B.E. / B.Tech. Degree Examination, April / May 2010.

B.E. (CIVIL) – MA2264 – Numerical Methods.

Part - A.

1. Sufficient condition for convergence of an iterative metho for $f(x) = 0$; written as $x = g(x)$ is

$$|g'(x)| < 1.$$

2. Let A be given matrix.

Choose the eigen Vector $X_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then fid AX_0 = smallest eigen value $\lambda \times X_1$

⋮

3. Divided difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		3		
			5	
1	4			
				1
		18		
3	40		9	
		45		
4	85			

4. Cubic Spline :

$$f(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}\left[(x_i - x)\left(y_{i-1} - \frac{h^2}{6} M_{i-1}\right)\right.$$

$$\left. + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6} M_i\right)\right].$$

where $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$.

5. Romberg's Integration formula

$$I_h = I_2 + \frac{1}{3}(I_2 - I_1)$$

6. Given $f(x) = \frac{1}{1+x^2}$

By Gaussian two - point quadrature formula

$$\int_{-1}^{1} \frac{dx}{1+x^2} = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 0.866025 + 0.866025$$

$$= 1.73205$$

7. Given $\frac{dy}{dx} = x+y$, $y(0)=1$. $\therefore f(x,y) = x+y$

$h = 0.2$

$x_0 = 0 \rightarrow y_0 = 1$

$x_1 = .2 \rightarrow y_1 = ?$

$x_2 = .4 \rightarrow y_2 = ?$

By Euler's method,

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 1 + .2(0+1) = 1.2$$

$$y_2 = y_1 + h\, f(x_1, y_1) = 1.2 + .2(.2 + 1.2)$$

$$= 1.48$$

8. Adams Predictor - Corrector formula

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'\right]$$

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9 y_{n+1}' + 19 y_n' - 5 y_{n-1}' + y_{n-2}'\right]$$

9. Explicit formula to solve one dimensional wave equation

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}.$$

10. Standard five point formula:

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} \right]$$

Part - B

11. (a) (i) Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method.

Solution:

consider $f(x) = x^4 - x - 10 = 0$.

$$\left.\begin{aligned} f(0) &= -10 \\ f(1) &= -10 \\ f(2) &= 4 \end{aligned}\right\} \text{change of sign.}$$

$\therefore$ The root lies between 1 & 2.

$\therefore$ Let $x_0 = \frac{1+2}{2} = 1.5$

Newton's Iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^4 - x_n - 10)}{4x_n^3 - 1}$$

$$= \frac{4x_n^4 - \cancel{x_n} - x_n^4 + \cancel{x_n} + 10}{4x_n^3 - 1}$$

$$\therefore x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1}$$

$x_1 = 2.015$

$x_2 = 1.87409$

$x_3 = 1.85587$

$x_4 = 1.85558$

$x_5 = 1.85558$

$\therefore$ The one real root of the equation is $x = 1.85558$

(ii) Use Gauss-Seidal iterative method to obtain the solution of equations.

$9x - y + 2z = 9$, $x + 10y - 2z = 15$, $2x - 2y - 13z = -17$.

Solution:

$$x = \frac{1}{9}[9 + y - 2z]$$

$$y = \frac{1}{10}[15 - x + 2z]$$

$$z = \frac{1}{13}[17 + 2x - 2y]$$

Let $x = y = z = 0$.

Iteration	x	y	z
1	1	1.4	1.24615
2	0.87863	1.66137	1.18727
3	0.92076	1.64538	1.19621
4	0.91700	1.64754	1.19530
5	0.91744	1.64732	1.19540.

$\therefore$ The solution is $x = 0.917$, $y = 1.647$, $z = 1.195$.

[OR]

(b) (i) Find the inverse of the matrix by Gauss-Jordan method: $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Consider the augmented Matrix $(A \mid I)$

$$= \left[\begin{array}{ccc|ccc} 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 7 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -5 & -8 & -3 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 5 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 3 & -4 & -1 \\ 0 & 0 & 1 & \frac{-3}{2} & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

$\therefore$ The inverse of the matrix A is given by

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 3 & -4 & -1 \\ -3/2 & 5/2 & 1/2 \end{bmatrix}$$

(ii) Find the dominant eigen value and the corresponding eigen value of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an approximate eigen vector.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} \approx 4.12\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072\begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072X_7$$

$$A X_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 3.9982 X_8$$

$$A X_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Since 7^{th} & 8^{th} iterations are equal,

dominant eigen value = 4

& the corresponding eigen vector = $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

12. (a) (i) Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ & $f(4) = 12$. Hence find $f(2)$.

Solution:

x :	0	1	3	4
y :	-12	0	6	12

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$

$y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$

By Lagrange's formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \times (-12) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 12$$

$$= -\frac{1}{12}(x^3 - 8x^2 + 19x - 12)(-12) - \frac{1}{6}(x^3 - 5x^2 + 4x) \times 6 + \frac{1}{12}(x^3 - 4x^2 + 3x) \times 12$$

$$Ax_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 3.9982 x_8$$

$$Ax_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Since 7^{th} & 8^{th} iterations are equal,

dominant eigen value $= 4$

& the corresponding eigen vector $= \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

12. (a) (i) Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ & $f(4) = 12$. Hence find $f(2)$.

Solution:

x :	0	1	3	4
y :	-12	0	6	12

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$

$y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$

By Lagrange's formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \times (-12) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 12$$

$$= -\frac{1}{12}(x^3 - 8x^2 + 19x - 12)(-12) - \frac{1}{6}(x^3 - 5x^2 + 4x) \times 6$$
$$+ \frac{1}{12}(x^3 - 4x^2 + 3x) \times 12$$

$\therefore f(x) = x^3 - 7x^2 + 18x - 12$

$\therefore f(2) = 2^3 - 7(2)^2 + 18 \times 12 - 12$

$\boxed{f(2) = 184}$

(ii) Find the function $f(x)$ from the following table using Newton's divided difference formula:

x:	0	1	2	4	5	7
$f(x)$:	0	0	-12	0	600	7308

Solution:

Divided difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	0					
		0				
1	0		-6			
		-12		~~8~~ 3		
2	-12		6		9	
		6		48		1
4	0		198		16	
		600		144		
5	600		918			
		3354				
7	7308					

By Newton's divided differences.

$$y = f(x) = f(x_0) + (x - x_0)\Delta f(x) + (x - x_0)(x - x_1)\Delta^2 f(x) + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 f(x) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\Delta^5 f(x)$$

$$= \cancel{0} + \cancel{(x-0)(0)} + (x-0)(x-1)(-6) + (x-0)(x-1)(x-2) \times 3 + (x-0)(x-1)(x-2)(x-4)\,9 + (x-0)(x-1)(x-2)(x-4)(x-5) \times 1$$

$$= -6x^2 + 6x + 3x^3 - 9x^2 + 6x + 9x^4 - 63x^3 + 126x^2 - 72x + x^5 - 12x^4 + 49x^3 - 78x^2 + 40x$$

$$f(x) = x^5 - 3x^4 - 11x^3 - \overset{33}{\cancel{81}}x^2 - 20x$$

(OR)

(b) (i) If $f(0)=1$, $f(1)=2$, $f(2)=33$ and $f(3)=244$. Find a cubic spline approximation, assuming $M(0)=M(3)=0$. Also, find $f(2.5)$.

Solution:

Here $h=1$ & $n=3$.

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}\left[y_{i-1}+2y_i+y_{i+1}\right], \quad i=1,2.$$

$$\therefore M_0+4M_1+M_2=6(y_0-2y_1+y_2)$$

$$\& \quad M_1+4M_2+M_3=6(y_1-2y_2+y_3)$$

Putting $M_0=M_3=0$.

$$4M_1+M_2=180$$

$$M_1+4M_2=1080$$

On solving, $M_1=-24$, $M_2=276$.

By cubic spline approximation,

$$f(x)=\frac{1}{6h}\left[(x_i-x)^3M_{i-1}+(x-x_{i-1})^3M_i\right]+\frac{1}{h}(x_i-x)\left(y_{i-1}-\frac{h^2}{6}M_{i-1}\right)+\frac{1}{h}(x-x_{i-1})\left(y_i-\frac{h^2}{6}M_i\right)$$

Putting $i=1$,

$$f(x)=\frac{1}{6}\left((1-x)^3(0)+x^3(-24)\right]+(1-x)\left(1-\frac{1}{6}(0)\right)+(x\left(2-\frac{(-24)}{6}\right)$$

$$=-4x^3+(1-x)+6x=-4x^3+5x+1$$

Putting $i=2$,

$$f(x)=\frac{1}{6}\left[(2-x)^3(-24)+(x-1)^3(276)\right]+(2-x)\left[2-\frac{-24}{6}\right.+(x-1)\left(33-\frac{276}{6}\right)$$

$$=50x^3-162x^2+167x-53$$

Putting $i=3$,

$$f(x)=\frac{1}{6}\left((3-x)^3(276)\right)+(3-x)(33-46)+(x-2)($$

$f(x) = -46x^3 + 414x^2 - 985x + 715$

$y(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715$

$f(2.5) = 121.25$

(ii) Given the following table, find the numbers of students whose weight is between 60 and 70 lbs:

Weight (in lbs) x:	0-40	40-60	60-80	80-100	100-120
No. of students :	250	120	100	70	50

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
" 60	370		-20		
		100		-10	
" 80	470		-30		20
		70		10	
" 100	540		-20		
		50			
" 120	590				

$$n = \frac{x - x_0}{h} = \frac{70-40}{20} = 1.5$$

$$y(x) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\Delta^5 y_0 +$$

$$y(70) = 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6} \times (-10)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24} \times 20$$

$$= 250 + 180 - 7.5 + 0.625 + 0.46875$$

$y(70) = 424$

$\therefore$ No. of students whose weight is between 60 & 70

$= y(70) - y(60)$

$= 424 - 370 = 54$ students.

13. (a) (i) Given the following data, find $y'(6)$ and the maximum value of y

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

Solution:

Divided difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

By Newton's divided difference formula,

$$y(x) = f(x_0) + (x-x_0)\Delta f(x) + (x-x_0)(x-x_1)\Delta^2 f(x) + \cdots$$

$$= 4 + (x-0)\times 11 + (x-0)(x-2)\times 7 + (x-0)(x-2)(x$$

$$y(x) = x^3 + 2x^2 + 3x + 4$$

$$y'(x) = 3x^2 + 4x + 3$$

$$\therefore y'(6) = 3(6)^2 + 4(6) + 3$$

$$y'(6) = 135$$

$$x = \frac{-4 \pm \sqrt{16-36}}{6}$$

$$= \frac{-4 \pm i\sqrt{20}}{6}$$

$y(x)$ is maximum if $y'(x) = 0$.

$$\therefore 3x^2 + 4x + 3 = 0 \Rightarrow x = \frac{-4 \pm i\sqrt{20}}{6}$$

$\therefore$ The roots are imaginary.

$\therefore$ There is no extremum value in this range.

(ii) Evaluate $\int_1^{1.2}\int_1^{1.4} \frac{dx\,dy}{x+y}$ by trapezoidal formula by taking $h=k=0.1$

Solution:

Here $f(x,y) = \frac{1}{x+y}$

y \ x	1	1.1	1.2	1.3	1.4
1	0.5	.4762	.4545	.4348	.4167
1.1	.4762	.4545	.4348	.4167	.4
1.2	.4545	.4348	.4167	.4	.3846

By Trapezoidal formula,

$$\int_1^{1.2}\int_1^{1.4} \frac{dx\,dy}{x+y} = \frac{hk}{4}\Big[4 \text{ corner values} + 2\,(\text{Sum of the values in the rectangular boxes}) + 4\,(\text{remaining values})\Big]$$

$$= \frac{(0.1)(0.1)}{4}\Big[.5 + .4167 + .3846 + .4545 + 2(.4762 + .4545 + .4348 + .4762 + .4 + .4348 + .4167 + .4) + 4(.4545 + .4348 + .4167)\Big]$$

$$= 0.0025\big[1.7558 + 5.4932 + 5.224\big]$$

$$= 3.1183$$

(OR)

b)(i) Using Romberg's integration evaluate $\int_0^1 \frac{dx}{1+x^2}$

Solution

$f(x) = \frac{1}{1+x^2}$

(i) when $h = \frac{1}{2} = 0.5$

x :	0	0.5	1
$y = f(x)$:	1 (y_0)	.8 (y_1)	0.5 (y_2)

By Trapezoidal Rule,

$$I_1 = \frac{h}{2}\left[y_0 + y_2 + 2(y_1)\right]$$

$$= \frac{0.5}{2}\left[1 + .5 + 2(.8)\right]$$

$$I_1 = .7750$$

(ii) when $h = \frac{1}{4} = 0.25$

x :	0	0.25	.5	0.75	1
$y = f(x)$:	1	.9412	.8	.64	.5
	y_0	y_1	y_2	y_3	y_4

By Trapezoidal Rule,

$$I_2 = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{.5}{2}\left[(1 + .5) + 2(.9412 + .8 + .64)\right]$$

$$I_2 = 1.5656.$$

By Romberg's integration,

$$I = I_2 + \frac{1}{3}(I_2 - I_1)$$

$$= 1.5656 + \frac{1}{3}(1.5656 - .7750)$$

$$I = 1.8291$$

(iii) when $h = \frac{1}{8} = .125$

x :	0	.125	.25	.375	.5	.625	.75	.875	1
y :	1	.9846	.9412	.8767	.8	.7191	.64	.5664	.5

By Trapezoidal Rule,

$$I_3 = \frac{0.5}{2}\left[1.5 + 2(5.528)\right]$$

$$= 3.139$$

By Romberg's integration,

$$I = I_3 + \frac{1}{3}(I_3 - I_2)$$

$$= 2.0901$$

(ii) The velocity v of a particle at a distance s from a point on its path is given by the table below:

S(meters) :	0	10	20	30	40	50	60
v (m/sec) :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's 1/3 Rule & Simpson's 3/8 Rule.

Solution:

~~x~~ : ~~y~~

~~0~~

~~10~~

We know that $\frac{ds}{dt} = v$

$$ds = v\,dt$$

$$dt = \frac{1}{v}\,ds.$$

$$\int dt = \int \frac{1}{v}\,ds.$$

$x = s$:	0	10	20	30	40	50	60
$y = \frac{1}{v}$:	0.02127	0.01723	0.01563	0.01538	0.01639	0.01923	0.0263

By Simpson's $\frac{1}{3}$ Rule,

$$\int_0^{60} y\,dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{10}{3}\left[(0.02127 + 0.0263) + 2(0.01563 + 0.01639) + 4(0.01724 + 0.01538 + 0.01923)\right]$$

$$= 1.063 \text{ secs}.$$

By Simpsons 3/8 Rule,

$$\int_0^{60} y\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= 1.064 \text{ secs}.$$

14.(a) (i) Use Taylor series method to find $y(0.1)$ & $y(0.2)$ given that $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ correct to 4 decimal accuracy.

Solution:

Given $y' = 3e^x + 2y$, $x_0 = 0$, $y_0 = 0$.

$y' = 3e^x + 2y$, $y_0' = 3e^{x_0} + 2(y_0) = 3$

$y'' = 3e^x + 2y'$, $y_0'' = 3e^{x_0} + 2(y_0') = 9$

$y''' = 3e^x + 2y''$, $y_0''' = 3e^{x_0} + 2y_0'' = 21$

By Taylor series.

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$= 0 + (x)(3) + \frac{x^2}{2} \times 9 + \frac{x^3}{\cancel{6}\,2} \times \overset{7}{\cancel{21}} + \cdots$$

$$y(0.1) = 3(0.1) + \frac{(0.1)^2}{2} \times 9 + \frac{7}{2}(0.1)^3 + \cdots$$

$$\boxed{y(0.1) = 0.3485}$$

$$y(0.2) = 3(.2) + 4.5(0.1)^2 + 3.5(0.1)^3 + \cdots$$

$$\boxed{y(0.2) = 0.6485}$$

(ii) Use Milne's formula to find $y(0.4)$, given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$ & $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$ & $y(0.3) = 1.21$.

Solution: $y' = (1+x^2)y^2/2$, $h = 0.1$

$x_0 = 0 \longrightarrow y_0 = 1$ $\qquad$ $x_4 = .4 \longrightarrow y_4 = ?$

$x_1 = .1 \longrightarrow y_1 = 1.06$

$x_2 = .2 \longrightarrow y_2 = 1.12$

$x_3 = .3 \longrightarrow y_3 = 1.21$

Using Milne's Predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

Put $n=3$, $y_{4,p} = y_0 + \frac{4(0.1)}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$

Consider $y'_0 = \frac{1}{2}(1+x^2)y^2$

$$y'_1 = \tfrac{1}{2}(1+x_1^2)y_1^2 = 0.5674$$

$$y'_2 = \tfrac{1}{2}(1+x_2^2)y_2^2 = 0.6522$$

$$y'_3 = \tfrac{1}{2}(1+x_3^2)y_3^2 = 0.7979$$

$$\therefore y_{4,p} = 1 + \frac{4(.1)}{3}\left[2\times .5674 - .6522 + 2\times .7979\right]$$

$$\boxed{y_{4,p} = 1.2791}$$

Using Milne's Corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n-1}\right)$$

$$y'_4 = \tfrac{1}{2}(1+x_4^2)y_4^2$$
$$= 0.946$$

Put $n=3$, $y_{4,c} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)$

$$= 1.12 + \frac{(0.1)}{3}\left(.6522 + 4(.7979) + .946\right)$$

$$\boxed{y_{4,c} = 1.2797}$$

$$\therefore y(0.4) = y_4 = 1.2797$$

15. (a) Deduce the standard five point formula for $\nabla^2 u = 0$. Hence, solve it over the square region given by the boundary conditions as in the figure below.

	u_1	u_2
	u_3	u_4

Consider $\nabla^2 u = 0$.

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

By finite difference approximation,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0.$$

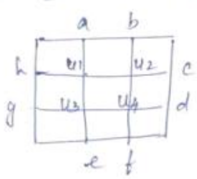
For a square mesh $h = k$.

$$\therefore\ 4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}$$

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right].$$

This is called standard fire point formula.

Consider



By standard fire point formula,

$$u_1 = \frac{1}{4}(u_2 + u_3 + a + h)$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + b + c)$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + e + g)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + d + f).$$

(ii) Obtain the Crank - Nicholson finite difference method by taking $\lambda = \frac{kc^2}{h^2} = 1$. Hence, find $u(x,t)$ in the rod for two time steps for the heat eqnation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(x,0) = \sin(\pi x)$, $u(0,t) = 0$, $u(1,t) = 0$ Take $h = 0.2$

Soln:

Consider the heat eqnation $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

By forward difference approximation,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2}$$

$$\therefore \alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = \alpha^2 \left(\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \right) \quad \text{--- (1)}$$

By Backward difference approximation,

$$\frac{\partial u}{\partial t} = \frac{u_{i,j} - u_{i,j-1}}{k}$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \frac{u_{i,j} - u_{i,j-1}}{k} = \alpha^2 \left(\frac{u_{i+1,j+1} + u_{i-1,j+1} - 2u_{i,j+1}}{h^2} \right) \quad \text{--- (2)}$$

$$(1) + (2) \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{k} = \frac{\alpha^2}{h^2} \left(u_{i+1,j} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j+1} + u_{i-1,j+1} - 2u_{i,j+1} \right)$$

$$u_{i,j+1} = u_{i,j} + \frac{\lambda}{2} \left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1} - 2u_{i,j+1} \right], \quad \lambda = \frac{k\alpha^2}{h^2}$$

When $\lambda = 1$,

$$u_{i,j+1} = \frac{1}{4} \left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j} \right]$$

Consider $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $a^2 = 1 \Rightarrow a = 1$

$h = 0.2$ & $\lambda = 1 \Rightarrow \frac{k a^2}{h^2} = 1 \Rightarrow k = h^2 = \frac{1}{25}$

$u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \sin(\pi x)$

x / t	0	.2	.4	.6	.8	1
0	0	.5878	.9511	.9511	.5878	0
$\frac{1}{25}$	0	u_1	u_2	u_3	u_4	0
$\frac{2}{25}$	0	u_5	u_6	u_7	u_8	0

By Crank-Nicholson method,

$u_1 = \frac{1}{4}(.9511 + u_2)$ $\Rightarrow 4u_1 - u_2 = .9511$

$u_2 = \frac{1}{4}(u_1 + u_3 + .5878 + .9511)$ $\quad -u_1 + 4u_2 - u_3 = 1.5$

$u_3 = \frac{1}{4}(u_2 + u_4 + .9511 + .5878)$ $\quad 4u_1 - u_2 = .9511$

$u_4 = \frac{1}{4}(u_3 + .9511)$ $\quad -\cancel{4u_1} + 16u_2 - 4u_3 = 6.1$

① $\longleftarrow 15u_2 - 4u_3 = 7.1067$

② $\longleftarrow -u_2 + 4u_3 - u_4 = 1.53$

③ $\longleftarrow -u_3 + 4u_4 = .9511$

On solving $u_2 = 0.6461$, $u_3 = 0.6461$, $u_4 = .3993$.
& $u_1 =$

$u_5 = \frac{1}{4}(u_6 + 0.6461)$

$u_6 = \frac{1}{4}(u_5 + u_7 + .6461 + \quad)$

$u_7 = \frac{1}{4}(u_6 + u_8 + .6461 + .3993)$

$u_8 = \frac{1}{4}(u_7 + .6461)$