NM DEC 2002

B.E / B.Tech. Degree Examination, December 2002.

Numerical Methods.

Part - A

1. If $g(x)$ is continuous in $[a,b]$, then under what condition the iterative method $x = g(x)$ has a unique solution in $[a,b]$?

$$|g'(x)| < 1 \quad \text{in } [a,b]$$

2. Compare Gauss-Jacobi and Gauss-Seidal methods for solving linear systems of the form $Ax = B$.

Gauss elimination is direct method

Gauss seidal method is iterative method

3. Construct a linear interpolating polynomial given the points (x_0, y_0) and (x_1, y_1).

$$y(x) = \frac{(x - x_1)}{(x_0 - x_1)} \times y_0 + \frac{(x - x_0)}{(x_1 - x_0)} \times y_1$$

4. Give inverse Lagrange's interpolation formula.

$$x = f(y) = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)} \times x_0 + \ldots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})} \times x_n.$$

5. Find the error in the derivative of $f(x) = \cos x$ by computing directly and using the approximation $f'(x) = \frac{f(x+h) - f(x-h)}{2h}$ at $x = 0.8$ choosing $h = 0.01$

Here $f(x) = \cos x$

$f'(x) = -\sin x$ & $f'(0.8) = -\sin(0.8)$

$=$

$f(0.8)$ = Given $f(x) = \dfrac{f(x+h) - f(x-h)}{2h}$

$$\therefore f(0.8) = \frac{f(.81) - f(.79)}{0.02}$$

$$= -0.000211782$$

$\therefore$ Error =

6. What are the errors in Trapezoidal and Simpson's Rules of numerical integration?

Trapezoidal Rule : $E < \dfrac{(b-a)h^2}{12} y''(\xi)$

Simpson's Rule : $E < \dfrac{-(b-a)h^4}{180} y^{iv}(\xi)$

7. What is Predictor - Correct method?

By using a formula we will predict the value which is to be found and we will apply one more formula to correct the predicted value.

Eg: Milnes method, Adams method.

8. What do we mean by saying that a method is self - starting? Not - self - starting?

Iteration method is self - starting since we can take value which lies in the given interval [a, b] in which the root lies. But Milne's method is not self - starting. Since we should know any 4 values prior to the value which we need.

9. State standard five point formula for solving $u_{xx} + u_{yy} = 0$.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i,j+1} + u_{i+1,j} + u_{i,j-1} \right].$$

10. For what value of λ, the explicit method of solving the hyperbolic equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t^2}$ is stable?. where $\lambda = \frac{c\Delta t}{\Delta x}$?

when $\frac{c\Delta t}{\Delta x} > 1$

Part - B

11. (a) Find the smallest positive root of the equation $x e^{-2x} = \frac{1}{2}\sin x$ correct to 3 decimal places using Newton - Raphson method.

Solution:

$$f(x) = x e^{-2x} - \frac{1}{2}\sin x$$

$$f'(x) = -2xe^{-2x} + e^{-2x} - \frac{1}{2}\cos x.$$

$f(0) = 0$

$f(1) = -0.2858$

$f(2) = -0.419$

$f(3) = -0.063$

$f(4) = 0.379.$

$\therefore$ The root lies between 3 & 4.

By Newton - Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n e^{-2x_n} - \frac{1}{2}\sin x_n)}{-2x_n e^{-2x_n} + e^{-2x_n} - \frac{1}{2}\cos x_n}$$

$$x_{n+1} = \frac{-2x_n^2 e^{-2x_n} - \frac{1}{2}x_n\cos x_n + \frac{1}{2}\sin x_n}{-2x_n e^{-2x_n} + e^{-2x_n} - 1/2\cos x_n}$$

Let $x_0 = 3.5$

Then $x_1 = 3.130$

$x_2 = 3.130$

$\therefore$ The root is $\boxed{x = 3.130}$

(OR)

(b) Find all eigen values of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by Jacobi method. (Apply only 3 iterations).

Solution:

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ a_{12} is the largest off-diagonal element

$$\therefore S_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right) = \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}\ \pi/2 = \pi/4$$

$$\therefore S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore B_1 = S_1^{-1} A S_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \\ 1/\sqrt{2} & & \\ 0 & & \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 & -1/\sqrt{2} \\ 0 & 3 & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 2 \end{bmatrix}$$

To find B_2.

The largest off-diagonal element in B_1 is $\frac{-1}{\sqrt{2}}$.

$$\therefore S_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right) = \frac{1}{2} \tan^{-1}(\sqrt{2})$$

$$= 27^\circ 21'$$

$$S_2 = \begin{bmatrix} 0.88808 & 0 & -0.45969 \\ 0 & 1 & 0 \\ 0.45969 & 0 & 0.88808 \end{bmatrix}$$

$$\& \quad B_2 = S_2^{-1} B_1 S_2 = \begin{bmatrix} +0.633398 & -0.32505 & 0 \\ -0.32505 & 3 & -0.62797 \\ 0 & -0.62797 & 2.36603 \end{bmatrix}$$

To find B_3:

The largest off-diagonal element is $a_{32} = -0.62797$.

$$\therefore S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{32}}{a_{33} - a_{22}} \right) = 31^\circ 36'$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.85165 & -0.52410 \\ 0 & 0.52410 & 0.85165 \end{bmatrix}$$

$$B_3 = S_3^{-1} B_2 S_3$$

$$= \begin{bmatrix} -0.63398 & -0.27683 & 0.17036 \\ -0.27682 & 2.26523 & -0.56595 \\ 0.17035 & 0.56595 & 3.10079 \end{bmatrix}$$

$\therefore$ The eigen values are 0.56595, 2.26523, 3.10074

and the eigen vectors are the columns of

$$S_1 S_2 S_3 = \begin{bmatrix} 0.5 & 0.707 & 0.5 \\ 0.707 & 0 & -0.707 \\ 0.5 & -0.707 & 0.5 \end{bmatrix}$$

12. (a) Find the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$.

x :	-1	0	1	2
y :	-1	1	3	35

Solution:

Here $h=1$, $M_0 = M_3 = 0$.

$x_0 = -1$, $x_1 = 0$ $x_2 = 1$ $x_3 = 2$

$y_0 = -1$ $y_1 = 1$ $y_2 = 3$ $y_3 = 35$

$$M_{i-1} + 4M_i + M_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$$

$i = 1 \Rightarrow$ $4m_1 + m_2 = 0$

$i = 2 \Rightarrow$ $m_1 + 4m_2 = 180$

Solving, $m_1 = -12$, $m_2 = 48$.

The cubic spline in $x_{i-1} \le x \le x_i$, is

$$y(x) = \frac{1}{6}\left[(x_i - x)^3 m_{i-1} + (x - x_{i-1}) m_i\right]$$
$$+ (x_i - x)\left(y_{i-1} - \frac{1}{6} m_{i-1}\right) + (x - x_{i-1})\left(y_i - \frac{1}{6} m_i\right).$$

$i = 1 \Rightarrow$ $$y(x) = \frac{1}{6}\left((x+1)^2(-12) + (0 - x)(-1)\right]$$
$$+ (x+1)\left(1 + \frac{12}{6}\right)$$

$$y(x) = -2x^3 - 6x^2 - 2x + 1 \quad \text{in } -1 \le x \le 0.$$

$i = 2 \Rightarrow$ $$y(x) = \frac{1}{6}\left[(x_2 - x)^3 m_1 + (x - x_1)^3 m_2\right]$$
$$+ (x_2 - x)\left(y_1 - \frac{1}{6} m_1\right) + (x - x_1)\left(y_2 - \frac{1}{6} m_2\right)$$

$$y(x) = \frac{1}{6}\left[(1-x)^3(-12) + (x-0)(48)\right]$$
$$+ (1-x)\left(1 + \frac{12}{6}\right) + (x-0)\left(3 - \frac{48}{6}\right).$$

$$y(x) = 10x^3 - 6x^2 - 2x + 1 \quad \text{in} \quad 0 \le x \le 1$$

$i = 3 \Rightarrow$ $$y(x) = \frac{1}{6}\left[(2-x)^3(48) + (x-1)^3(0)\right.$$
$$+ (2-x)\left(3 - \frac{1}{6} \times 48\right) + (x-1)\left(35 - \frac{1}{6}(0)\right)).$$

$$y(x) = -8x^3 + 48x^2 - 56x + 19 \quad \text{in } 1 \le x \le 2.$$

(b) (i) Use Newton's Backward difference formula to construct an interpolating polynomial of degree 3 for the data:

$f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475$

$f(-0.25) = 0.3349375$ & $f(0) = 1.101$. Hence find $f(-1/3)$.

Solution:

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-0.75	-0.718125			
		0.0470625		
-0.5	-0.02475		0.312625	
		0.3596875		0.09375
-0.25	$+0.3349375$		0.406375	$\nabla^3 y_n$
		0.7660625	$\nabla^2 y_n$	
0 x_n	1.101 y_n	∇y_n		

By Newton's backward difference formula,

$$y(x) = y_n + n\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \cdots$$

$$n = \frac{x - x_n}{h} = \frac{-\frac{1}{3} - 0}{0.25} = -1.3333$$

$$\therefore y\left(\tfrac{-1}{3}\right) = 1.101 + (-1.3333) \times (0.7660625) + (-1.3333)(-1.3333+1) \times (0.406375)/2 + (-1.3333)(-1.3333+1)(-1.3333+2) \times (0.09375)/6.$$

$$y\left(\tfrac{-1}{3}\right) = -0.169890426.$$

(ii) Using Lagrange's interpolation formula find $f(x)$ from the following table:

x:	1	2	3	4	7
$y = f(x)$:	2	4	8	16	128

and hence find $f(6)$

Solution: Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$, $y_0 = 2$, $y_1 = 4$, $y_2 = 8$, $y_3 = 16$, $y_4 = 128$.

By Lagrange's formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

$$= \frac{(x-2)(x-3)(x-4)(x-7)}{(1-2)(1-3)(1-4)(1-7)} \times 2 + \frac{(x-1)(x-3)(x-4)(x-7)}{(2-1)(2-3)(2-4)(2-7)}$$

$$+ \frac{(x-1)(x-2)(x-4)(x-7)}{(3-1)(3-2)(3-4)(3-7)} \times 8 + \frac{(x-1)(x-2)(x-3)(x-7)}{(4-1)(4-2)(4-3)(4-7)}$$

$$+ \frac{(x-1)(x-2)(x-3)(x-4)}{(7-1)(7-2)(7-3)(7-4)} \times 128$$

$$= \frac{1}{18}(x^4 - 16x^3 + 89x^2 - 206x + 168) - \frac{2}{5}(x^4 - 15x^3 + 75x^2 - 145x$$

$$+ (x^4 - 14x^3 + 63x^2 - 106x + 56) - \frac{8}{9}(x^4 - 13x^3 + 53x^2 - 83x + 42)$$

$$+ \frac{16}{45}(x^4 - 10x^3 + 35x^2 - 50x + 24)$$

$$y(x) = \frac{1}{90}[11x^4 - 80x^3 + 295x^2 - 310x + 264)$$

$$y(6) = 66.67.$$

18. (a)(i) Consider the following table of data:

x:	0.2	0.4	0.6	0.8	1
$f(x)$:	0.9798622	0.91771	0.8080348	0.6386093	0.3483735

Find $f'(0.25)$ using Newton's forward difference approximation, $f'(0.6)$ by Stirling's approximation and $f'(0.95)$ using Newton's backward difference approximation.

Solution:

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 0.2	y_0 0.9798622				
		Δy_0 −0.0620912			
0.4	0.917771		$\Delta^2 y_0$ −0.047645		
		−0.1097362		$\Delta^3 y_0$ −0.0120443	
0.6	0.8080348		−0.0596893		$\Delta^4 y_0$ −0.0490767 $\nabla^4 y_n$
		−0.1694255		−0.061121 $\nabla^3 y_n$	
0.8	0.6386093		−0.1208103 $\nabla^2 y_n$		
		−0.2902358 ∇y_n			
x_n 1	y_n 0.3483735				

By Newton's forward formula for derivative

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2n-1}{2}\Delta^2 y_0 + \frac{3n^2-6n+2}{6}\Delta^3 y_0 + \cdots\right]$$

$$\boxed{n = \frac{x - x_0}{h} = \frac{.25 - .2}{.2} = 0.25}$$

$$y'(0.25) = \frac{1}{0.2}\left[-0.0620912 + \frac{2(0.25)-1}{2}(-0.047645) + \frac{3(0.25)^2 - 6(0.25)+2}{6}(-0.0120443) + \cdots\right]$$

$$y'(0.25) = -0.257799$$

By Backward difference formula,

$$y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2n+1}{2}\nabla^2 y_n + \frac{3n^2+6n+2}{6}\nabla^3 y_n + \frac{2n^3+9n^2+11n+3}{}\nabla^4 y_n \cdots\right]$$

$$n = \frac{x - x_n}{h} = \frac{0.95 - 1}{.2} = -0.25$$

$$y'(0.95) = \frac{1}{0.2}\Big[-0.2902358 + \frac{2(-0.25)+1}{2}(-0.1208103)$$

$$+ \frac{3(-0.25)^2 + 6(-0.25)+2}{6}(-0.061121) +$$

$$\frac{2(-0.25)^3 + 9(-0.25)^2 + 11(-0.25)+3}{12} \times (-0.0490767)\Big]$$

$$y'(0.95) = -1.63764$$

(OR)

(b) (i) Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ using the three point Gaussian Quadrature.

<u>Solution:</u> Here $a = 0.2$, $b = 1.5$

Let $x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t = \frac{1.7 + 1.3t}{2}$

$$dx = \frac{1.3}{2}dt = 0.65\,dt$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} e^{-\left(\frac{1.7+1.3t}{2}\right)^2}(0.65)\,dt$$

$$\therefore f(t) = e^{-\frac{(1.7+1.3t)^2}{4}}$$

$$f(0) = 0.48555$$

$$f(-\sqrt{3/5}) = 0.8868$$

$$f(\sqrt{3/5}) = 0.16013.$$

$\therefore$ By Three point formula

$$\int_{0.2}^{1.5} e^{-x^2}\,dx = \frac{5}{9}\left[f(-\sqrt{3/5}) + f(\sqrt{3/5})\right] + \frac{8}{9}f(0)$$

$$= \frac{5}{9}(0.8868 + 0.16013) + \frac{8}{9}(0.48555)$$

$$= 1.01303.$$

(ii) Using Trapezoidal Rule evaluate $\int_{1.4}^{2}\int_{1}^{1.5} \ln(x+2y)\,dy\,dx$ choosing $\Delta x = 0.15$ and $\Delta y = 0.25$

Solution:

x \ y	1	0.25	0.5
1.4	1.224	1.361	1.482
1.55	1.266	1.398	1.515
1.7	1.308	1.435	1.547
1.85	1.348	1.47	1.578
2	1.386	1.504	1.609

By Trapezoidal Rule,

$$I = \frac{\cancel{hk}^{\,0.15\times .25}}{4}\Big[1.224+1.386+1.609+1.482+2(1.361+1.504+1.266+1.308+1.348+1.515+1.547+1.578)+4(1.398+1.435+1.47)\Big]$$

$$= 0.45434$$

14.(a) Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 0.5$) (i) Using Modified Euler's method find $y(0.2)$
(ii) Using 4^{th} order Range-kutta method find $y(0.4)$ & $y(0.6)$
(iii) Using Adam's method, find $y(0.8)$

Solution:

Here $f(x,y) = y - x^2 + 1$

$x_0 = 0$, $y_0 = 0.5$

(i) By Modified Euler's method,

$$y_{n+1} = y_n + h\, f\left(x_n + h/2,\ y_n + h/2\, f(x_n, y_n)\right)$$

$$n=0 \Rightarrow\ y_1 = y_0 + h\, f\left(x_0 + h/2,\ y_0 + h/2\, f(x_0, y_0)\right)$$

$$= 0.5 + 0.2\, f\left(\frac{.2}{2},\ 0.5 + \frac{0.2}{2}(1.5)\right)$$

$$y(0.2) = 0.828$$

(ii) By Range-Kutta method,

$$y_2 = y_1 + \Delta y_1, \quad \Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_1, y_1) = 0.2\, f(0.2, 0.828)$$
$$= 0.3576$$

$$k_2 = h\, f\left(x_1 + h/2,\ y_1 + \frac{k_1}{2}\right) = 0.2\, f(0.3, 1.0068)$$
$$= 0.38336$$

$$k_3 = h\, f\left(x_1 + h/2,\ y_1 + \frac{k_2}{2}\right) = 0.2\, f(0.3, 1.0968)$$
$$= 0.385946$$

$$k_4 = h f(x_1 + h,\ y_1 + k_3) = 0.2 f(0.4, 1.21394)$$
$$= 0.41079.$$

$$\Delta y_1 = 0.3845$$
$$y(0.4) = y_2 = y_1 + \Delta y_1 = 1.2125$$

& $y_3 = y_2 + \Delta y_2$, $\Delta y_2 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = h\, f(x_2, y_2) = 0.2 f(0.4, 1.2125) = 0.4105$$

$$k_2 = h\, f\left(x_2 + h/2,\ y_2 + \frac{k_1}{2}\right) = 0.2 f(0.5, 1.41775)$$
$$= 0.43355$$

$k_3 = hf(x_2 + h/2,\ y_2 + \frac{k_2}{2})$

$= 0.2 f(0.5, 1.42928)$

$= 0.43586$

$k_4 = hf(x_2 + h,\ y_2 + k_3) = 0.2 f(0.6, 1.64836)$

$= 0.45767$

$\Delta y = 0.43449$

$y_3 = y(0.6) = 1.64699.$

(iii) By Adam's Predictor - Corrector formula,

$y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$

Consider $y' = y - x^2 + 1$

$y_0' = y_0 - x_0^2 + 1 = 1.5$

$y_1' = y_1 - x_1^2 + 1 = 1.788$

$y_2' = y_2 - x_2^2 + 1 = 2.0525$

$y_3' = y_3 - x_3^2 + 1 = 2.28699$

$y_{4,p} = 1.64699 + \frac{0.2}{24}[55 \times 2.28699 - 59 \times 2.0525 + 37 \times 1.788 - 9 \times 1.5]$

$y_{4,p} = 2.12466$

$y_{4,c} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$

$y_4' = y_4 - x_4^2 + 1$
$= 2.48466$

$= 1.64699 + \frac{0.2}{24}[9 \times 2.48466 + 19 \times 2.28699 - 5 \times 2.0525 + 1.788]$

$= 2.1246$

$\therefore\ y(0.4) = 2.1246.$

(OR)

(b) Consider the second order initial value problem

$$y'' - 2y' + 2y = e^{2t}\sin t, \text{ with } y(0) = -0.4 \ \& \ y'(0) = -0.6$$

(i) Using Taylor series, find $y(0.1)$

(ii) Using 4^{th} order Runge-kutta method, find $y(0.2)$.

Solution:

Consider $y'' = 2y' - 2y + e^{2t}\sin t$, $y(0) = -0.4$ & $y'(0) = -0.6$ —— ①

Let $y' = z \Rightarrow y'' = \frac{dz}{dx}$.

| $y' = z$
| $y'' = z'$
| $y''' = z''$

$\therefore$ ① $\Rightarrow$ $\frac{dz}{dx} = 2z + 2y + e^{2t}\sin t$

& $z' = 2z + 2y + e^{2t}\sin t$

$z'' = e^{2t}\cos t + 2e^{2t}\sin t + 2z' + 2y'$

$\therefore$ By Taylor Series for y_1

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$y_0'' = e^{2t_0}\sin t_0 + 2z_0 + 2y_0 = -2.0$$

$$y_0''' = e^{2t_0}\cos t_0 + 2e^{2t_0}\sin t_0 + 2z_0' + 2z_0$$

$$= -4.2.$$

$$\therefore y_1 = -0.4 + (0.1)(-0.6) + \frac{(0.1)^2}{2!}(-2) + \frac{(0.1)^3}{3!}(-4.2) + \cdots$$

$y_1 = -0.4707$.

$y(0.1) = -0.4707$

(ii) To find $y(0.2)$ using R-K. method.

Let $\frac{dy}{dt} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{dz}{dx}$

$y_2 = y_1 + \Delta y_1$ $\qquad\qquad$ $z_2 = z_1 + \Delta z_1$

$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ $\qquad\qquad$ $\Delta z_1 = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$k_1 = hf(x_1, y_1, z_1)$
$= 0.2\, z_1 = (0.2)(-0.6)$
$= -0.12$

$l_1 = hf(x_1, y_1, z_1)$
$= 0.1(-0.6 + 0.8)$
$= 0.04$

$k_2 = hf\left(x_1 + h/2, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$
$= 0.1 f(0.15, -0.5307, -0.58)$
$= -0.116$

$l_2 = hg\left(x_1 + h/2, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$
$= 0.09237$

$k_3 = hf\left(x_p + h/2, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$
$= -0.11076$

$l_3 = hg\left(x_1 + h/2, y_1 + \frac{k_1}{2}, z_1 + l_2/2\right)$
$= 0.09682$

$k_4 = hf(x_1 + h, y_1 + k_3, z_1 + l_3)$
$= -0.100636$

$K_4 = hg(x_1 + h/2, y_1 + k_3, z_1 + l_3)$
$= 0.162944$

$\Delta y_1 = -0.11236$

$\Delta z_1 = 0.09689$

$y_2 = y_1 + \Delta y_1$
$y(0.2) = -0.51236$

$z(0.2) = -0.50311$

Q5.(a) Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$; $u(0,t) = u(2,t) = 0$, and $u(x,0) = \sin\frac{\pi x}{2}$, $0 \le x \le 2$ using $\Delta x = 0.5$, $\Delta t = 0.25$ for one time step by Crank-Nicholson method.

Solution:

Given $u_{xx} = u_t$, $h = 0.5$, $a = 1$, $k = 0.25$

$\therefore \lambda = \frac{k}{ah^2} = 1.$

By Crank-Nicholson formula,

$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$

x / t	0	0.5	1	1.5	2
0	0	.7071	1	.7071	0
0.25	0	u_1	u_2	u_3	0
.5	0	—	—		0

$\therefore \quad u_1 = \frac{1}{4}(0+u_2+0+1) \Rightarrow 4u_1 - u_2 = 1 \quad$ —(1)

$u_2 = \frac{1}{4}(u_1+u_3+.7071+.7071)$

$\Rightarrow -u_1 + 4u_2 - u_3 = 1.4142 \quad$ —(2)

$u_3 = \frac{1}{4}[u_2+1+0+0] \Rightarrow -u_2 + 4u_3 = 1 \quad$ —(3)

Solving (1), (2) & (3),

$u_1 = 0.3867, \quad u_2 = 0.5468, \quad u_3 = 0.3867$

(OR)

(b) Approximate the solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1$, $t > 0$, $u(0,t) = u(1,t) = 0$, $t > 0$ and $u(x,0) = \sin \pi x$, $0 \le x \le 1$ and $\frac{\partial u}{\partial t}(x,0) = 0$, $0 \le x \le 1$ with $\Delta x = 0.25$ and $\Delta t = 0.25$ for 3 time steps.

Solution:

$h = \frac{1}{4}, \quad k = 1/4$

By Explicit formula,

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

& $u_t = 0 \Rightarrow$ $u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2}$

$u(0,t)=0$ $\quad$ $u(1,t)=0$

t \ x	0	0.25	.5	.75	1	
0	0	0.7071	1	.7071	0	$\leftarrow u(x,0)=\sin\pi x$
0.25	0	0.5	0.7071	0.5	0	
.5	0	0	0	0	0	
0.75	0	−0.5	−0.7071	−0.5	0	
1	0	−0.7071	−1	−0.7071	0	