B.E / B.Tech Degree Examination April / May 2008

MA 1251 - Numerical Methods.

Part A

1) Find an iterative formula for finding $\sqrt{N}$ where N is a real number, using Newton-Raphson formula.

Soln:

$x = \sqrt{N} \Rightarrow x^2 - N = 0$

$\therefore f(x) = x^2 - N, \quad f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

2) State the condition for the convergence of Gauss-Seidel iterative method for solving system of equations.

Soln: Coefficient matrix should be diagonally dominant.

3) Obtain a divided difference table for the following data

x	5	7	11	13	17
y	150	392	1452	2366	5202

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

4) Write the Newton's forward difference interpolation formula.

$$y(x) = y_0 + n\,\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

5) State Newton's backward difference formula to find $\left(\frac{dy}{dx}\right)_{x=x_n}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=x_n}$.

Soln:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

6 State two point Gaussian quadrature formula.

$$\int_{-1}^{1} f(t)\,dt = f(1/\sqrt{3}) + f(-1/\sqrt{3})$$

7 Find $y(1.1)$ using Euler's method from $\frac{dy}{dx} = x^2+y^2$, $y(1) = 1$

Soln. $y(1) = y(1.1) = y_0 + h\,f(x_0, y_0) = 1 + (0.1)(1^2+1^2)$

$$\boxed{y(1.1) = 1.2}$$

8 State Adam's predictor-corrector formula for solving initial value problem.

$$y_{n+1,p} = y_{n-1} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

9 State finite difference approximation for $\frac{d^2y}{dx^2}$ and state the order of truncation error.

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad \text{Error} = O(h^2).$$

⑨

(b) State explicit finite difference Scheme for one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Soln:

$$u_{i,j+1} = \lambda (u_{i+1,j} + u_{i-1,j}) + (1-2\lambda) u_{i,j}$$

Part B

11)a (i) find a real root of the equation $x^3-2x-5=0$ by the method of false position correct to 3 decimal places.

Out of syllabus.

(ii) Apply Gauss-Seidal method to solve the system of equations

$$20x+y-2z=17, \quad 3x+20y-z=-18, \quad 2x-3y+20z=25$$

Soln:

$$x = \frac{1}{20}[17-y+2z]$$

$$y = \frac{1}{20}[-18-3x+z]$$

$$z = \frac{1}{20}[25-2x+3y]$$

Let $x=y=z=0$

Iteration	x	y	z
1	0.85	-1.0275	1.0109
2	1.0025	-0.9998	0.9998
3	1	-1	1

$\therefore x=1, y=-1, z=1$

(or)

(b)(i) Using Gauss-Jordan method, find the inverse of the matrix $\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$.

Soln:

$$(A,I) = \begin{bmatrix} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 & 1/8 & 0 & 0 \\ 0 & +6 & -4 & 1/2 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/3 & 3/8 & 1/12 & 0 \\ 0 & 1 & -2/3 & 1/12 & 1/6 & 0 \\ 0 & 0 & 16/3 & 1/3 & 2/3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 0 & 1/8 & 1/4 & 1/2 \\ 0 & 0 & 0 & 3/16 & 1/8 & 3/16 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{16}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 8 \\ 1 & 2 & 3 \end{bmatrix}}$$

(ii) Find the largest eigen value and the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Soln:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7X_2$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$= 3.5714\, X_4$$

(5)

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$\therefore$ Largest eigen value = 4 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

12)
a) The following value of x and y are given

x	1	2	3	4
y	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$

Soln:

Here $M_0 = M_3 = 0$,

$$M_{i-1} - 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$

$i = 1 \Rightarrow \quad M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$

$$4M_1 + M_2 = 12$$

$i = 2 \Rightarrow \quad M_1 + 4m_2 + m_3 = 6(y_1 - 2y_2 + y_3)$

$$m_1 + 4m_2 = 18$$

On solving $m_1 = 2, m_2 = 4$.

$$S_i(x) = \frac{1}{6h}\left[(x_{i+1} - x)^3 m_i + (x - x_{i-1})^3 m_{i-1}\right] + \frac{1}{h}(x_{i+1} - x)\left(y_i - \frac{h^2}{6}M_i\right) + \frac{1}{h}(x - x_i)\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right)$$

$i = 0 \Rightarrow$

$$S_1(x) = \frac{(x_1 - x)^2 M_0}{6} + \frac{(x - x_0)^3}{6}M_1 + (x_1 - x)\left(y_0 - \frac{m_0}{6}\right) + (x - x_0)\left(y - \frac{M_2}{6}\right)$$

$$\boxed{S_1(x) = \frac{1}{3}(x^3 - 3x^2 + 5x)}, \quad 1 \le x \le 2$$

$i=1 \Rightarrow S_2(x) = \frac{(x_2-x)^3}{6h} m_1 + \frac{(x-x_1)^3 m_2}{6h} + \frac{(x_2-x)}{h}\left(y_1 - \frac{h^2}{6} m_1\right)$

$+ \frac{(x-x_1)}{h}\left(y_2 - \frac{h^2}{6} m_2\right)$

$= \frac{1}{3}(x^3 - 3x^2 + 5x), \quad 2 \le x \le 3.$

$i=2 \Rightarrow S_3(x) = \frac{(x_3-x)^3}{6h} m_2 + \frac{(x-x_2)^3}{6h} m_3$

$+ \frac{(x_3-x)}{h}\left(y_2 - \frac{h^2}{6} m_2\right) + \frac{(x-x_2)}{h}\left(y_3 - \frac{h^2}{6} m_3\right)$

$= \frac{1}{3}(-2x^3 + 24x^2 - 76x + 81), \quad 3 \le x \le 4$

$s_1(1.5) = y(1.5) = \frac{1}{3}\left[(1.5)^3 - 3(1.5)^2 + 5(1.5)\right]$

$$\boxed{y(1.5) = \frac{11}{8}}$$

(or)

3) (i) Use Lagrange's formula to fit a polynomial to the data

x :	-1	0	2	3
y :	-8	3	1	12.

Soln:

$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \dots$

$= \frac{(x-0)(x-2)(x-3)}{(-1)(-3)(-4)}(-8) + \frac{(x+1)(x-2)(x-3)}{1(-2)(-3)} \times 3$

$+ \frac{(x+1)(x-0)(x-3)}{(3)(2)(1)} 1 + \frac{(x+1)(x)(x-2)}{(4)(3)(1)} \times 12$

$y(x) = 2x^3 - 6x^2 + 3x + 3$

(ii) The following data are taken from the steam table

Temp° c :	140	150	160	170	180
Pressure kg f/cm² :	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature $t = 142^\circ$ & $t = 175^\circ$.

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Newton's forward difference formula.

$$y(x) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 \cdots$$

where $n = \frac{x - x_0}{h} = \frac{142^\circ - 140^\circ}{10^\circ} = \frac{1^\circ}{5^0} = 0.2$

$$y(x) = 3.685 + (0.2)(1.169) + \frac{(0.2)(0.2-1)}{2}(0.279)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{6}(0.047) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24}(0.002)$$

$$\boxed{y(142) = 3.898}$$

Backward difference formula

$$y(x) = y_n + n\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \cdots \quad \text{where } n = \frac{x - x_n}{h} = \frac{175 - 180}{10}$$

$$= -0.5$$

$$= 10.225 + (-0.5)(2.149) + \frac{(-0.5)(-0.5+1)}{2}(0.375)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(0.049)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(0.002)$$

$\therefore y(1.75) = 9.1$

3b)
(i) Find $f'(4)$ and $f''(4)$ from the table

x :	0	2	3	5
$f(x)$:	8	6	20	108

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	8			
2	6	-1		
3	20	14	5	1
5	108	44	10	

Newton's divided difference formula,

$y(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \dots$

$= 8 + (x-0)(-1) + (x-0)(x-2)5 + x(x-2)(x-3)\,1$

$y(x) = x^3 - 5x + 8$

$y'(x) = 3x^2 - 5 \Rightarrow y'(4) = 43$

$y''(x) = 6x \Rightarrow y''(4) = 24$

(ii) Using Romberg's method evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking $h = 0.5$ & 0.2

Soln:

$y = \frac{1}{1+x^2}$

(i) when $h = 0.5$

x	0	0.5	1
y	1	0.8	0.5

By Trapezoidal rule $\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_2) + 2y_1)\right]$

$= 0.775$

(ii) when $h=0.25$

x:	0	0.25	0.5	0.75	1
y:	1	0.9412	0.8	0.64	0.5

By Trapezoidal Rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$=\frac{0.25}{2}\left[(1+0.5)+2(0.9412+0.8+0.64)\right]$$

$$=0.7828.$$

∴ By Romberg's Integration

$$I = I_2+\frac{1}{3}(I_2-I_1) = 0.7828+\frac{1}{3}\left[0.7828-0.775\right]$$

$$\boxed{I = 0.7858}$$

b) (ii) Given that

x	1.1	1.2	1.3	1.4	1.5
y	8.403	8.781	9.29	9.451	9.75

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=1.1$.

Soln :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.1	8.403				
		0.378			
1.2	8.781		−0.03		
		0.348		0.004	
1.3	9.29		−0.026		0
		0.322		0.004	
1.4	9.451		−0.023		
		0.299			
1.5	9.75				

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004)\right]$$

$$= 3.946$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.03 - 0.004 + \frac{11}{12}(0)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.1} = -3.545$$

(ii) Evaluate $\int_0^1 \frac{dt}{1+t}$ by Gaussian formula with 3 points.

Soln:

Let $t = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)x$

$t = \frac{1}{2} + \frac{x}{2} \Rightarrow dt = \frac{dx}{2}$

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx}{2\left(1+\frac{1}{2}+\frac{x}{2}\right)} = \int_{-1}^{1} \frac{dx}{2\left(\frac{3+x}{2}\right)} = \int_{-1}^{1} \frac{dx}{3+x}$$

$\therefore f(x) = \frac{1}{3+x}$

$f(0) = \frac{1}{3} = 0.333$

$f\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 0.265$

$f\left(-\sqrt{\frac{3}{5}}\right) = 0.449$

$$\int_{-1}^{1} \frac{dx}{x+3} = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(\frac{-\sqrt{3}}{\sqrt{5}}\right) + f\left(\sqrt{3/5}\right)\right]$$

$$= 0.6931$$

14) a) Apply Runge-kutta method to find approximate value y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$. ③

Soln:

Given $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$y = y_0 + \Delta y_0$, $\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(x_0, y_0) = 0.1 f(0, 1) = 0.1$

$k_2 = h f(x_0 + h/2, y_0 + h/2) = 0.1 f(0.05, 1.05) = 0.1152$

$k_3 = h f(x_0 + h/2, y_0 + k_2/2) = 0.1 f(0.05, 0.0576) = 0.1168$

$k_4 = h f(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 1.1168) = 0.1347$

$\Delta y_0 = \frac{1}{6}[0.1 + 2(.1152) + 2(.1168) + (0.1347)]$

$= 0.1165$

$y(0.1) = y_0 + \Delta y_0 = 1 + 0.1165 = 1.1165$ $\boxed{y_1 = 1.1165}$

$y_2 = y_1 + \Delta y_1$

$k_1 = h f(x_1, y_1) = 0.1 f(0.1, 1.1165) = 0.1347$

$k_2 = h[f(x_1 + h/2, y_1 + k_1/2)] = 0.1 f(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1347}{2})$

$= 0.1551$

$k_3 = h f(x_1 + h/2, y_1 + \frac{k_2}{2}) = 0.1 f(.1 + \frac{.1}{2}, 1.1165 + \frac{.1551}{2})$

$= 0.1576$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.1 f(0.1 + 0.1, 1.1165 + 0.1576)$

$= 0.1823$

$\Delta y_1 = \frac{1}{6}[0.1347 + 2(0.1551) + 2(.1576) + 0.1823]$

$= 0.1571$

$y_2 = y_1 + 0.1571$

$\boxed{y(0.2) = 1.2736}$

(or)

b)(i) Find by Taylor series method, the values of y at $x=0.1$ and $x=0.2$ to four decimal places from $\frac{dy}{dx} = x^2y-1$, $y(0)=1$.

Soln: By Taylor series,

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \dots$$

$y' = x^2y-1$	$y_0' = x_0^2y_0 - 1 = -1$
$y'' = 2xy + x^2y'$	$y_0'' = 0$
$y''' = 2xy' + 2y + x^2y'' + 2xy'$	$y_0''' = 2$
$y^{iv} = 2y' + 2xy'' + x^2y''' + 4y' + 4xy''$	$y_0^{iv} = -6$

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \dots$$

$\boxed{y(0.1) = 0.9003}$

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \dots$$

$\boxed{y(0.2) = 0.8023}$

(ii) Given $\frac{dy}{dx} = xy + y^2$, $y(0)=1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$. Evaluate $y(0.4)$ by using Milne's method.

Soln:

Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$y' = xy + y^2$$

$$y_1' = x_1 y_1 + y_1^2 = 1.3592$$

$$y_2' = x_2 y_2 + y_2^2 = 1.8869$$

$$y_3' = x_3 y_3 + y_3^2 = 2.7162$$

$$y_{4,p} = 1 + \frac{4}{3}\times(0.1)\left(2(1.3592) - 1.8869 + 2(2.7162)\right)$$

$$\boxed{y_{4,p} = 1.8352}$$

By corrector formula, $y_4' = 4.102$

$$y_{4,c} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 1.2773 + \frac{(0.1)}{3}\left(1.8869 + 4(2.7162) + 4.102\right)$$

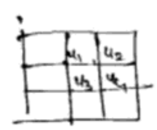
$$\boxed{y_{4,c} = 1.8391}$$

15) a)

(i) Solve the eqn $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0, y=0, x=3, y=3$ with $u=0$ on the boundary and mesh length is 1.

Soln:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2+j^2+10)$$



$$4u_1 - u_2 - u_3 = 150 \quad \text{(1)}$$

$$-u_1 + 4u_2 - u_4 = 180 \quad \text{(2)}$$

$$-u_1 + 4u_3 - u_4 = 120 \quad \text{(3)}$$

$$-u_2 - u_3 + 4u_4 = 150 \quad \text{(4)}$$

① ⇒ $4u_1 - u_2 - u_3 = 150$

②×4 ⇒ $-4u_1 + 16u_2 - 4u_4 = 720$

$15u_2 - u_3 - 4u_4 = 870$ — ⑤

② - ③ ⇒ $4u_2 - 4u_3 = 60$ — ⑥

Solving ④, ⑤ & ⑥

$u_2 = 82.5, \ u_3 = 67.5, \ u_4 = 75$

① ⇒ $u_1 = 75$.

(iii) State implicit finite difference scheme for one dimensional heat equation.

Soln:

Consider the 1D heat eqn $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ — ①

Using finite difference forward scheme

① ⇒ $\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha^2 \left(\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \right)$ — ②

By using Backward difference scheme

$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{\alpha^2}{h^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

If j is replace by j+1

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{\alpha^2}{h^2} (u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) \text{ — ③}$$

Taking average of ② & ③

$$u_{i,j+1} = u_{i,j} + \frac{\alpha^2 k}{2h^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1} - 2u_{i,j+1})$$

$$u_{i,j+1} = u_{i,j} + \frac{\lambda}{2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1} - 2u_{i,j+1}) \text{ where } \lambda = \frac{\alpha^2 k}{h^2}.$$

(8)

$$\frac{\lambda}{2} u_{i+1,j+1} + \frac{\lambda}{2} u_{i-1,j+1} - (\lambda+1) u_{i,j+1}$$

$$= -\frac{\lambda}{2} u_{i+1,j} - \frac{\lambda}{2} u_{i-1,j} + (\lambda-1) u_{i,j}$$

This is called Crank - Nicholson's implicit finite difference scheme.

b)

(i) Find the values of $u(x,t)$ satisfying the parabolic eqn $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t)=0$, $u(8,t)=0$ and $u(x,0) = 4x - \frac{x^2}{2}$ at the points $x = i$, $i = 0,1,2,\dots 7$ and $t = \frac{1}{8} j$, $j = 0,1,2,3$.

Soln : $a = 1/2$

Bender - Schmidt formula.

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$$

$t \backslash x$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0.

(ii) Solve the equation $y'' = x + y$ with conditions $y(0) = y(1) = 0$ by finite difference method taking $h = 0.25$

Soln :

$$y'' = x + y$$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} - y_k = x_k$$

$k=1 \Rightarrow \dfrac{y_0 - 2y_1 + y_2}{(0.25)^2} - y_1 = x_1$

$-2.0625\, y_1 + y_2 = 0.0156$ —— ①

$k=2 \Rightarrow \dfrac{y_1 - 2y_2 + y_3}{(0.25)^2} - y_2 = x_2$

$y_1 - 2.0625 y_2 + y_3 = 0.0313$ —— ②

$k=3 \Rightarrow \dfrac{y_2 - 2y_3 + y_4}{(0.25)^2} - y_3 = x_3$

$y_2 - 2.0625\, y_3 = 0.0469$ —— ③

solving ①, ② & ③ we get

$y_1 = -0.0349$

$y_2 = -0.0564$

$y_3 = -0.0501$.