AU- Chennai - May/June 2007 ①

Numerical Methods.

Part - A

1 State the formula for the method of false position to determine a root of $f(x)=0$.

(out of syllabus)

2) State the sufficient condition on $\phi(x)$ for the convergence of an iterative method for $f(x)=0$ written as $x=\phi(x)$

$$|\phi'(x)|<1 \text{ in } [a,b]$$

3) Show that the divided differences are symmetrical in their arguments.

Soln:

$$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1,x_0)$$

4) State Newton's backward difference interpolation formula.

$$y(x) = y_n + n\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \cdots$$

where $n = \frac{x-x_n}{h}$.

5) State Romberg's method integration formula to find the value of $I=\int_a^b f(x)\,dx$ using h & $h/2$.

Soln: $I = I_2 + \frac{1}{3}(I_2 - I_1)$ where $I_1 = I_h$, $I_2 = I_{h/2}$.

6) Write down the simpson's 3/8 rule of integration given (n+1) data.

$$I = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+y_9+\cdots+y_{n-3}) + 3(y_1+y_2+y_4+y_5+\cdots)\right]$$

7) State Taylor Series formula to find $y(x_1)$ for solving

$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$

$$y(x_1) = y_0 + (x_1 - x_0) y'(x_0) + \frac{(x_1 - x_0)^2}{2!} y''(x_0) + \frac{(x_1 - x_0)^3}{3!} y'''(x_0) + \ldots$$

8) State Milne's predictor - corrector formula.

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

9) Name atleast two numerical methods that are used to solve one dimensional diffusion eqn

1) Bender - Schmidt method

2) Crank - Nicholson method

10) Write down Laplace eqn and its finite difference analogue and the standard five - point formula

Soln:

Laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Finite difference analogue

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Standard five point formula

$$u_{i,j} = \frac{1}{4}\left[u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j}\right]$$

Part - B

11) a) (i) Prove the quadratic convergence of Newton - Raphson method. Find a positive root of $f(x) = x^3 - 5x + 3 = 0$ using this method.

Soln:

Let α be the root of $f(x) = 0$

Let $e_i = x_i - \alpha$ be the error at the i^{th} stage of iteration where x_i is the approximate root at the i^{th} iteration

Consider $f(x) = 0$

$$\Rightarrow x = \varphi(x)$$

Then $x_{i+1} = \varphi(x_i)$

$$x_{i+1} = \varphi(\alpha + e_i)$$

$$= \varphi(\alpha) + \frac{e_i}{1!}\varphi'(\alpha) + \frac{e_i^2}{2!}\varphi''(\alpha) + \dots$$

$$e_{i+1} = x_{i+1} - \alpha = \frac{e_i}{1!}\varphi'(\alpha) + \frac{e_i^2}{2!}\varphi''(\alpha) + \dots$$

In Newton - Raphson method $\varphi(x) = x - \frac{f(x)}{f'(x)}$, $\varphi'(\alpha) = 0$ &

$$\varphi''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$$

$$\therefore e_{i+1} = \frac{e_i^2 f''(\alpha)}{2 f'(\alpha)}, \text{ omitting higher powers of } e_i$$

$\therefore$ The convergence is second order

Consider $f(x) = x^3 - 5x + 3 = 0$

$$f'(x) = 3x^2 - 5$$

By Newton's formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$f(0) = 3$$

$$f(1) = -1$$

$\therefore$ The roots lies b/w 0 & 1.

7) State Taylor Series formula to find $y(x_1)$ for solving

$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

$$y(x_1) = y_0 + (x_1 - x_0) y'(x_0) + \frac{(x_1 - x_0)^2}{2!} y''(x_0) + \frac{(x_1 - x_0)^3}{3!} y'''(x_0) + \dots$$

8) State Milne's predictor - corrector formula.

$$y_{n+1, p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$y_{n+1, c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

9) Name atleast two numerical methods that are used to solve one dimensional diffusion eqn

1) Bender - Schmidt method

2) Crank - Nicholson method

10) write down Laplace eqn and its finite difference analog and the Standard five - point formula.

Soln:

Laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Finite difference analogue

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

Standard five point formula

$$u_{i,j} = \frac{1}{4}\left[u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j}\right]$$

Part - B

11) a) (i) Prove the quadratic convergence of Newton - Raphson method. Find a positive root of $f(x) = x^3 - 5x + 3 = 0$ using this method.

Soln:

Let α be the root of $f(x) = 0$

Let $e_i = x_i - \alpha$ be the error at the i^{th} stage of iteration where x_i is the approximate root at the i^{th} iteration

Consider $f(x) = 0$

$\Rightarrow x = \varphi(x)$

Then $x_{i+1} = \varphi(x_i)$

$x_{i+1} = \varphi(\alpha + e_i)$

$= \varphi(\alpha) + \frac{e_i}{1!}\varphi'(\alpha) + \frac{e_i^2}{2!}\varphi''(\alpha) + \dots$

$e_{i+1} = x_{i+1} - \alpha = \frac{e_i}{1!}\varphi'(\alpha) + \frac{e_i^2}{2!}\varphi''(\alpha) + \dots$

In Newton - Raphson method $\varphi(x) = x - \frac{f(x)}{f'(x)}$, $\varphi'(\alpha) = 0$ &

$\varphi''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)}$

$\therefore e_{i+1} = \frac{e_i^2 f''(\alpha)}{2 f'(\alpha)}$, omitting higher powers of e_i

$\therefore$ The convergence is second order

Consider $f(x) = x^3 - 5x + 3 = 0$

$f'(x) = 3x^2 - 5$

By Newtons formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(0) = 3$

$f(1) = -1$

$\therefore$ The roots lies b/w 0 & 1.

Let $x_0 = 0.5$

$$x_{n+1} = x_n - \frac{(x_n^3 - 5x_n + 3)}{3x_n^2 - 5}$$

$$= \frac{3x_n^3 - 5x_n - x_n^3 + 5x_n^2 - 3}{3x_n^2 - 5}$$

$$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 5}$$

Let $x_0 = 0.5$

$x_1 = 0.6470$

$x_2 = 0.6566$

$x_3 = 0.6566$

$\therefore$ The root is $x = 0.6566$.

ii) a)

(ii) Solve the following system by Gauss-Seidal iteration

$28x + 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$.

Soln.:

$$x = \frac{1}{28}[32 - 4y + z]$$

$$y = \frac{1}{17}[35 - 2x - 4z]$$

$$z = \frac{1}{10}[24 - x - 3y]$$

Let $x = y = z = 0$

Iteration	x	y	z
1	1.1429	1.9244	1.8084
2	0.9325	1.5236	1.8497
3	0.9913	1.5070	1.8488
4	0.9936	1.5069	1.8486
5	0.9936	1.5069	1.8486

$\therefore$ The soln is $x = 0.9936$, $y = 1.5069$, $z = 1.8486$.

(or)

8

11) b)
(i) Find the inverse of the matrix by Gauss-Jordan method.

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

Soln:

$$[A|I] = \begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & 4 & 1 & -2 & 0 \\ 0 & 9 & -6 & 1 & 0 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 20 & 0 & 14 & 6 & -2 & 0 \\ 0 & 30 & 0 & 50 & 60 & -144 \\ 0 & 0 & 6 & 14 & -18 & -36 \end{bmatrix}$$

$$\sim \begin{bmatrix} 120 & 0 & 0 & -160 & 240 & 504 \\ 0 & 30 & 0 & 50 & 60 & -144 \\ 0 & 0 & 6 & 14 & -18 & -36 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1.333 & 2 & 4.2 \\ 0 & 1 & 0 & 1.666 & 0.5 & -1.2 \\ 0 & 0 & 1 & 2.333 & -3 & -6 \end{bmatrix}$$

$\therefore$ The inverse of $A = A^{-1} = \begin{bmatrix} -1.333 & 2 & 4.2 \\ 1.666 & 0.5 & -1.2 \\ 2.333 & -3 & -6 \end{bmatrix}$

11) b)
(ii) Find the dominant eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Soln:

Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1x_2$$

$$Ax_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7x_3$$

$$Ax_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714\begin{bmatrix} 1 \\ 0.52 \\ 1 \end{bmatrix} = 3.5714\,x_4$$

$$Ax_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 2 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\,x_5$$

$$Ax_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\,x_6$$

$$Ax_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072\begin{bmatrix} 1 \\ 0.49997 \\ 0 \end{bmatrix}$$

$$= 4.0072\,x_7$$

$$Ax_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.49997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$Ax_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value = 4 & the corresponding eigen vector = $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

(4)

12) a)

(i) If $f(0)=0$, $f(1)=0$, $f(2)=-12$, $f(4)=0$, $f(5)=600$ + $f(7)=7308$ find a polynomial that satisfy the data using Newton's divided difference interpolation formula. Hence find $f(6)$.

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		0				
1	0		-6			
		-12		3		
2	-12		6		9	
		6		48		1
4	0		198		16	
		600		144		
5	600		918			
		3354				
7	7308					

By Newton's divided difference formula

$$y(x) = (x-0)(x-1)(-6) + (x-0)(x-1)(x-2)\times 3 + (x)(x-1)(x-2)(x-4)\times 9$$
$$+ x(x-1)(x-2)(x-4)(x-5)\times 1$$

$$y(x) = x^5 - 4x^3 + 23x^2 - 20$$

$$\boxed{y(6) = 7720}$$

12) a)

(ii) Given the following table, find $f(2.5)$ using cubic spline functions

i	0	1	2	3
x_i	1	2	3	4
$f(x_i)$	0.5	0.3333	0.25	0.2

Soln:

Let $M_0 = M_3 = 0$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right], \; i = 1, 2$$

$i=1 \Rightarrow M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$

$$4M_1 + M_2 = 0.5004 \quad \text{---} (1)$$

$i=2 \Rightarrow M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$

$$M_1 + 4M_2 = 0.1998 \quad \text{---} (2)$$

Solving (1) & (2), $M_1 = 0.12012$

$M_2 = 0.01992$

and $S_i(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right]$

$+ (x_i - x)\left(y_{i-1} - \frac{h^2}{6} M_{i-1}\right) + (x - x_{i-1})\left(y_i - \frac{h^2}{6} M_i\right)$

$i=1 \Rightarrow S_1(x) = \frac{1}{6}(x-1)^3(0.12012) + 0.5(2-x) + (x-1)(0.31328)$

in $1 \le x \le 2$.

$i=2 \Rightarrow$

$S_2(x) = \frac{1}{6}\left((3-x)^3(0.12012) + (x-2)^3(0.01992)\right)$

$+ (3-x)(0.31328) + (x-2)(0.24668)$ in $2 \le x \le 3$

$i=3 \Rightarrow$

$S_3(x) = \frac{1}{6}\left((4-x)^3(0.01992)\right) + (4-x)(0.24668) + (x-3)(0.2)$

in $3 \le x \le 4$.

(or)

12) b)

(i) Find the ~~hom~~ Lagrange's polynomial of degree 3 to fit $y(0) = -12$, $y(1) = 0$, $y(3) = 6$ and $y(4) = 12$. Hence find $y(5)$

Soln: $x_0 = 0, \; x_1 = 1, \; x_2 = 3, \; x_3 = 4$

$y_0 = -12, \; y_1 = 0, \; y_2 = 6, \; y_3 = 12$

By Lagrange's formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \cdots$$

(5)

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 12$$

$y(x) = x^3 - 7x^2 + 18x - 12$

$\boxed{y(2) = 4}$

12)
b)(ii) Find a polynomial of degree two for the data by Newton's forward difference method

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29

Soln.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		0
6	22		1	
		7		
7	29			

$n = \frac{x - x_0}{h} = \frac{x-0}{1} = x$.

By Newton's forward formula

$$y(x) = y_0 + n(\Delta y_0) + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(1) + \frac{x(x-1)(x-2)}{6}(0) + \cdots$$

$y(x) = \frac{1}{2}(x^2 + x + 2)$.

12) a)

(i) Find $f'(6)$ and the maximum value of $y = f(x)$ given the data

x :	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	992

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	992				

By Newton's divided difference formula

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \ldots$$

$$= 4 + (x-0)11 + (x-0)(x-2)7 + (x-0)(x-2)(x-3)1 + \ldots$$

$$y(x) = x^3 + 2x^2 + 3x + 4$$

$$y'(x) = 3x^2 + 4x + 3$$

$$y'(6) = 135$$

y is maximum if $y'(x) = 0$.

(ie) $3x^2 + 4x + 3 = 0$

$$x = \frac{-4 \pm \sqrt{16-36}}{8} = \frac{-4 \pm i\sqrt{20}}{8}$$

The roots are imaginary

$\therefore$ No maximum exists in the given range.

(ii) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg method by taking $h = 0.5, 0.25, 0.125$ successively.

Refer.

(or)

13) b)

(i) Evaluate $I=\int_0^1 \frac{dx}{1+x}$ using three point Gauss quadrature formula.

Refer

(ii) Use Trapezoidal rule to evaluate $I=\int_1^2\int_1^2 \frac{dx\,dy}{x+y}$ taking 4 sub intervals.

Refer.

14) a)

(i) Solve $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0)=2$ by Euler's modified method and find the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$ taking $h=0.2$.

Soln:

By Euler's modified method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n) + f(x_n+h,\ y_n+hf(x_n,y_n))\right]$$

Here $h=0.2$, $x_1=0.2$, $x_2=0.4$

$y_0=2$, $f(x,y)=\log(x+y)$

$$n=0 \Rightarrow y_1 = y_0+\frac{1}{2}\left[f(x_0,y_0)+f(x_0+h,\ y_0+hf(x_0,y_0))\right]$$

$$f(x_0,y_0) = f(0,2) = 0.3010$$

$$f(x_0+h,\ y_0+hf(x_0,y_0)) = f(0.2,\ 2+0.2(0.3010))$$

$$= 0.3541$$

$$y_1 = y(0.2) = 2+\frac{0.2}{2}\left[0.301+0.3541\right] = 2.0655$$

$$n=1 \Rightarrow y_2 = y_1+\frac{h}{2}\left[f(x_1,y_1)+f(x_1+h,\ y_1+hf(x_1,y_1))\right]$$

$$f(x_1,y_1) = f(0.2,\ 2.0655) = 0.3552$$

$$f(x_1+h,\ y_1+hf(x_1,y_1)) = f(0.4,\ 2.0655+0.2(0.3552))$$

$$= 0.4042$$

$$y_2 = 2.0655 + \frac{0.2}{2}[0.3552 + 0.4042]$$

$$\boxed{y_2 = 2.1415}$$

$n=2 \Rightarrow \quad y_3 = y_2 + \frac{h}{2}\left[f(x_2,y_2) + f(x_2+h,\ y_2 + hf(x_2,y_2))\right]$

$$f(x_2, y_2) = f(0.4,\ 2.1415) = 0.4051$$

$$f(x_2+h,\ y_2+hf(x_2,y_2)) = f(0.6,\ 2.1415+0.2(0.4051))$$

$$= 0.4506.$$

$$y_3 = y(0.6) = 2.1415 + \frac{0.2}{2}[0.4051 + 0.4506]$$

$$\boxed{y_3 = 2.2272}$$

(ii) Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, find the value of $y(0.1)$ by Runge-Kutta method of fourth order.

Soln:

$$y'' + xy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\frac{dy}{dx} = z = f_1(x,y,z) \text{ and } \frac{dz}{dy} = -xz - y = f_2(x,y,z) \text{ with}$$

$$x_0 = 0,\ y_0 = 1,\ z_0 = 0$$

$$k_1 = hf_1(x_0, y_0, z_0) = 0.1 f_1(0,1,0) = 0$$

$$l_1 = hf_2(x_0, y_0, z_0) = 0.1 f_2(0,1,0) = 0$$

$k_2 = hf_1(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$

$= 0.1 f_1(0.05,\ 1,\ -0.05)$

$= -0.005$

$l_2 = hf_2(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$

$= 0.1 f_2(0.05,\ 1,\ -0.05)$

$= -0.09975$

$K_3 = hf_1(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= 0.1\, f_1(0.05, 0.9975, -0.0499)$

$= -0.00499$

$l_3 = hf_2(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= 0.1\, f_2(0.05, 0.9975, -0.0499)$

$= -0.995$

$K_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.1\, f_1(0.1, 0.99511, -0.0995)$

$= -0.00995$

$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= 0.1\, f_2(0.1, 0.99511, -0.099)$

$= -0.0985$

$y(0.1) = y_0 + \frac{1}{6}(K_1 + 2(k_1 + k_3) + k_4)$

$\boxed{y(0.1) = 0.9950}$

$z_1 = z_0 + \Delta z_0$

$= z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$=$

(or)

14) b)

(i) Solve $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ using Milne's predictor corrector formula find $y(0.4)$ using Taylor series to find $y(0.1)$, $y(0.2)$, $y(0.3)$.

Soln:

$y' = f(x, y) = xy + y^2$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.4$

$y'' = y + xy' + 2yy'$, $y_0'' = y_0 + x_0 y_0' + 2y_0 y_0' = 3$

$y''' = y' + xy'' + y' + 2yy'' + 2y'^2$, $y_0''' = 2y_0' + xy_0'' + 2y_0 y_0'' + 2y_0'^2 = 10$

$y' = xy + y^2 \Rightarrow y_0' = x_0 y_0 + y_0^2 = 1$

By Taylor series method

$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \dots$

$$y(x) = 1 + \frac{(x-0)}{1}(1) + \frac{(x-0)^2}{2}(3) + \frac{(x-0)^3}{6} \times 10 + \dots$$

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \dots$$

$$y(0.1) = 1 + (0.1) + \frac{3}{2}(0.1)^2 + \frac{5}{3}(0.1)^3 + \dots$$

$$\boxed{y(0.1) = 1.1167}$$

$$y(0.2) = 1 + (0.2) + \frac{3}{2}(0.2)^2 + \frac{5}{3}(0.2)^3 + \dots$$

$$\boxed{y(0.2) = 1.2767}$$

$$y(0.3) = 1 + (0.3) + \frac{3}{2}(0.3)^2 + \frac{5}{2}(0.3)^3 + \dots$$

$$\boxed{y(0.3) = 1.5023}$$

By Milne's method

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$y' = xy + y^2$$

$$y_1' = x_1y_1 + y_1^2 = 1.3587$$

$$y_2' = x_2y_2 + y_2^2 = 1.8853$$

$$y_3' = x_3y_3 + y_3^2 = 2.7076$$

$$y_{4,p} = 1 + 4\frac{(0.1)}{3}(2(1.3587) - 1.8853 + 2(2.7076))$$

$$\boxed{y_{4,p} = 1.83297}$$

$$y_{4,c} = y_2 + h/3\,(y_2' + 4y_3' + y_4')$$

$$y_4' = x_4y_4 + y_4^2 = 1.83292$$

$$y_{4,c} = 1.2767 + \frac{0.1}{3}(1.8853 + 4(2.7076 + 1.83297))$$

$$\boxed{y_{4,c} = 1.83698}$$

b) ⑧

(ii) Compute $y(0.2)$ given $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0)=1$ by R-K method of fourth order, taking $h=0.2$.

Soln:

$f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$, $x_1=0.2$, $h=0.2$

$y_1 = y_0 + \Delta y_0$, $\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(x_0, y_0) = 0.2 f(0,1) = 0.2$

$k_2 = h f(x_0 + h/2, y_0 + k_1/2) = 0.2 f(0.1, 1.1) = 0.1967$

$k_3 = h f(x_0 + h/2, y_0 + k_2/2) = 0.2 f(0.1, 1.09835) = 0.1967$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1967) = 0.1891$

$\Delta y_0 = \frac{1}{6}(0.2 + 2(0.1967) + 2(0.1967) + 0.1891) = 0.1959$

$y_1 = 1 + 0.1959 \Rightarrow y_1 = y(0.2) = 1.19589$

$\boxed{y(0.2) = 1.1959}$

(OR)

15 a)

(i) Solve the boundary value problem $y'' = xy$, subject to the conditions $y(0)=1$, $y(1)=1$ taking $h=1/3$ by finite difference method.

Soln: $h = 1/3$, $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$

$y_0 = 1$, $y_1 = ?$, $y_2 = ?$, $y_3 = ?$

Consider $y'' = xy$

$y_k'' = x_k y_k \Rightarrow y_k'' - x_k y_k = 0$

By finite difference approximate,

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} - x_k y_k = 0$$

$$y_{k+1} - (2 + h^2 x_k) y_k + y_{k-1} = 0$$

$k = 1 \Rightarrow \quad y_2 - \left(2 + \frac{1}{9} x_1\right) y_1 + y_0 = 0 \Rightarrow y_2 - \frac{55}{27} y_1 = -1$ — ①

$k = 2 \Rightarrow \quad y_3 - \left(2 + \frac{1}{9} x_2\right) y_2 + y_1 = 0 \Rightarrow y_1 - \frac{56}{27} y_2 = -1$ — ②

Solving ① & ② we get

$$y_1 = 0.95321$$

$$y_2 = 0.94173.$$

(ii) Using Bender-Schmidt formula, solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 0$ $u(5,t) = 0$, $u(x,0) = x^2(25 - x^2)$. Assume $\Delta x = 1$. Find $u(x,t)$ upto $t = 5$.

Soln:

$$a = 1, \ h = 1, \quad k = \frac{ah^2}{2} = \frac{1}{2}$$

By Bender Schmidt formula, $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}$

$t \backslash x$	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	59	0
1.5	0	39	60	67.5	33.75	0
2	0	30	53.25	49.5	24.75	0
2.5	0	26.625	39.75	42.5	21.75	0
3	0	19.875	35.0625	32.25	16.125	0
3.5	0	17.532	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0
4.5	0	11.4843	17.0625	18.5859	10.5465	0
5	0	8.5312	15.0351	13.8047	9.2929	0

OR.

b) i) Use Crank - Nicholson Scheme to solve $\frac{\partial^2 u}{\partial x^2} = 16\frac{\partial u}{\partial t}$, $0<x<1$ and $t>0$ given $u(x,0)=0$, $u(0,t)=0$ and $u(1,t) = 100t$. Compute $u(x,t)$ for one time step taking $\Delta x = 1/4$.

Soln:

Here $h = 1/4$.

Let $\lambda = 1$.

By Crank - Nicholson formula,

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1}\right]$$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

$u_1 = \frac{u_2}{4} \Rightarrow 4u_1 - u_2 = 0$

$u_2 = \frac{1}{4}(u_1+u_3) \Rightarrow -u_1 - 4u_2 - u_3 = 0$

$u_3 = \frac{1}{4}(u_2+100) \Rightarrow -u_2 + 4u_3 = 100$

Solving $u_1 = 1.7857$, $u_2 = 7.1429$, $u_3 = 26.7857$.

b)

(ii) Evaluate $u(x,t)$ at the pivotal points of the equation $16\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t)=0$, $u(5,t)=0$, $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = x^2(5-x)$ taking $\Delta x = 1$ and upto $t = 1.25$

Soln:

$a = 4$, $h = 1$, $k = \frac{h}{a} = 0.25$

t \ x	0	1	2	3	4	5
0	0	4	12	18	16	0
0.25	0	6	11	14	9	0
0.5	0	7	8	2	−2	0
0.75	0	2	−2	−8	−7	0
1	0	−9	−14	−11	−6	0
1.25	0	−16	−18	−12	−4	0