NM MAY JUNE2007-08

B.E./B.Tech. Degree Examination, May/June 2007.

MA 1251 – Numerical Methods.

Part – A

1. What is the criterion for the convergence of Newton-Raphson method?

Newton-Raphson method converges only when

$$|f(x) \cdot f''(x)| < |f'(x)|^2$$

2. Write down the condition for the convergence of Gauss-Seidal iteration method.

The coefficient matrix should be diagonally dominant.

3. If $f(x) = \frac{1}{x^2}$, find $f(a,b)$ & $f(a,b,c)$ by using divided differences.

$$f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a} = \frac{\left(\frac{a^2-b^2}{a^2b^2}\right)}{(b-a)} = \frac{\left(\frac{(a+b)(a-b)}{a^2b^2}\right)}{b-a}$$

$$= \frac{-(a+b)}{a^2b^2}$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{-\frac{(b+c)}{b^2c^2}+\frac{(a+b)}{a^2b^2}}{(c-a)} = \frac{\frac{-a^2(b+c)+c^2(a+b)}{a^2b^2c^2}}{(c-a)}$$

$$= \frac{-a^2b-a^2c+ac^2+bc^2}{(c-a)(a^2b^2c^2)}$$

4. Using Lagrange's interpolation, find the polynomial through $(0,0)$, $(1,1)$ and $(2,2)$.

Solution: Here $x_0=0,\ x_1=1,\ x_2=2$

$$y_0=0,\ y_1=1,\ y_2=2.$$

By Lagrange's formula,
$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$= \frac{(x-1)(x-2)}{(-1)(-2)} \times 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \times 2$$

$$y = -x^2+2x+x^2-x \implies \boxed{y = x}$$

5. State the formula of Simpson's $\frac{3}{8}$th Rule:

$$\text{Simpson's } \frac{3}{8} \text{ Rule} = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\right]$$

6. Write Newton's forward difference formula to find the derivatives $\left(\frac{dy}{dx}\right)_{x=x_0}$ & $\left(\frac{d^2y}{dx^2}\right)_{x=x_0}$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots \right]$$

7. Write Range-Kutta 4th order formula to solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.

$$y_{n+1} = y_n + \Delta y_n, \quad \Delta y_n = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + h/2,\ y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f\left(x_n + h/2,\ y_n + \frac{k_2}{2}\right)$$
$$k_4 = h\,f(x_n + h,\ y_n + k_3)$$

8. Write Taylor's series formula to solve $y' = f(x,y)$, $y(x_0) = y_0$

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

9. Write down one dimensional wave equation & its boundary condition.

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

with boundary conditions : $u(0,t) = 0$, $u(l,t) = 0$

and initial conditions : $u(x,0) = f(x)$, $u_t(x,0) = 0$.

10. State the explicit formula for the one dimensional wave equation with $1-\lambda^2 a^2 = 0$ where $\lambda = \frac{k}{h}$ and $a^2 = T/m$

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}.$$

Part - B

11. (a) (i) Obtain the positive root of $2x^3 - 3x - 6 = 0$ that lies between 1 and 2 by using Newton - Raphson method.

Solution:

$$f(x) = 2x^3 - 3x - 6 = 0$$

$$f(0) = -6$$

$$\left.\begin{aligned} f(1) &= -7 \\ f(2) &= 4 \end{aligned}\right\} \text{change of sign}$$

$\therefore$ The root lies between 1 & 2.

Let $x_0 = \frac{1+2}{2} = 1.5$.

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(2x_n^3 - 3x_n - 6)}{6x_n^2 - 3}$$

$$= \frac{6x_n^3 - \cancel{3x_n} - 2x_n^3 + \cancel{3x_n} + 6}{6x_n^2 - 3}$$

$$x_{n+1} = \frac{4x_n^3 + 6}{6x_n^2 - 3}$$

Let $x_0 = 1.5$

$x_1 = 1.85714$

$x_2 = 1.78711$

$x_3 = 1.78378$

$x_4 = 1.78377$

$x_5 = 1.78377$.

$\therefore$ The root is $\boxed{x = 1.78377}$

(ii) Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ by Gauss-Jordan method.

Solution: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

The augmented matrix is

$$(A/I) = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -7 & 4 & -4 & 1 & 0 \\ 0 & -5 & 5 & -2 & 0 & 0 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} & \frac{-1}{7} & 0 \\ 0 & 0 & & & & \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & -1 & \frac{5}{6} & \frac{-7}{6} \end{bmatrix}$$

$\therefore$ The inverse of the matrix A is $A^{-1} = \begin{bmatrix} 0 & 1/6 & 1/6 \\ 0 & 1/3 & -2/3 \\ -1 & 5/6 & -7/6 \end{bmatrix}$

(OR)

b. (i) By using Gauss-Seidal method, Solve the following system of equation $6x+3y+12z=35$, $8x-3y+2z=20$, $4x+11y-z=33$.

Solution:

$$x = \frac{1}{8}[20+3y-2z]$$

$$y = \frac{1}{11}[33-4x+z]$$

$$z = \frac{1}{12}[35-6x-3y].$$

Let $x=y=z=0$

Iteration	x	y	z
1	2.5	2.273	1.098
2	3.078	1.981	0.880
3	3.023	1.981	0.910
4	3.015	1.986	0.913

$\therefore$ The solution is $x = 3$, $y = 1.9$, $z = 0.9$.

(i) Find, by power method, the largest eigen value & the eigen vector of the matrix $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{bmatrix} = 25.1778\begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix} = 25.1826\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.726 \end{bmatrix} = 25.1821\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$\therefore$ The eigen ~~vector~~ value is 25.1821

& the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$

12.(a) (i) Using Newton's divided difference formula find $f(x)$ & $f(6)$ from the following data

x:	1	2	7	8
$f(x)$:	1	5	5	4

Solution: Divided difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
			-0.667	
2	5			0.071
		0		
			-0.167	
7	5			
		-1		
8	4			

By Newton's formula for divided differences.

$$f(x) = f(x_0) + (x-x_0)\Delta f(x) + (x-x_0)(x-x_1)\Delta^2 f(x) + \cdots$$

$$f(x) = 1 + (x-1)\times 4 + (x-1)(x-2)(-0.667) + (x-1)(x-2)(x-7)\times 0.071 + \cdots$$

$$f(6) = 1 + (6-1)\times 4 - (5)(4)(0.667) + 5\times 4\times 1\times 0 + \cdots$$

$$= 1 + 20 - 13.34 - 1.42$$

$$\boxed{f(6) = 6.24}$$

(ii) From the following table, find the value of $\tan 45^\circ 15'$ by Newton's forward interpolation formula.

x° :	45	46	47	48	49	5
$\tan x^\circ$:	1	1.03553	1.07237	1.11061	1.15037	1.1

Solution:

Here $x_0 = 45^\circ$, $y_0 = 1.0$, $h = 1^\circ$

$$n = \frac{x - x_0}{h} = \frac{45^\circ 15' - 45^\circ}{1^\circ} = 15' = \frac{1}{4}^\circ$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	1				
		0.03553			
46	1.03553		0.00131		
		0.03684		0.00009	
47	1.07237		0.00140		0.00003
		0.03824		0.00012	
48	1.11061		0.00152		-0.00002
		0.03976		0.0001	
49	1.15037		0.00162		
		0.04138			
50	1.19175				

$\Delta^5 y$

-0.00005

By Newton's forward formula,

$$y(x) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots$$

$$y(45^\circ 15') = 1 + \frac{1}{4}(0.03553) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}(0.00131) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}(0.0009 + \dots$$

$$\boxed{\therefore y(45^\circ 15') = 1.008765}$$

(OR)

(b) (i) Fit the cubic spline for the data,

x:	0	1	2	3
$f(x)$:	1	2	9	28

Solution:

Here $h=1$

The cubic spline in the interval $x_{i-1} \le x \le x_i$ is given by

$$y(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right] + (x - x_{i-1})\left(y_i - \frac{1}{6}M_i\right) \quad \text{--- (1)}.$$

& $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3$

$y_0 = 1,\ y_1 = 2,\ y_2 = 9,\ y_3 = 28$

And $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right], i=1$

Let $M_0 = M_3 = 0$.

$i=1 \Rightarrow 4M_1 + M_2 = 36$

$i=2 \Rightarrow M_1 + 4M_2 = 72$.

On solving $M_1 = \frac{24}{5}$ & $M_2 = \frac{84}{5}$.

Sub. in ①, we get

$i=1 \Rightarrow y(x) = \frac{1}{6}\left(\cancelto{0}{(1-x)^3(0)} + x^3\left(\frac{\overset{4}{\cancel{24}}}{5}\right)\right) + (1-x) + x\left(2-\right.$

$$y(x) = \frac{4}{5}x^3 + \frac{x}{5} + 1 \quad \text{in} \quad 0 \le x \le 1$$

$i=2 \Rightarrow y(x) = \frac{1}{\cancel{6}}\left[(2-x)^3\left(\frac{\overset{4}{\cancel{24}}}{5}\right) + (x-1)^3\left(\frac{\overset{14}{\cancel{84}}}{5}\right)\right]$

$+ (2-x)\left(2 - \frac{1}{\cancel{6}}\left(\frac{\overset{4}{\cancel{24}}}{5}\right)\right) + (x-1)\left[9 - \frac{1}{\cancel{6}} \times \frac{\overset{14}{\cancel{84}}}{5}\right]$

$= \frac{1}{5}\left(4\left(8 - x^3 - 12x + 6x^2\right) + 14\left(x^3 - 1 - 3x^2 + 3\right.\right.$

$+ (2-x)\left(\frac{6}{5}\right) + (x-1)\left(31\right)$

$$y(x) = \frac{1}{5}\left[10x^3 - 18x^2 + 19x - 1\right] \quad \text{in} \quad 1 \le x \le 2.$$

$i=3 \Rightarrow y(x) = \frac{1}{6}\left[(3-x)^3 \times \frac{\overset{14}{\cancel{84}}}{5} + \cancelto{0}{(x-2)^3(0)}\right]$

$+ (3-x)\left(9 - \frac{1}{\cancel{6}} \times \frac{\overset{14}{\cancel{84}}}{5}\right) + (x-2)\left(28\right)\Big]$

$= \frac{14}{5}\left(9 - x^3 - 27x + 9x^2\right) + (3-x)\frac{(31)}{5} + (x-2)($

$$y(x) = \frac{1}{5}\left(-14x^3 + 126x^2 - 269x + 191\right).$$

13.(a) (i) Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x+y}$ with $h=k=0.2$ by using trapezoidal Rule.

Solution:

$$f(x,y) = \frac{1}{x+y}, \quad h=k=0.2$$

x / y	1	1.2	1.4	1.6	1.8	2
1	.5	.455	0.417	.385	.357	.333
1.2	.455	.417	.385	.357	.333	.313
1.4	.417	.385	.357	.333	.313	.294
1.6	.385	.357	.333	.313	.294	.278
1.8	.357	.333	.313	.294	.278	.263
2	.333	.313	.294	.278	.263	.25

By Trapezoidal Rule,

$$\int_1^2\int_1^2 \frac{dx\,dy}{x+y} = \frac{hk}{2}\Big[\text{Sum of the corner values} + 2\,(\text{sum of the values in } \square \text{ box}) + 4\,(\text{sum of remaining values})\Big]$$

$$= \frac{.2 \times .2}{2}\Big[1.416 + 2\,(5.524) + 4\,(5.395)\Big]$$

$$= 0.681$$

(ii) From the following table, find the value of x for which $f(x)$ is maximum. Also find the maximum value.

x:	60	75	90	105	120
$f(x)$:	28.2	38.2	43.2	40.9	37.7

Solution.

$f(x)$ is maximum if $f'(x)=0$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
60	28.2 (y_0)	10 (Δy_0)	-5 ($\Delta^2 y_0$)	-2.3 ($\Delta^3 y_0$)	8.7 ($\Delta^4 y_0$)
75	38.2	5	-7.3	6.4	
90	43.2	-2.3	-0.9		
105	40.9	-3.2			
120	37.7				

$y_x = \frac{x - x_0}{\Delta^4 y^{4}}$

By Newtons forward difference formula,

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\Delta y_0 + \frac{(x-x_0)(x-x_0-1)}{2!}\Delta^2 y_0 + \frac{(x-x_0)(x-x_0-2)(x_0-x_0-3)}{3!}\Delta^3 y_0 + \cdots$$

$$= 28.2 + (x-60)\,10 + \frac{(x-60)(x-61)}{2}\times(-5) + \frac{(x-60)(x-61)(x-62)}{6}(-2.3) + \frac{(x-60)(x-61)(x-62)(x-63)}{24}(8.7) + \cdots$$

$$y(x) = 28.2 + 10x - 600 - 2.5x^2 + 302.5x - 9150$$

$y(x)$ is maximum if $y'(x) = 0$.

$$10 - 2.5(2x) = 0$$

$$\Rightarrow 5x = 10$$

$$\boxed{x = 2}$$

$y(x)$ attains its maximum at $x = 2$.

(b) (i) Using Romberg's Rule, evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places by taking $h = 0.5, .25, 0.125$

Solution:

$$y = \frac{1}{1+x}$$

(i) when $h = 0.5$

x:	0	.5	1
y:	1	.666	.5

By Trapezoidal Rule,

$$I_1 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

$$= \frac{0.5}{2}\left[1 + .5 + 2(.666)\right]$$

$$= 0.7083.$$

(ii) when $h = 0.25$

x:	0	.25	.5	.75	1
y:	1	0.8	.6666	.5714	.5

$$I_2 = \int_0^1 \frac{dx}{1+x} = \frac{0.25}{2}\left[1 + .5 + 2(0.8 + .6666 + .5714)\right]$$

$$= 0.697.$$

(iii) when $h = 0.125$

x:	0	.125	.25	.375	.5	.625	.75	.875	1
y:	1	.8889	.8	.7273	.6667	.6154	.5714	.5333	.5

$$I_3 = \int_0^1 \frac{dx}{1+x} = \frac{0.125}{2}\left[1 + .5 + 2(.8889 + .8 + .7273 + .6667 + .6154 + .5714 + .5333 + .5)\right]$$

$$= 0.6941$$

(b) (i) Using Romberg's Rule, evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places by taking $h = 0.5, .25, 0.125$

Solution:

$$y = \frac{1}{1+x}$$

(i) when $h = 0.5$

x:	0	.5	1
y:	1	.666	.5

By Trapezoidal Rule,

$$I_1 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= \frac{0.5}{2}\left[1 + .5 + 2(.666)\right]$$

$$= 0.7083.$$

(ii) when $h = 0.25$

x:	0	.25	.5	.75	1
y:	1	0.8	.6666	.5714	.5

$$I_2 = \int_0^1 \frac{dx}{1+x} = \frac{0.25}{2}\left[1 + .5 + 2(0.8 + .6666 + .5714)\right]$$

$$= 0.697.$$

(iii) when $h = 0.125$

x:	0	.125	.25	.375	.5	.625	.75	.875	1
y:	1	.8889	.8	.7273	.6667	.6154	.5714	.5333	.5

$$I_3 = \int_0^1 \frac{dx}{1+x} = \frac{0.125}{2}\left[1 + .5 + 2(.8889 + .8 + .7273 + .6667 + .6154 + .5714 + .5333 + .5)\right]$$

$$= 0.6941$$

By Romberg integration,

$$I = I_3 + \frac{1}{3}(I_2 - I_1)$$

$$= 0.6931$$

$$I = I_3 + \frac{1}{3}(I_2 - I_1)$$

$$= \underline{0.6931}$$

(ii) By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by using Simpson's 1/3 Rule. Is it possible to evaluate the same by Simpson's $\frac{3}{8}$ Rule. Justify your answer.

Soln: $h = \frac{b-a}{n} = \frac{\pi}{10}$.

x:	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$
$y = \sin x$:	0	.309	.5858	.809	.9511	1	.9511	.809	.5850
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

x:	$\frac{9\pi}{10}$	π
y:	.307	0
	y_9	y_{10}

By Simpson's 1/3 Rule,

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3}\left[y_0 + y_{10} + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)\right]$$

$$= 2.00091$$

It is not possible to apply Simpson's $\frac{3}{8}$ Rule, because the number of subintervals is not multiple of 3.

14.(a) (i) Using Taylor series method, find y when $x = 1.1$ and 1.2 from $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$ (4 decimal places).

Solution:

Given $y' = xy^{1/3}$, $x_0 = 1$, $y_0 = 1$, $h = 0.1$, $y_0' = x_0 y_0^{1/3} = 1$

$$y'' = \frac{1}{3} y^{-2/3} y' x + y^{1/3} \longrightarrow y_0'' = \frac{1}{3} y_0^{-2/3} y_0' x_0 + y_0^{1/3}$$

$$= \frac{4}{3}$$

$\cancel{y''' \Rightarrow x^2 y'' + 4xy'}$

$$y''' = \frac{1}{3}\left[y^{-2/3} y' + y^{-2/3} x y'' + y' x \left(\frac{-2}{3}\right) y^{-5/3} \right] + \frac{1}{3} y^{-2/3} y'$$

$$y_0''' = \frac{1}{3}\left[y_0^{-2/3} y_0' + y_0^{-2/3} x_0 y_0'' + y_0' x_0 \left(-\tfrac{2}{3}\right) y_0^{-5/3} \right] + \tfrac{1}{3} y_0^{-2/3} y_0'$$

$$= \frac{8}{9}$$

$\therefore$ By Taylor series,

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

$$= 1 + (x-1) \times 1 + \frac{(x-1)^2}{2} \times \frac{4}{3} + \frac{(x - 1)^3}{6} \times \frac{8}{9} + \cdots$$

$$y(x) = 1 + (x-1) + \frac{2}{3}(x-1)^2 + \frac{4}{27}(x-1)^3 + \cdots$$

$$y(1.1) = 1 + (1.1 - 1) + \tfrac{2}{3}(1.1 - 1)^2 + \frac{4}{27}(1.1 - 1)^3 + \cdots$$

$$\boxed{y(1.1) = 1.10681}$$

$$y(1.2) = 1 + (1.2 - 1) + \tfrac{2}{3}(1.2 - 1)^2 + \frac{4}{27}(1.2 - 1)^3 + \cdots$$

$$\boxed{y(1.2) = 1.22772}$$

(ii) By using Adam's method find y when $x = 0.4$, given $\frac{dy}{dx} = \frac{xy}{2}$, $y(0) = 1$, $y(0.1) = 1.01$, $y(0.2) = 1.02$, $y(0.3) = 1.023$.

Solution: Here $y' = \frac{xy}{2}$, $h = 0.1$

$x_0 = 0 \longrightarrow y_0 = 1$

$x_1 = 0.1 \longrightarrow y_1 = 1.01$

$x_2 = 0.2 \longrightarrow y_2 = 1.022$

$x_3 = 0.3 \longrightarrow y_3 = 1.023$

$x_4 = 0.4 \longrightarrow y_4 = ?$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

Put $n = 3$, $y_{4,p} = y_3 + \frac{0.1}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

Consider $y' = \frac{xy}{2}$

$y_0' = \frac{x_0 y_0}{2} = 0$

$y_1' = \frac{x_1 y_1}{2} = 0.0505$

$y_2' = \frac{x_2 y_2}{2} = 0.1022$

$y_3' = \frac{x_3 y_3}{2} = 0.1535$

$$\therefore y_{4,p} = 1.023 + \frac{0.1}{24}\left[55 \times .1535 - 59 \times .1022 + 37 \times .0505 - 9 \times 0\right]$$

$y_{4,p} = \cancel{0.2081}\ 1.0408$

By Adam's corrector formula,

$$y_{n+1,c} = y_n + \frac{h}{24}\left[y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

Put $n=3$, $y_{4,c} = y_3 + \frac{0.1}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$

$$y_4' = \frac{x_4 y_4}{2} = 0.2081$$

$$\therefore y_{4,c} = 1.023 + \frac{0.1}{24}\left(9(.2081) + 19(.1535) - 5(.1022) + 0.0505\right)$$

$$y_{4,c} = 1.0410$$

$$\therefore \boxed{y(0.4) = 1.0410}$$

(b) (i) Using Range-kutta method of 4^{th} order, solve Take $h=0.2$

$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, given $y(0)=1$ at $x=0.2$.

Solution:

$f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

By Range-kutta method,

$$y_{n+1} = y_n + \Delta y_n$$

$$\Delta y_n = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Putting $x=0$, $y_1 = y_0 + \Delta y_0$

$$k_1 = h f(x_0, y_0) = 0.2 f(0,1) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f\left(0.1, 1 + \frac{0.2}{2}\right)$$

$$= 0.2 f(0.1, 1.1)$$

$$k_2 = 0.19672$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 f\left(0.05, 1 + \frac{0.19672}{2}\right)$$

$$= 0.1967$$

$K_4 = h\, f(x_0+h,\ y_0+k_3)$

$= 0.1\, f(0.2,\ 1+0.1967)$

$= 0.1\, f(0.2,\ 1.1967)$

$K_4 = 0.19598.$

$\Delta y_0 = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}(0.2+2(1.19672)+2(0.1967)+0.1891)$

$\Delta y_0 = 0.19598$

$\therefore\ y_1 = y(0.2) = y_0 + \Delta y_0 = 1 + .19598$

$\boxed{y(0.2) = 1.19598}$

(ii) Find the value of y when $x=0.1$ & 0.2, given $\frac{dy}{dx} = x^2+y^2$ with $y=1$ when $x=0$. Use Modified Euler's method.

Solution:

$f(x,y) = x^2+y^2, \quad x_0 = 0, \quad y_0 = 1., \ h=0.1$

By Modified Euler's method,

$y_{n+1} = y_n + h f\left(x_n + h/2,\ y_n + h/2\, f(x_n, y_n)\right)$

$n=0 \Rightarrow\ y_1 = y_0 + h\, f\left(x_0 + h/2,\ y_0 + h/2\, f(x_0,y_0)\right)$

$f(x_0,y_0) = x_0^2 + y_0^2 = 1$

$f\left(x_0+h/2,\ y_0 + h/2\, f(x_0,y_0)\right)$

$= f\left(\frac{0.1}{2},\ 1+\frac{0.1}{2}(1)\right) = f(0.05,\ 1.05)$

$= (0.05)^2 + (1.05)^2 = 1.105$

$\therefore\ y_1 = 1 + (0.1)(1.105) = 1.1105$

$n = 1 \Rightarrow\ y_2 = y_1 + h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right)$

$f(x_1, y_1) = x_1^2 + y_1^2 = (0.1)^2 + (1.1105)^2 = 1.2432$

$f\left(x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right) = f\left(0.1 + \frac{0.1}{2},\ 1.1105 + \frac{0.1}{2}(1.2432)\right)$

$= f(0.15,\ 1.17266)$

$= (0.15)^2 + (1.17266)^2 = 1.3976$

$\therefore\ y_2 = 1.1105 + 0.1(1.3976)$

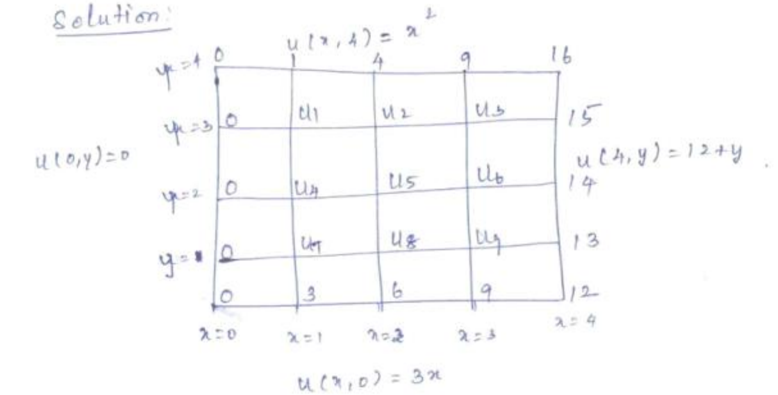
$y(0.2) = y_2 = 1.25026.$

15.(a) Solve the Laplace's equation over the square mesh of side 4 units, satisfying the boundary conditions:

$u(0, y) = 0, \quad 0 \le y \le 4, \quad u(4, y) = 12 + y, \quad 0 \le y \le 4$

$u(x, 0) = 3x, \quad 0 \le x \le 4, \quad u(x, 4) = x^2, \quad 0 \le x \le 4.$

<u>Solution:</u>



Rough values:

$u_5 = \frac{1}{4}(4+6+0+14) = 6$ (SFPF)

$u_1 = \frac{1}{4}(0+6+4+0) = 2.5$ (DFPF)

$u_3 = \frac{1}{4}(16+6+14+4) = 10$ (DFPF)

$u_7 = \frac{1}{4}(0+6+0+6) = 3$ (DFPF)

$u_9 = \frac{1}{4}(6+14+6+12) = 9.5$ (DFPF)

$u_2 = \frac{1}{4}(4+6+2.5+10) = 5.625$ (SFPF)

$u_4 = \frac{1}{4}(0+6+2.5+3) = 3.125$ (SFPF)

$u_6 = \frac{1}{4}(6+14+10+9.5) = 9.875$ (SFPF)

$u_8 = \frac{1}{4}(6+6+3+9.5) = 6.125$ (SFPF)

Using SFPF,

$u_1 = \frac{1}{4}(u_2+u_4) = 2.4375 \Rightarrow$ Error

$u_2 = \frac{1}{4}(4+u_1+u_3+u_5)$

$u_3 = \frac{1}{4}(24+u_2+u_6)$

$u_4 = \frac{1}{4}(u_1+u_5+u_7)$

$u_5 = \frac{1}{4}(u_2+u_4+u_6+u_8)$

$u_6 = \frac{1}{4}(14+u_3+u_5+u_9)$

$u_7 = \frac{1}{4}(3+u_4+u_8)$

$u_8 = \frac{1}{4}(6+u_5+u_7+u_9)$

$u_9 = \frac{1}{4}(22+u_6+u_8)$.

First iteration:

$u_1 = 2.4375$ $\quad$ $u_6 = 9.8721$

$u_2 = 5.6094$ $\quad$ $u_7 = 2.9946$

$u_3 = 9.8711$ $\quad$ $u_8 = 6.153$

$u_4 = 2.8594$ $\quad$ $u_9 = 9.5063$

$u_5 = 6.1172$

Second iteration:

$u_1 = 2.3672$ $\quad$ $u_7 = 3.0057$

$u_2 = 5.5888$ $\quad$ $u_8 = 6.1582$

$u_3 = 9.8652$ $\quad$ $u_9 = 9.5078$

$u_4 = 2.8698$

$u_5 = 6.1209$

$u_6 = 9.8731$

Third iteration:

$u_1 = 2.37$ $\quad$ $u_6 = 9.88$

$u_2 = 5.59$ $\quad$ $u_7 = 3.01$

$u_3 = 9.87$ $\quad$ $u_8 = 6.16$

$u_4 = 2.88$ $\quad$ $u_9 = 9.51$

$u_5 = 6.13$

$\therefore$ The solution is

$u_1 = 2.37$, $u_2 = 5.59$, $u_3 = 9.87$, $u_4 = 2.88$

$u_5 = 6.13$, $u_6 = 9.88$, $u_7 = 3.01$, $u_8 = 6.16$, $u_9 = 9.51$

(OR)

(b) (i) Derive Bender - Schmidt for solving $u_{xx} - \alpha u_t = 0$ with boundary conditions $u(0,t) = T_0$, $u(l,t) = T_l$ and $u(x,0) = f(x)$ for $0 < x < l$. Also find corresponding recurrence equation.

Solution:

consider $u_{xx} = \alpha u_t$ —— ①

By central difference approximation

$$\left.\begin{aligned} u_{xx} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \\ u_t &= \frac{u_{i,j+1} - u_{i,j}}{k} \end{aligned}\right\} \text{—— ②}$$

Using ② in ①,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{\alpha}{k}\left(u_{i,j+1} - u_{i,j}\right)$$

$$\frac{k}{\alpha h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = u_{i,j+1} - u_{i,j}$$

$$\lambda\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = u_{i,j+1} - u_{i,j}, \quad \text{where } \lambda = \frac{k}{\alpha h^2}$$

$$u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j}.$$

This is called schmidt relation —— ③

when $\lambda = 1/2$, eqn ③ reduces to

$$u_{i,j+1} = \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right).$$

This is called Bender- schmidt recurrence relation.

(ii) By finite difference method, solve $\frac{d^2y}{dx^2} + x^2y = 0$ with Boundary conditions $y(0) = 0$ and $y(1) = 1$, $h = .25$

Solution:

Given $y'' + x^2y = 0$

By finite differences,

$$y'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + x_k^2 y_k = 0.$$

$$y_{k+1} + y_k(-2 + x_k^2 h^2) + y_{k-1} = 0.$$

$k=1$, $y_2 + y_1(-2 + x_1^2(\frac{1}{16}) + y_0 = 0$

$k=2$, $y_3 + y_2(-2 + x_2^2(\frac{1}{16}) + y_1 = 0$

$k=3$, $y_4 + y_3(-2 + x_3^2(\frac{1}{16}) + y_2 = 0$

$\Rightarrow$ $31.937\, y_1 + 16y_2 = 0$

$16y_1 - 31.75y_2 + 16y_3 = 0$

$16y_2 - 30.875y_3 = -16.$

On solving $y_1 = 0.2617$, $y_2 = 0.5223$, $y_3 = 0.7748$

$\therefore$ The solution is

$y(0.25) = 0.2617$
$y(0.5) = 0.5223$
$y(0.75) = 0.7748$