NM MAY 2007

B.E / B.Tech Degree Examination, May - 2007.

Numerical Methods

Part - A

1. State the condition for convergence of Gauss - Seidal method

The coefficient matrix should be diagonally dominant

2. Find an iterative formula to find $\sqrt{N}$, where N is a positive integer.

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

3. Obtain the interpolation quadratic polynomial for the given data by using Newton's forward difference formula

x:	0	2	4	6
y:	-3	5	21	45

Solution:

$$n = \frac{x - x_0}{h} = \frac{x}{2}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-3			
		8		
2	5		8	
		16		0
4	21		8	
		24		
6	45			

$$\therefore y(x) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \cdots$$

$$= -3 + \frac{x}{2}(8) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2} - 1\right)}{2} \times (8) + \cdots$$

$$\boxed{y(x) = x^2 + 2x - 3}$$

4. Find the second degree polynomial fitting the following data

x:	1	2	4
y:	4	5	13

Solution:

Since the intervals are unequal.

By Lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

where $x_0 = 1,\ x_1 = 2,\ x_2 = 4$

$y_0 = 4,\ y_1 = 5,\ y_2 = 13$

$$\therefore y(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)}(4) + \frac{(x-1)(x-4)}{(2-1)(2-4)} \times 5 + \frac{(x-1)(x-2)}{(4-1)(4-2)} \times 13$$

$$y(x) = x^2 - 2x + 5$$

5. Write down Newton-Cotes quadrature formula.

$$\int_{x_0}^{x_0+nh=x_n} y(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right]$$

6. What is the condition for Simpson's $\frac{3}{8}$ Rule and State the formula:

Condition: Number of subintervals must be a multiple of 3.

formula: $$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots)\right]$$

7. What is the condition to apply Adams-Bashforth method?

To apply predictor formula we need previously calculated 4 values.

And the predicted value will be used to ~~calted~~ calculate y'_{n+1}.

8. What do you mean by error ~~ana~~ in error analysis?

Error = True value − Approximate value.

9. What is the error for solving Laplace and Poisson's equation's equations by finite difference method?

Order of error $= O(h^2)$ = order of h^2.

10. Write down Bender-Schmidt relation for one dimensional heat equation.

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right]$$

Part - B

11.(i) Solve the given system of equations by using Gauss - Seidal method.

$20x - y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.

Solution:

$$x = \frac{1}{20}[17 + y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

Let $x = y = z = 0$

Iteration	x	y	z
1	0.85	-1.0275	1.010875
2	0.8997125	-0.984413	1.012366
3	0.90202	-0.98468	1.0120
4	0.90197	-0.98469	1.0121
5	0.90197	-0.98469	1.0121

∴ The solution is $x = 0.90197$, $y = -0.98469$, $z = 1.0121$.

(ii) Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ by Power method.

Solution:

Let $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.231 \\ 0.692 \\ 1 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1.307 \\ 6.077 \\ 12.537 \end{bmatrix} = 12.537\begin{bmatrix} 0.104 \\ 0.485 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 0.559 \\ 5.282 \\ 11.836 \end{bmatrix} = 11.836\begin{bmatrix} 0.047 \\ 0.446 \\ 1 \end{bmatrix}$$

$$Ax_4 = \begin{bmatrix} 0.385 \\ 5.033 \\ 11.737 \end{bmatrix} = 11.737 \begin{bmatrix} 0.032 \\ 0.429 \\ 1 \end{bmatrix}$$

$$Ax_5 = \begin{bmatrix} 0.319 \\ 4.954 \\ 11.684 \end{bmatrix} = 11.684 \begin{bmatrix} 0.027 \\ 0.423 \\ 1 \end{bmatrix}$$

$$Ax_6 = \begin{bmatrix} 0.296 \\ 4.907 \\ 11.665 \end{bmatrix} = 11.665 \begin{bmatrix} 0.025 \\ 0.421 \\ 1 \end{bmatrix}$$

$$Ax_7 = \begin{bmatrix} 0.288 \\ 4.917 \\ 11.659 \end{bmatrix} = 11.659 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$$

$$Ax_8 = \begin{bmatrix} 0.291 \\ 4.919 \\ 11.663 \end{bmatrix} = 11.663 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$$

$\therefore$ The largest eigen value $= 11.663$

eigen vector $= \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$

12.(a) (i) From the following data, estimate, the number of persons earning weekly wages between 60 and 70 Rupees.

Wage (in Rs):	below 40	40-60	60-80	80-100	100-120
No. of Persons (in thousands):	250	370	470	540	590

Solution:

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
below 40	250				
		370	100		
" 60	620			-30	10
		470 ~~840~~	70	-20	
" 80	1090				
		540	50		
" 100	1130				
		590			
" 120	1220				

The number of Persons whose weekly wages between 60 and 70 is = No. of persons whose weekly wages below 70 − below 60.

By Newton's forward formula,

$$y(n) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

where $n = \frac{x - x_0}{h} = \frac{70-40}{10} = 1.5$

$$y(70) = 250 + 1.5(370) + \frac{1.5(1.5-1)}{2} \times 100 + \frac{(1.5)(1.5-1)(1.5-2)}{6} \times (-30)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{24} \times (10)$$

$y(70) = 423.5937$

But $y(60) = 370$.

$\therefore$ No. of persons $= 423.5937 - 370$

$= 53.5937$ thousands.

(ii) Find the cubic ~~spline~~ polynomial from the following table using divided difference and hence find $f(4)$.

x:	0	1	2	5
$y = f(x)$:	2	3	12	147

Solution:

Divided difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

By Newtons divided difference formula

$y(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \dots$

$y(x) = 2 + x(1) + x(x-1)4 + x(x-1)(x-2)(1)$

$y(x) = x^3 + x^2 - x + 2$

$y(4) = 78$

(OR)

(b) (i) For the given values evaluate $f(9)$ using Lagranges formula.

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Solution:

Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$, $y_1 = 392$, $y_2 = 1452$, $y_3 = 2366$, $y_4 = 5202$

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \dots + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

$$y(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452) + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$\boxed{y(9) = 810.001}$$

(ii) Fit the cubic spline for the data:

x:	1	2	3
y:	-6	-1	16.

Hence evaluate $y(1.5)$ given that $y_0'' = y_2'' = 0$.

Solution: Given that $M_0 = M_2 = 0$. $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$y_0 = -6$, $y_1 = -1$, $y_2 = 16$.

& $h = 1$.

We have

$$M_{i-1} + 4m_i + m_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \; i = 1$$

$i = 1 \Rightarrow$ $$m_0 + 4m_1 + m_2 = 6(y_0 - 2y_1 + y_2)$$

$$\Rightarrow m_1 = 18$$

And $$y(x) = \frac{1}{6}\left((x_i - x)^3 m_{i-1} + (x - x_{i-1})^3 m_i\right) +$$

$$(x_i - x)\left(y_{i-1} - \frac{1}{6}m_{i-1}\right) + (x - x_{i-1})\left(y_i - \frac{1}{6}m_i\right), \; i = 1, 2$$

$i = 1 \Rightarrow$ $$y(x) = \frac{1}{6}\left(\cancel{(x_1 - x)^3 m_0}^{\,0} + (x - x_0)^3 m_1\right)$$

$$+ (x_1 - x)\left(y_0 - \frac{1}{6}\cancel{m_0}^{\,0}\right) + (x - x_0)\left(y_1 - \frac{1}{6}m_1\right)$$

$$= \frac{1}{6}(x-1)^3(18) + (2-x)(-6) + (x-1)\left(-1 - \frac{1}{6}(18)\right)$$

$$y(x) = 3x^3 - 9x^2 + 11x - 11 \quad \text{in} \quad 1 \le x \le 2.$$

$i = 2 \Rightarrow$ $$y_2(x) = \frac{1}{6}\left((x_2 - x)^3 m_1 + \cancel{(x - x_1)^3 m_2}^{\,0}\right)$$

$$+ (x_2 - x)\left(y_1 - \frac{1}{6}m_1\right) + (x - x_1)\left(y_2 - \frac{1}{6}\cancel{m_2}^{\,0}\right)$$

$$= \frac{1}{6}\left((3-x)^3 \times 18\right) + (3-x)\left(-1 - \frac{1}{6}(18)\right) + (x-2)(16)$$

$$y(x) = -3x^3 + 27x^2 - 61x + 37 \quad \text{in} \quad 2 \le x \le 3.$$

$$y(1.5) = -4.625.$$

13. (a)(i) obtain the value of $f'(0.04)$ using an appropriate formula for the given data.

x:	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Solution:

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.01	0.1023				
		0.0024			
0.02	0.1047		0		
		0.0024		0.0001	
0.03	0.1071		0.0001		−0.0001
		0.0025		0	
0.04	0.1096		0.0001		0.0001
		0.0026		−0.0001	
0.05	0.1122		0.0		
		0.0026			
0.06	0.1148				

By Newtons formula,

$$y'(n) = \frac{1}{h}\left(\Delta y_0 + \frac{2n-1}{2}\Delta^2 y_0 + \frac{3n^2-6n+2}{6}\Delta^3 y_0 + \frac{2n^3-9n^2+11n-3}{12}\Delta^4 y_0 + \cdots\right)$$

where $n = \dfrac{x-x_0}{h} = \dfrac{0.04-0.01}{0.01} = 3$

$$y'(0.04) = \frac{1}{0.01}\left[0.0024 + \frac{(6-1)}{2}(0) + \frac{(3\times 3^2-6\times 3+2)}{6}(0.0001) + \frac{(2\times 3^3-9\times 3^2+11\times 3-3)}{12}\times(-0.0001)\right]$$

$$y'(0.04) = 0.24.$$

(ii) Using Trapezoidal Rule evaluate $\int_1^2\int_1^2 \left(\frac{1}{x+y}\right) dx dy$ taking 4 sub intervals.

Solution:

Let $h = k = 0.5$

y \ x	1	1.5	2
1	0.5	$\boxed{0.4}$	0.33
1.5	$\boxed{0.4}$	0.33	$\boxed{0.285}$
2	0.33	$\boxed{0.285}$	0.25

By Trapezoidal Rule,

$$\int_1^2\int_1^2 \frac{1}{x+y}\, dx dy = \frac{0.5 \times 0.5}{4}\Big(.5 + .33 + .33 + .25 + 2(0.4 + 0.4 + 0.285 + 0.285) + 4(0.33)\Big).$$

$$= 0.3418$$

(OR)

(b) (i) For the given data

x :	1	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$:	7.989	8.403	8.781	9.129	9.451	9.75	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at 1.1.

Solution: Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7.989				
		0.414			
1.1	8.403		-0.36		
		0.378		0.006	
1.2	8.781		0.03		-0.002
		0.348		0.004	
1.3	9.129		0.026		0.001
		0.322		0.003	
1.4	9.451		0.023		0.002
		0.299		0.005	
1.5	9.75				

By Newton's formula.

$$\frac{dy}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{2n-1}{2}\Delta^2 y_0 + \frac{3n^2-6n+2}{6}\Delta^3 y_0 \right.$$

$$\left. + \frac{2n^3-9n^2+11n-3}{12}\Delta^4 y_0 + \cdots \right)$$

$$n = \frac{x-x_0}{h} = \frac{1.1-1}{0.1} = 1$$

$$\left.\frac{dy}{dx}\right)_{x=1.1} = 4.34$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (n-1)\Delta^3 y_0 + \frac{(6n^2-18n+11)}{12}\Delta^4 y_0 + \cdots \right.$$

$$= -4.3833$$

(ii) Use three point Gauss formula to evaluate $\int_1^2 \frac{1}{x}\,dx$

<u>Solution:</u> Here $a=1$, $b=2$

Let $x = \frac{3}{2} + \frac{1}{2}t$

$dx = \frac{dt}{2}$

$$\therefore \int_1^2 \frac{1}{x}\,dx = \int_{-1}^{1} \frac{1}{\left(\frac{3}{2}+\frac{1}{2}t\right)}\frac{dt}{2} = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

$$\therefore f(t) = f(t) = \frac{1}{3+t}$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = 0.4493$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 0.2649$$

$$\therefore \int_1^2 \frac{1}{x}\,dx = 0.4493 + 0.2649$$

$$= 0.7142$$

(14) (a) Using Range - Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = xy + y^2$ $y(0) = 1$ and also find the solution at $x = 0.4$ using Milne's method.

Solution: Given $f(x,y) = xy + y^2$

$x_0 = 0, \; y_0 = 1, \; h = 0.1$

To find $y(x_1 = 0.1)$

$y_1 = y_0 + \Delta y_0, \quad \Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1$

$k_2 = hf(x_0 + h/2, y_0 + \frac{k_1}{2}) = 0.1155$

$k_3 = hf(x_0 + h/2, y_0 + \frac{k_2}{2}) = 0.11717$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.13597$

$\Delta y_0 = 0.11686$

$y_1 = 1.11686.$

To find $y_2 = y(0.2)$

$y_2 = y_1 + \Delta y_1, \quad \Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_1, y_1) = 0.1359$

$k_2 = hf(x_1 + h/2, y_1 + \frac{k_1}{2}) = 0.1581$

$k_3 = hf(x_1 + h/2, y_1 + \frac{k_2}{2}) = 0.1609$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1888$

$\Delta y_1 = 0.16045$

$y_2 = y(0.2) = 1.2773.$

To find $y(0.3)$

$$y_3 = y(0.3) = y_2 + \Delta y_2 \quad , \quad \Delta y_2 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = hf(x_2, y_2) = 0.1886$

$k_2 = hf(x_2 + h/2, \; y_2 + \frac{k_1}{2}) = 0.2224$

$k_3 = hf(x_2 + h/2, \; y_2 + \frac{k_2}{2}) = 0.2275$

$k_4 = hf(x_2 + h, \; y_2 + k_3) = 0.2715$

$\Delta y_2 = 0.22665$

$y_3 = y(0.3) = 1.5039$.

$y' = xy + y^2$

$y_1' = x_1 y_1 + y_1^2 = 1.359$

$y_2' = x_2 y_2 + y_2^2 = 1.8869$

$y_3' = x_3 y_3 + y_3^2 = 2.7129$

$y_4' = x_4 y_4 + y_4^2 = 4.0971$

Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}(2 \times 1.359 - 1.8869 + 5.4258)$$

$y_{4,p} = 1.834$

Corrector formula:

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7129 + 4.0971)$$

$$= 1.838$$

$$\boxed{y(0.4) = 1.838}$$

15. (a) By Crank-Nicholso method solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$ for two time steps..

Solution:

Given $u_{xx} = u_t$

$a = 1$, let $h = 1/4$ ~~then $k = \frac{1}{16}$~~

Let $\lambda = 1$

$\frac{k}{ah^2} = 1$

$k = ah^2 = 1 \times (\frac{1}{4})^2$

$\therefore k = \frac{1}{16}$.

By Crank-Nicholson method (when $\lambda = 1$)

$$u_{i,j+1} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_i + u_{i+1,j+1} + u_{i-1,j+1} \right)$$

$u(0,t) = 0$ $\qquad$ $u(1,t) = t$

x	0	0.25	.5	.75	1
t					$\leftarrow u(x,0) = 0$.
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$

$u_1 = \frac{1}{4}(u_2) \Rightarrow 4u_1 - u_2 = 0$ —— (1)

$u_2 = \frac{1}{4}(u_1 + u_3) \Rightarrow -u_1 + 4u_2 - u_3 = 0$ —— (2)

$u_3 = \frac{1}{4}\left(u_2 + \frac{1}{16}\right) \Rightarrow -u_2 + 4u_3 = 1/16$ —— (3)

Solving (1), (2) and (3),

$u_1 = 0.001116$, $u_2 = 0.004464$, $u_3 = 0.01674$.

$u_4 = \frac{1}{4}(u_5 + u_2) \Rightarrow -u_2 + 4u_4 - u_5 = 0$ —— (4)

$u_5 = \frac{1}{4}(u_4 + u_6 + u_1 + u_3) \Rightarrow 4u_5 - u_4 - u_6 - u_1 - u_3 = 0$ —— (5)

$u_6 = \frac{1}{4}\left(u_5 + u_2 + 1/16 + \frac{2}{16}\right) \Rightarrow 4u_6 - u_5 - u_2 = \frac{3}{16}$ —— (6)

Solving (4), (5) & (6).

$u_4 = 0.005899$, $u_5 = 0.019132$, $u_6 = 0.052771$.

(OR)

(b) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$ given $u(x,0) = 0$.

$\frac{\partial u}{\partial t}(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = 100 \sin \pi t$.

Compute $u(x,t)$ for 4 time steps with $h = 0.25$

Solution:

Given $u(0,t) = 0$, $a = 1$

$h = 0.25$, $k = \frac{h}{a} = \frac{0.25}{1} = 0.25$

By Explicit formula,

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$$

$u(0,t) = 0$; $u(1,t) = 100 \sin \pi t$

t \ x	0	0.25	.5	.75	1
0	0	0	0	0	0
0.25	0	0	0	0	70.71
.5	0	0	0	70.71	100
.75	0	~~70.71~~ 0	70.71	100	70.71
1	0	70.71	100	70.71	0

$\leftarrow u(x,0) = 0$

May/June 2007
April May 2008
April May 10
ND 11