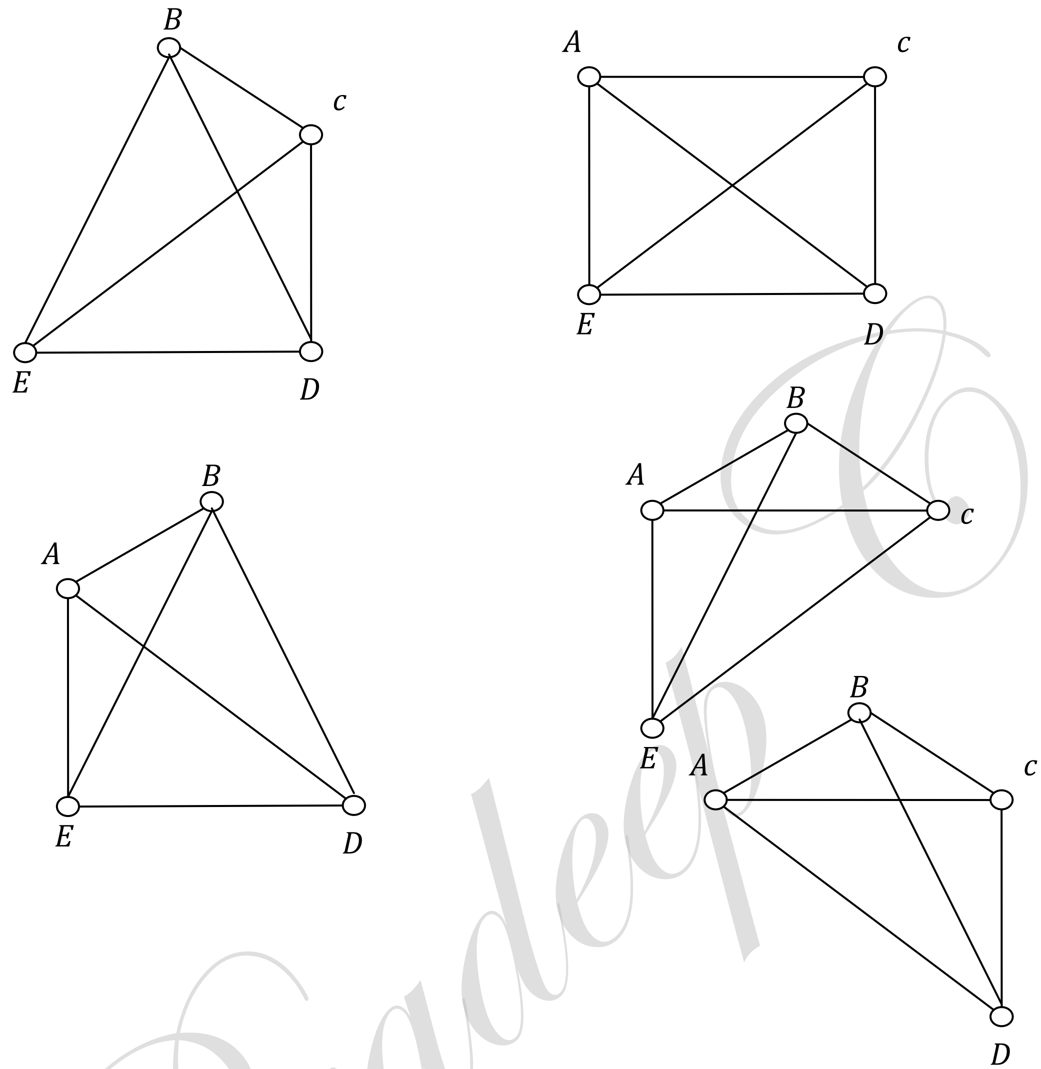
13. a) i) Draw the complete graph K_5 with vertices A, B, C, D and E. Draw all complete sub graph of K_5 with 4 vertices.

Solution:

A complete graph with five vertices K_5 is shown below



Complete sub graph of K_5 with 4 vertices are



ii) If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k .

Solution:

Let $G(V, E)$ be a graph with n vertices and e edges.

Let $v_1, v_2, \dots, v_n$ be the n vertices.

Given that all the vertices of G are each of degree k.

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = \cdots = \deg(v_k) = k$$

By handshaking theorem,

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \cdots + \deg(v_n) = 2e$$

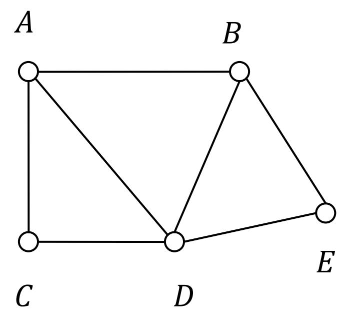
$$k + k + k + \cdots ntimes = 2e$$

$$nk = 2e$$

$$e = k\left(\frac{n}{2}\right)$$

∴ The number of edges of the graph G is a multiple of k .

b) i) Draw the graph with 5 vertices, A, B, C, D, E such that $deg(A) = 3$, B is an odd vertex, $deg(C) = 2$ and $D$and E are adjacent.



ii) The adjacency matrices of two pairs of graph as given below. Examine the isomorphism of G and H by finding a permutation matrix. $\mathbf{A_G} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}, \mathbf{A_H} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$

Solution:
We know that two simple graphs G_1 and G_2 are isomorphic iff their adjacency matrices A_1 and A_2 are related by

$$PA_1P^T = A_2$$

[A matrix whose rows are the rows of the unit matrix, but not necessarily in their natural order, is called Permutation matrix.]

$$\mathrm{A_G} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathrm{A_H} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$PA_GP^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} A_H$$

$$PA_GP^T = A_H$$

∴ The two graphs G and H are isomorphic.

14.a) i) If $(\boldsymbol{G},*)$is an abelian group, show that $(\boldsymbol{a} * \boldsymbol{b})^2 = \boldsymbol{a}^2 * \boldsymbol{b}^2$.
Proof:

$$(a * b)^2 = (a * b) * (a * b)$$
$$= a * (b * a) * b \quad [Associative\ law]$$
$$= a * (a * b) * b \ [Commutative\ law]$$
$$= (a * a) * (b * b) \ [Associative\ law]$$
$$(a * b)^2 = a^2 * b^2$$

ii) Show that $(Z, +, \times)$ is an integral domain where Z is the set of all integers.

Proof:
Closure:

$$\forall a,b \in Z \Rightarrow a+b \in Z$$
$$\forall a,b \in Z \Rightarrow a \times b \in Z$$

$\therefore Z$ is closed under $+$ and $\times$.
Associative:

$$\forall a,b,c \in Z \Rightarrow (a+b)+c = a+(b+c)$$
$$\forall a,b \in Z \Rightarrow (a \times b)\times c = a \times (b \times c)$$

$\therefore Z$ is associative under $+$ and $\times$.
Identity:
Let $e \in Z$ be the identity element.

$$\forall a \in Z, a+e=e+a=a \Rightarrow a+e=a \Rightarrow e=0$$

$\therefore 0 \in Z$ is the identity element with respect to the binary operation $+$.

$$\forall a \in Z, a \times e = e \times a = a \Rightarrow a \times e = a \Rightarrow e = 1$$

$\therefore 1 \in Z$ is the identity element with respect to the binary operation $+$.
Inverse:
Let $b \in Z$ be the inverse element of $a \in Z$.

$$a+b=b+a=0 \Rightarrow a+b=0 \Rightarrow b=-a \in Z$$

$-a \in Z$ is the inverse of $a \in Z$
$\therefore$ Every element has its inverse in Z under binary operation $+$.
Commutative:

$$\forall a,b \in Z, a+b=b+a$$
$$\forall a,b \in Z, a \times b = b \times a$$

$\therefore Z$ is Commutative under $+$ and $\times$.
Distributive:

$$\forall a,b,c \in Z, a \times (b+c) = a \times b + a \times c$$

$\therefore \times$ is distributive over $+$.

$$\forall a,b \in Z, a \times b = 0 \Rightarrow a = 0 \ or \ b = 0$$

$\therefore Z$ has no zero divisors.
$\therefore (Z,+,\times)$ is an integral domain.

b) i) State and Prove Lagrange's theorem.
Statement:
The order of a subgroup of a finite group is a divisor of the order of the group.
Proof:
Let aH and bH be two left cosets of the subgroup $\{H,*\}$ in the group $\{G,*\}$.
Let the two cosets aH and bH be not disjoint.
Then let c be an element common to aH and bH i.e., $c \in aH \cap bH$

$$\because c \in aH, c = a * h_1, for\ some\ h_1 \in H \ldots (1)$$
$$\because c \in bH, c = b * h_2, for\ some\ h_2 \in H \ldots (2)$$

From (1) and (2), we have

$$a * h_1 = b * h_2$$
$$a = b * h_2 * h_1^{-1} \ldots (3)$$

Let x be an element in aH
$x = a * h_3, for\ some\ h_3 \in H$

$$= b * h_2 * h_1^{-1} * h_3, using\ (3)$$

Since H is a subgroup, $h_2 * h_1^{-1} * h_3 \in H$
Hence, (3) means $x \in bH$

Thus, any element in aH is also an element in bH. $\therefore\ aH \subseteq bH$
Similarly, we can prove that $bH \subseteq aH$
Hence $aH = bH$
Thus, if aH and bH are disjoint, they are identical.
The two cosets aH and bH are disjoint or identical. …(4)
Now every element $a \in G$ belongs to one and only one left coset of H in G,
For,
$a = ae \in aH, since\ e \in H \Rightarrow a \in aH$
$a \notin bH$, since aH and bH are disjoint i.e., a belongs to one and only left coset of H in G i.e., aH … (5)
From (4) and (5), we see that the set of left cosets of H in G form the partition of G. Now let the order of H be m.
Let $H = \{h_1, h_2, \dots, h_m\}, where\ h_i's$ are distinct
Then $aH = \{ah_1, ah_2, \dots, ah_m\}$
The elements of aH are also distinct, for, $ah_i = ah_j \Rightarrow h_i = h_j$, which is not true.
Thus H and aH have the same number of elements, namely m.
In fact every coset of H in G has exactly m elements.
Now let the order of the group $\{G,*\}$ be n, i.e., there are n elements in G
Let the number of distinct left cosets of H in G be p.
$\therefore$ The total number of elements of all the left cosets $= pm$ = the total number of elements of G. i.e., $n = pm$
i.e., m, the order of H is adivisor of n, the order of G.

ii) If $(\boldsymbol{Z}, +)$ and $(\boldsymbol{E}, +)$ where $\boldsymbol{Z}$ is the set all integers and $\boldsymbol{E}$ is the set all even integers, show that the two semi groups $(\boldsymbol{Z}, +)$ and $(\boldsymbol{E}, +)$ are isomorphic.

Proof:
Let $f: (Z, +) \rightarrow (E, +)$ be the mapping between the two semi groups $(Z, +)$ and $(E, +)$ defined by

$$f(x) = 2x, \forall x \in Z$$

f is one to one:

$$f(x) = f(y)$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

$\therefore\ f$ is one to one.
f is onto:
Let $f(x) = y \Rightarrow y = 2x \Rightarrow x = \frac{y}{2} \in Z \quad [\because\ y\ is\ an\ even\ number]$
$\therefore\ \forall x \in E$ there is a preimage $\frac{x}{2} \in Z$.
$\therefore\ f$ is onto.
f is homomorphism:

$$\forall x, y \in Z, f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$$
$$f(x+y) = f(x) + f(y)$$

$\therefore\ f$ is homomorphism.
$\therefore\ f$ is isomorphism.
$\therefore$ The two semi groups $(Z, +)$ and $(E, +)$ are isomorphic.

15. a) i) Show that $(\boldsymbol{N}, \leq)$ is a partially ordered set where $\boldsymbol{N}$ is set of all positive integers and $\leq$ is defined by $\boldsymbol{m} \leq \boldsymbol{n}$ iff $\boldsymbol{n} - \boldsymbol{m}$ is a non-negative integer.

Proof:

Let R be the relation $m \leq n \; iff \; n - m$ is a non-negative integer.

$i) \; \forall x \in N, (x - x) = 0 \; is \; also \; a \; non \; negative \; integer \Rightarrow (x, x) \in R$

$\therefore R$ is reflexive.

$ii) \forall x, y \in N,$

$(x, y) \in R \; \& \; (y, x) \in R$

$\Rightarrow \; (x - y) \; is \; a \; non \; negative \; integer \; \& \; (y - x) \; is \; a \; non \; negative \; integer$

It is possible only if $x - y = 0 \Rightarrow x = y$

$(x, y) \in R \; \& \; (y, x) \in R \Rightarrow x = y$

$\therefore R$ is Anti Symmetric.

$iii) \forall x, y, z \in N, (x, y) \in R \; and \; (y, z) \in R$

$x - z = (x - y) + (y - z)$

Since sum of two non-negative integer is also a non-negative integer.

$\Rightarrow (x - z) is \; also \; a \; non - negative \; integer \Rightarrow (x, z) \in R$

$(x, y) \in R \; and \; (y, z) \in R \Rightarrow (x, z) \in R$

$\therefore R$ is Transitive.

$\therefore (\boldsymbol{N}, \leq)$ is a partially ordered set.

ii) In a Boolean algebra, prove that $(\boldsymbol{a} \wedge \boldsymbol{b})' = \boldsymbol{a}' \vee \boldsymbol{b}'$.

Solution: Let $a, b \in (B, \wedge, \oplus, ', 0, 1)$

To prove $(a \wedge b)' = a' \vee b'$

$$\begin{aligned}(a \wedge b) \vee (a' \vee b') &= \big(a \vee (a' \vee b')\big) \wedge \big(b \vee (a' \vee b')\big)\\ &= \big(a \vee (a' \vee b')\big) \wedge \big((a' \vee b') \vee b\big)\\ &= \big((a \vee a') \vee b'\big) \wedge \big(a' \vee (b' \vee b)\big)\\ &= (1 \vee b') \wedge (a' \vee 1) = 1 \wedge 1\\ (a \wedge b) \vee (a' \vee b') &= 1 \ldots (1)\end{aligned}$$

$$\begin{aligned}(a \wedge b) \wedge (a' \vee b') &= \big((a \wedge b) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)\\ &= \big((b \wedge a) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)\\ &= \big(b \wedge (a \wedge a')\big) \vee \big(a \wedge (b \wedge b')\big)\\ &= (b \wedge 0) \vee (a \wedge 0) = 0 \vee 0\\ (a \wedge b) \wedge (a' \vee b') &= 0 \ldots (2)\end{aligned}$$

$From \; (1) \; and \; (2) \; we \; get,$

$$(a \wedge b)' = a' \vee b'$$

b) i) In a Lattice $(\boldsymbol{L}, \leq)$, prove that $\boldsymbol{x} \vee (\boldsymbol{y} \wedge \boldsymbol{z}) \leq (\boldsymbol{x} \vee \boldsymbol{y}) \wedge (\boldsymbol{x} \vee \boldsymbol{z})$.

Proof:

From the definition of LUB,

$$(x \vee y) \geq x \; \& \; (x \vee z) \geq x \Rightarrow (x \vee y) \wedge (x \vee z) \geq x \ldots (1)$$

$$y \wedge z \leq y \leq x \vee y \ldots (2)$$

$$y \wedge z \leq z \leq x \vee z \ldots (3)$$

From (2) and (3), we get

$$(x \vee y) \wedge (x \vee z) \geq y \wedge z \ldots (4)$$

From (1) and (4), we get

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

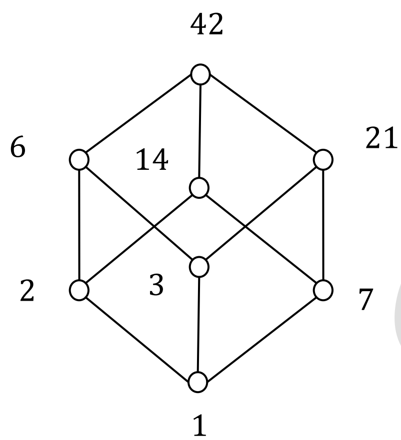
ii) If $\boldsymbol{S_{42}}$ is the set all divisors of 42 and $\boldsymbol{D}$ is the relation "divisor of" on $\boldsymbol{S_{42}}$, prove that $(\boldsymbol{S_{42}}, \boldsymbol{D})$ is a

complemented Lattice.

Solution:

$S_{42} = \{1,2,3,6,7,14,21,42\}$

The Hasse diagram for (S_{42}, D) is



$$1 \vee 42 = 42, 1 \wedge 42 = 1$$
$$2 \vee 21 = 42, 2 \wedge 21 = 1$$
$$3 \vee 14 = 42, 3 \wedge 14 = 1$$
$$7 \vee 6 = 42, 7 \wedge 6 = 1$$

The complement of 1 is 42, The complement of 42 is 1, The complement of 2 is 21, The complement of 21 is 2, The complement of 3 is 14, The complement of 14 is 3, The complement of 7 is 6, The complement of 6 is 7. Since all the elements in (S_{24}, D) has a complement,

$\therefore$ (S_{24}, D) is a complemented lattice.