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Regulations 2010.

SL ①

Part-A

1) Find the sum of the Fourier series for $f(x)=\begin{cases} x, & 0\le x<1 \\ 2, & 1<x<2 \end{cases}$ at $x=1$.

Soln:

$x=1$ is a point of discontinuous

Sum of the fourier series $=\dfrac{f(1+)+f(1-)}{2}=\dfrac{1+2}{2}=\dfrac{3}{2}$

2) The cosine series for $f(x)=x\sin x$ for $0<x<\pi$ is given as

$x\sin x = 1-\frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n \cos nx}{n^2-1}$. Deduce that $1+2\left[\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots\right]=\frac{\pi}{2}$.

Soln:

$$x\sin x = 1-\frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

$x=\frac{\pi}{2}$ is a point of continuity

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1-\frac{\cos \pi/2}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos\left(\frac{n\pi}{2}\right).$$

$$\frac{\pi}{2} = 1-2\left[-\frac{1}{1\cdot3}+\frac{1}{3\cdot5}-\frac{1}{5\cdot7}+\cdots\right] \quad \left[\because \cos n\pi/2 = 0, \text{ when } n \text{ is odd}\right]$$

$$1+2\left[\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\cdots\right]=\frac{\pi}{2}$$

$$\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots = \frac{\pi-2}{4}.$$

3 Define Fourier transformation pair.

Soln:

The infinite Fourier Transform (or) complex Fourier Transform of of a function $f(x)$ is defined by

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

4) Find the Fourier sine transform of $\frac{1}{x}$.

Soln:

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F_S(1/x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}.$$

5) Form the p.d.e from $z = f(x+t) + g(x-t)$,

Soln:

$$z = f(x+t) + g(x-t) \text{ ——— } ①$$

$$p = \frac{\partial z}{\partial x} = f'(x+t) + g'(x-t) \text{ ——— } ②$$

$$q = \frac{\partial z}{\partial t} = f'(x+t) - g'(x-t) \text{ ——— } ③$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t) \text{ ——— } ④$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+t) + g''(x-t) \text{ — } ⑤$$

From ④ & ⑤ $\quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}.$

6) Find the complete integral of $q = 2px$.

Soln:

$q = 2px$

$F_1(x,p) = F_2(y,q)$ [Type ④]

$q = 2px = k$

$q = k \;\Big|\; 2px = k \Rightarrow p = \frac{k}{2x}$.

We know that $z = \int p\,dx + \int q\,dy = \frac{k}{2}\int \frac{dx}{x} + \int k\,dy$

$= \frac{k}{2}\log x + ky + b.$

7) State the governing equation for one dimensional heat equation and necessary to solve the problem.

(2).

Soln:

One dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

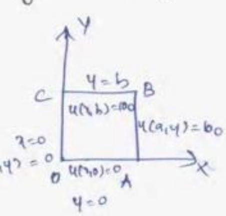
Solution of 1D heat eqn is

(i) $u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, e^{\alpha^2 p^2 t}$

(ii) $u(x,t) = (c_3 \cos px + c_4 \sin px)\, e^{-\alpha^2 p^2 t}$

(iii) $u(x,t) = (c_5 x + c_6)\, c_7$.

8] Write the boundary condition for the following problem. A rectangular plate is bounded by the line $x=0$, $y=0$, $x=a$ and $y=b$. Its surfaces are insulated. The temperature along $x=0$ and $y=0$ are kept at $0°c$ and the others at $100°c$.



Soln:

Let $u(x,y)$ be the temperature at any point in the plate. Then $u(x,y)$ satisfies the equation. From the given problem we get the following boundary conditions.

(i) $u(x,0) = 0$ for $0 < x < a$

(ii) $u(a,y) = 100$ for $0 < y < b$

(iii) $u(x,b) = 100$ for $0 < x < a$

(iv) $u(0,y) = 0$ for $0 < y < b$.

9. Find z-transformation of $\frac{a^n}{n!}$.

Soln:

$$Z[a^n f(n)] = F(z/a)$$

$$Z\left[\frac{a^n}{n!}\right] = \left[Z\left(\frac{1}{n!}\right)\right]_{z \to z/a} = \left[e^{1/z}\right]_{z \to z/a} = e^{\frac{1}{z/a}}.$$

10 Find $Z^{-1}\left[\frac{z}{(z-1)^2}\right]$

Soln:

$$Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n.$$

Part-B

11) a) Calculate the first 3 harmonics of the Fourier of $f(x)$ from the following data

x:	0	30	60	90	120	150	180	210	240	270	300	330
$f(x)$:	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0

x_0	Sin x	Cos x	Sin 2x	Cos 2x	Sin 3x	Cos 3x	y	y sin x	y cos x	y sin 2x	y cos 2x	y sin 3x	y cos 3x
0	0	1	0	1	0	1	1.8	0.00	1.80	0.00	1.80	0.00	1.80
30	0.50	0.87	0.87	0.50	1	0	1.10	0.55	0.96	0.96	0.55	1.1	0.00
60	0.87	0.50	0.87	-0.50	0	-1	0.30	0.26	0.15	0.26	-0.15	0.0	-0.30
90	1.00	0	0	-1	-1	0	0.16	0.16	0	0	0	-0.16	0
120	0.87	-0.50	-0.87	-0.50	0	1	0.50	0.43	-0.25	-0.43	-0.25	0.0	0.5
150	0.50	-0.87	-0.87	0.50	1	0	1.30	0.65	-1.13	-1.13	0.65	1.30	0.00
180	0	-1	0	1	0	-1	2.16	0.00	-2.16	0.00	2.16	0.00	-2.16
210	-0.50	-0.87	0.87	0.50	-1	0	1.25	-0.63	-1.09	1.09	0.63	-1.25	0.00
240	-0.87	-0.50	0.87	-0.50	0	1	1.30	-1.13	-0.65	1.13	-0.65	0.00	1.30
270	-1.00	0	0	-1.00	1	0	1.52	-1.52	0.00	0.00	-1.52	1.52	0.00
300	-0.87	0.50	-0.87	-0.50	0	-1	1.76	-1.53	0.88	-1.53	-0.88	0.00	-1.76
330	-0.50	0.87	-0.87	0.50	-1	0	2.00	-1.00	1.74	-1.74	1.00	-2.00	0
						Σ=	15.15	-3.76	0.25	-1.39	3.18	0.51	-0.62

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x.$$

$a_0 = 2\cdot\frac{\Sigma y}{12} = \frac{15.13}{6} = 2.53$

$a_1 = \frac{1}{6}\Sigma y\cos x = \frac{0.25}{6} = 0.44$

$a_2 = \frac{1}{6}\Sigma y\cos 2x = \frac{3.18}{6} = 0.53$

$a_3 = \frac{1}{6}\Sigma y\cos 3x = \frac{-0.62}{6} = -0.1$

$b_1 = \frac{1}{6}\Sigma y\sin x = -\frac{3.76}{6} = -0.63$

$b_2 = \frac{1}{6}\Sigma y\sin 2x = -\frac{1.39}{6} = 0.23$

$b_3 = \frac{1}{6}\Sigma y\sin 3x = \frac{0.51}{6} = 0.85$

Sub all values in eqn (1)

$$y = 1.26 + 0.04\cos x + 0.53\cos 2x - 0.1\cos 3x$$
$$- 0.63\sin x - 0.23\sin 2x + 0.085\sin 3x.$$

11 b) Find the Fourier series of the fn $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$ and hence evaluate $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$

soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\,dx\right] = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}(\sin(n+1)x - \sin(n-1)x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi(n^2-1)}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}}$$ Provided $n \neq 1$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{2\pi}\int_0^{\pi}(2\sin x\cos x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\sin 2x\,dx = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{1}{4\pi}[-1+1]$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]_0^{\pi}dx$$

$$\boxed{b_n = 0}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\boxed{b_1 = \frac{1}{2}}$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{1}{\pi} + \frac{\sin x}{2} + \sum_{n=2,4,6}^{\infty}\frac{-2}{\pi(n^2-1)}\cos nx$$

$$= \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right]$$

$x=0$ is a point of continuity, sum of F.S $= 0$

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots\right]$$

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = -\frac{1}{\pi}\times -\frac{\pi}{2}$$

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{1}{2}$$

12) a) Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a > 0 \end{cases}$ is

$2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$. Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$. Using Parseval's identity show that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

Soln:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx}\, dx + \int_{a}^{\infty} f(x)\, e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)(\cos sx + i\sin sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a (a^2 - x^2)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$\boxed{F[f(x)] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]}$$

Put $a = 1$, $F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$.

Inverse Fourier transform is $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$= \frac{1}{\sqrt{2\pi}}\cdot 2\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{1}{s^3}(\sin s - s\cos s)\, e^{-isx}\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\, ds \quad \left[\because \int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\, ds = 0 \text{ odd fn}\right]$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\, ds$$

Replacing s by t

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin t - t\cos t}{t^3}\cos tx\, dt.$$

Put $x=0$,

$$\frac{4}{\pi}\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = f(0) = 1$$

$$\therefore \int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$$

Using Parseval's identity $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\int_{-\infty}^{\infty}\left[\frac{2\sqrt{\frac{2}{\pi}}\,(\sin s - s\cos s)}{s^3}\right]^2 ds = \int_{-1}^{1}(1-x^2)^2 dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2 dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1+x^4-2x^2)dx = \frac{16}{15}$$

$$\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

(ie) $$\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$

12) b)

(i) Find Fourier Cosine Transformation of e^{-x^2}.

Soln:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-a^2x^2}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}(R.P \text{ of } e^{isx})\,dx$$

$$= R.P \text{ of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx+s^2/4a^2 - s^2/4a^2}\,dx.$$

$$= R.P \text{ of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2-isx-s^2/4a^2)}\cdot e^{-s^2/4a^2}\,dx$$

$$= R.P \text{ of } \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax-is/2a)^2}\,dx$$

Put $ax - is/2a = y$ | $x=-\infty,\ y=-\infty$

$a\,dx = dy$ | $x=\infty,\ y=\infty.$

(5)

$$= R.P \text{ of } \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy/a.$$

$$= R.P \text{ of } \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \cdot \sqrt{\pi}$$

$$\boxed{F_c[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}}$$

Put $a = 1$

$$\boxed{F_c[e^{-x^2}] = \frac{e^{-s^2/4}}{\sqrt{2}}}$$

11) b(ii) Find the Fourier sine transformation of $\frac{e^{-ax}}{x}$ where $a > 0$.

Soln:

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx \quad \text{— (1)}$$

Diff. on both sides w.r.to 's' we get

$$\frac{d}{ds} F_s(s) = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \sin sx\right) dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \frac{\cancel{x} \cos sx}{\cancel{x}}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$\boxed{\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2+a^2}}$$

Intg. w.r.to s we get

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2}\, ds = \sqrt{\frac{2}{\pi}} \cdot a \cdot \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right) + c$$

$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}(s/a) + c}$$

13) a(i) Solve $x(y^2-z^2)p + y(z^2-x^2)q - z(x^2-y^2) = 0$.

Soln:

$$x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$$

Lagrange's type $pP + qQ = R$.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Using Lagrange's multipliers x, y, z, we get each ratio is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$x^2 + y^2 + z^2 = c_1$$

$$\boxed{u = x^2 + y^2 + z^2}$$

Using Lagrange's multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get each of the above ratio is equal to

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0.$$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$xyz = c_2$$

$$\boxed{v = xyz}$$

$\therefore$ The soln of the given PDE is $\phi(u, v) = 0$

$$\phi(x^2+y^2+z^2,\ xyz) = 0.$$

13) a (ii) Solve $z^2(p^2+q^2) = x^2+y^2$.

Soln:

$$(zp)^2+(zq)^2 = x^2+y^2 \quad \text{——— (1)}$$

Here $m=1$

Put $Z_1 = z^{1+1} = z^2$

$$\frac{\partial Z_1}{\partial z} = 2z$$

$$\therefore \frac{\partial Z_1}{\partial x} = \frac{\partial Z_1}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad \frac{\partial Z_1}{\partial y} = \frac{\partial Z_1}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$P = 2zp \qquad Q = 2zq$$

$$zp = \frac{P}{2} \text{ — (2)} \qquad zq = \frac{Q}{2} \text{ ——— (3)}$$

Sub (2) & (3) in (1) we get

$$P^2+Q^2 = 4(x^2+y^2)$$

$$P^2-4x^2 = 4y^2-Q^2 \text{ (Type 4)}$$

$$P^2-4x^2 = 4y^2-Q^2 = 4a^2 \text{ (say)}$$

$$P^2-4x^2 = 4a^2 \qquad 4y^2-Q^2 = 4a^2$$

$$P = 2\sqrt{x^2+a^2} \qquad Q = 2\sqrt{y^2-a^2}$$

w.k.t $Z_1 = \int P\,dx + \int Q\,dy$

$$Z_1 = 2\int\sqrt{x^2+a^2}\,dx + 2\int\sqrt{y^2-a^2}\,dy$$

$$z^2 = x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\left[\sinh^{-1}(x/a) - \cosh^{-1}(y/a)\right]+b$$

13) b) Solve $(D^3-7DD'^2-6D'^3)z = \cos(x+2y)+x$.

Soln:

A.E is $m^3-7m-6=0$

$m = -1, -2, 3$

C.F $= f_1(y-x)+f_2(y-2x)+f_3(y+3x)$

$P = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^3-7DD'^2-6D'^3}\cos(x+2y).$$

$$= \frac{1}{-D-7D(-4)-6(-4)D'}\cos(x+2y) \quad , \quad D^2 \to -1, \; D'^2 \to -4.$$

$$= \frac{1}{27D+24D'} \cdot \frac{27D-24D'}{27D-24D'}\cos(x+2y)$$

$$= \frac{[27D-24D']}{729D^2-576D'^2}\cos(x+2y)$$

$$= \frac{-27\sin(x+2y)+48\sin(x+2y)}{1575}$$

$$= \frac{21}{1575}\sin(x+2y)$$

$$\boxed{P.I_1 = \frac{1}{75}\sin(x+2y)}$$

$$P.I_2 = \frac{1}{D^3-7DD'^2-6D'^3}x = \frac{1}{D^3\left[1-\frac{7D'^2}{D^2}-\frac{6D'^3}{D^3}\right]}x$$

$$= \frac{1}{D^3\left[1-\left(\frac{7D'^2}{D^2}+\frac{6D'^3}{D^3}\right)\right]^{-1}}x = \frac{1}{D^3}\left[1+\frac{7D'^2}{D^2}+\frac{6D'^3}{D^3}+\cdots\right](x)$$

$$= \frac{1}{D^3}(x) = \frac{1}{D^2}\left(\frac{x^2}{2}\right) = \frac{1}{D}\left(\frac{x^3}{6}\right) = \frac{x^4}{24}$$

$$y = f_1(y-x)+f_2(y-2x)+f_3(y+3x)+\frac{1}{75}\sin(x+2y)+\frac{x^4}{24}.$$

4) a) A String is stretched and fastened to two points $x=0$ and $x=l$ apart. Motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Soln:

the wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0,t) = 0$, for all $t>0$

(7)

(ii) $y(l,t)=0$ for all $t>0$

(iii) $\frac{\partial y}{\partial t}(x,0)=0$ ($\because$ initial velocity is zero)

(iv) $y(x,0) = k(lx-x^2)$.

The correct solution which satisfies our boundary conditions

$$y(x,t) = (C_1\cos px + C_2\sin px)(C_3\cos pat + C_4\sin pat) \quad \text{——} (1)$$

Apply condition (i) in (1) we get

$$y(0,t) = C_1(C_3\cos pat + C_4\sin pat) = 0$$

$\boxed{C_1=0}$ and $C_3\cos pat + C_4\sin pat \neq 0$

Put $C_1=0$ in (1) we get

$$y(x,t) = C_2\sin px(C_3\cos pat + C_4\sin pat) \quad \text{——} (2)$$

Applying condn. (ii) in (2) we get

$$y(l,t) = C_2\sin pl(C_3\cos pat + C_4\sin pat) = 0$$

$C_3\cos pat + C_4\sin pat \neq 0$

$\therefore$ Either $C_2=0$ (or) $\sin pl=0$.

Suppose if we take $C_2=0$ and already we have $C_1=0$ then we get a trivial soln.

$\therefore \sin pl = 0 \Rightarrow pl = n\pi \Rightarrow \boxed{p=\frac{n\pi}{l}}$

(2) $\Rightarrow$ $y(x,t) = C_2\sin\frac{n\pi x}{l}\left(C_3\cos\frac{n\pi at}{l} + C_4\sin\frac{n\pi at}{l}\right)$ —— (3)

Diff (3) p.w.r.to t

$$\frac{\partial y}{\partial t}(x,t) = C_2\sin\frac{n\pi x}{l}\left(-C_3\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + C_4\frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right)$$

Apply condl. (iii) we get

$$\frac{\partial y}{\partial t}(x,0) = C_2\sin\frac{n\pi x}{l}\left(C_4\frac{n\pi a}{l}\right) = 0$$

$C_2 \neq 0$, $\sin\frac{n\pi x}{l} \neq 0$ & $\frac{n\pi a}{l} \neq 0$ $\therefore \boxed{C_4=0}$

$\therefore$ ③ $\Rightarrow$

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$= c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \quad \text{where } c_n = c_2 c_3 \quad \text{—— ④}$$

$\therefore$ The most general soln of ④ can be written as

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi a t}{l}\right) \quad \text{—— ⑤}$$

Apply codn (iv) in ⑤ we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = k(lx - x^2).$$

Using H.R.F.S in $(0,l)$

(i.e.) $$k(lx-x^2) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\int_0^l k(lx-x^2)\sin\left(\frac{n\pi x}{l}\right) dx.$$

$$c_n = b_n = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} 0, & n \text{ is even} \\ \frac{8kl^2}{n^3\pi^3}, & n \text{ is odd.} \end{cases}$$

⑤ $\Rightarrow$

$$y(x,t) = \sum_{1,3,5}^{\infty} \frac{8kl^2}{\pi^3 n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}.$$

14) b) A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°c and 100°c respectively, until steady-state conditions prevail. The temp at A is suddenly raised to 90°c and at B is lowered to 60°c. Find the temp distribution in the bar thereafter.

Soln: The eqn of heat flow is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ —— ①

when the steady state conditions prevail,

① $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0\right]$

'u' is a fn of 'x' alone the above eqn becomes $\frac{d^2u}{dx^2} = 0$

$u(x) = ax + b$ —— ②

when the steady state condition exists the boundary codn. are

a) $u(0) = 50$

b) $u(l) = 100$

Apply (a) in ② we get

$u(0) = b = 50$.

$\therefore$ ② $\Rightarrow$

$u(x) = ax + 50$ —— ③

Apply cond/. (b) in ③ we get

$u(l) = al + 50 = 100 \Rightarrow al = 50 \Rightarrow \boxed{a = \frac{50}{l}}$ —— ④

Sub ④ in ③ we get

$u(x) = \frac{50x}{l} + 50$ —— ⑤

The temp A is raised to 90°c and the temp B is lowered to 60°c

$\therefore$ (ie), the steady state is changed to unsteady state.

$u(x,0) = \frac{50x}{l} + 50$

For unsteddy state we have the following boundary coditions

(i) $u(0,t) = 90$

(ii) $u(l,t) = 60$

(iii) $u(x,0) = \frac{50x}{l} + 50$

The correct soln of one dimensional heat flow eqn is

$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}$ —— (5a)

Sub (i) & (ii) in previous eqn we get

$u(0,t) = A e^{-\alpha^2 p^2 t} = 90$ —— (5b)

$u(l,t) = (A\cos pl + B\sin pl)\, e^{-\alpha^2 p^2 t} = 60$ —— (5c)

From (5b) and (5c) it is not possible to find the constants A and B. Since we have infinite number of values for A and B

$\therefore$ we split the soln $u(x,t)$ into two parts

$u(x,t) = u_s(x) + u_t(x,t)$ —— ⑥

To find $u_s(x)$:

$$u_s(x) = a_1 x + b_1 \quad \text{—} \ (7)$$

Applying the codn/. $u_s(0) = 50$ in (7) we get

$$u_s(0) = b_1 = 90$$

$$u_s(x) = a_1 x + 90 \quad \text{———} \ (8)$$

Apply the codn/. $u_s(l) = 60$ in (8) we get

$$u_s(l) = a_1 l + 90 = 60 \Rightarrow a_1 = \frac{-30}{l} \quad \text{—} \ (9)$$

Sub (9) in (8) we get

$$u_s(x) = -\frac{30x}{l} + 90 \quad \text{—} \ (10)$$

To find $u_t(x,t)$

$$u(x,t) = u_s(x) + u_t(x,t)$$

$$\therefore u_t(x,t) = u(x,t) - u_s(x) \quad \text{——} \ (11)$$

Put $x=0$ in (11).

$$u_t(0,t) = u(0,t) - u_s(0) = 90^\circ - 90^\circ = 0 \ \text{—} \ (12).$$

Put $x=l$ in (11).

$$u_t(l,t) = u(l,t) - u_s(l) = 60^\circ - 60^\circ = 0 \ \text{—} \ (13)$$

Put $t=0$ in (11).

$$u_t(x,0) = u(x,0) - u_s(x)$$

$$= \left(\frac{50x}{l} + 50\right) - \left(-\frac{30x}{l} + 90\right)$$

$$= \frac{80x}{l} - 40 \quad \text{——} \ (14)$$

Now for the fn $u_t(x,t)$ we have the following bdry/. codn/.

a) $u_t(0,t) = 0$

b) $u_t(l,t) = 0$

c) $u_t(x,0) = \frac{80x}{l} - 40$

The soln is

$$u_t(x,t) = (A\cos px + B\sin px)e^{-\alpha^2p^2t} \quad \text{—— (15)}$$

Apply codn/. a_1 & b_1 in (15)

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}} \quad \text{—— (16)}$$

Apply codn/. c_1 in (16) in we get

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{80x}{l} - 40$$

By H.R. F.S. series

$$\frac{80x}{l} - 40 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

$$b_n = \frac{2}{l}\int_0^l \left(\frac{80x}{l} - 40\right)\sin\frac{n\pi x}{l}\,dx.$$

$$= -\frac{80}{n\pi}\left[1+(-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ -\frac{160}{n\pi} & \text{if } n \text{ is even.} \end{cases}$$

$$u_t(x,t) = \sum_{n=2,4}^{\infty} -\frac{160}{n\pi}\sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

Put $l=10$, $u(x,t) = u_s(x) + u_t(x,t)$

$$= -3x + 90 + \sum_{n=2,4..}^{\infty} -\frac{160}{n\pi}\sin\frac{n\pi x}{10} e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$$

5 a) Using z-transform, solve $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ given that $y_0 = 3$ & $y_1 = -5$.

Soln:

$$y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$$

$$z[y_{n+2}] + 4z[y_{n+1}] - 5z[y_n] = 24z(n) - 8z(1)$$

$$z^2y(z) - z^2y(0) - zy(1) + 4[zy(z) - zy(0)] - 5y(z) = \frac{24z}{(z-1)^2} - \frac{8z}{z-1}$$

$$(z+5)(z-1)\,y(z) = \frac{32z - 8z^2}{(z-1)^2} + 3z^2 + 7z$$

$$y(z) = \frac{z[3z^3 + z^2 - 19z + 39]}{(z-1)^3(z+5)}$$

$$\frac{Y(z)}{z} = \frac{3z^3+z^2-19z+39}{(z+5)(z-1)^3}$$

Using partial fractions,

$$\frac{3z^3+z^2-19z+39}{(z+5)(z-1)^3} = \frac{A}{z+5} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3}$$

$A=1,\ B=2,\ C=-2,\ D=4.$

$$\frac{Y(z)}{z} = \frac{1}{z+5} + \frac{2}{z-1} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

$$y(n) = z^{-1}\left[\frac{z}{z+5}\right] + 2z^{-1}\left[\frac{z}{z-1}\right] - 2z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{4}{2}z^{-1}\left[\frac{2z}{(z-1)^3}\right]$$

$$= (-5)^n + 2 - 2n + 2n(n-1)$$

$$\boxed{y(n) = (-5)^n + 2n^2 - 4n + 2}$$

(08)

15) b(i) State and prove convolution theorem on z-transformation. Find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$.

Proof:

If $f(n)$ and $g(n)$ are two causal sequences,

$$Z[f(n) * g(n)] = Z[f(n)] \cdot Z[g(n)] = F(z) \cdot G(z)$$

Proof:

$$F(z)G(z) = \left[\sum_{n=0}^{\infty} f(n) z^{-n}\right]\left[\sum_{n=0}^{\infty} g(n) z^{-n}\right]$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f(k)\, g(n-k)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f(n-k)\, g(k)\right] z^{-n} \quad \text{—} \ (1)$$

By defn

$$Z[f(n) * g(n)] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f(n-k)\, g(k)\right] z^{-n} \quad \text{—} \ (2)$$

from (1) & (2)

$$Z[f(n) * g(n)] = F(z)G(z) = Z[f(n)] \cdot Z[g(n)].$$

(ii)

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

15) b (ii)

$$Z[a^n] = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

Let $F(z) = \frac{z}{z-a}$ and $G(z) = \frac{z}{z-b}$

$$Z^{-1}[F(z)G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)] = a^n * b^n$$

$$= \sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n\left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right] = b^n\left[\frac{(a/b)^{n+1} - 1}{a/b - 1}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

15) b (ii) If $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$, evaluate u_2 and u_3.

Soln:

$$U(z) = \frac{2z^2+5z+14}{(z-1)^4}$$

$$U(z)\, z^{n-1} = \frac{2z^2+5z+14}{(z-1)^4} z^{n-1} = \frac{z^n(2z+5+14z^{-1})}{(z-1)^4}.$$

$z=1$ is a pole of order 4.

$$\operatorname{Res}\left[z^{n-1}F(z)\right]_{z=1} = \lim_{z\to 1} \frac{1}{3!} \frac{d^3}{dz^3}\left[(z-1)^4 \cdot \frac{z^n(2z+5+14z^{-1})}{(z-1)^4}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d^3}{dz^3}\left[2z^{n+1} + 5z^n + 14z^{n-1}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d^2}{dz^2}\left[2(n+1)z^n + 5nz^{n-1} + 14(n-1)z^{n-2}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d}{dz}\left[2n(n+1)z^{n-1} + 5n(n-1)z^{n-2} + 14(n-1)(n-2)z^{n-3}\right]$$

$$= \frac{1}{6} \lim_{z\to 1} \frac{d}{dz}\left[2n(n+1)(n-1)z^{n-2} + 5n(n-1)(n-2)z^{n-3} + 14(n-1)(n-2)(n-3)z^{n-4}\right]$$

$$U(n) = \frac{1}{6}\left[2n(n^2-1) + 5(n^2-n)(n-2) + 14(n^2-3n+2)(n-3)\right]$$

$$u_2 = u(2) = \frac{1}{6}\left[4(3) + 5(0) + 14(0)\right] = 2$$

$$u_3 = u(3) = \frac{1}{6}\left[6(8) + 5(6)\right] = \frac{1}{6}\left[48+30\right] = 13$$