Anna University
November/December 2010
Part-A

1. If the random variable X has the distribution function

$$F(x) = \begin{cases} 1 - e^{-\alpha x} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$$

where α is the parameter, then find $P(1 \le X \le 2)$.

Solution:

$$P(1 \le X \le 2) = F(2) - F(1) = 1 - e^{-2\alpha} - (1 - e^{-\alpha}) = e^{-\alpha} - e^{-2\alpha}$$

2. Every week the average number of wrong-number phone calls received by a certain mail order house is seven. What is the probability that they will receive two wrong calls tomorrow?

Solution:

X follows Poisson distribution with mean $\lambda = 7$

The probability function of Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,\dots, \quad \lambda > 0$$

$$P(X = 2) = \frac{e^{-7}7^2}{2!} = \frac{49}{2}e^{-7}$$

3. If there is no linear correlation between two random variables X and Y, then what can you say about the regression lines?

Solution:

If there is no linear correlation between two random variables X and Y

$$r_{xy} = 0$$

The equation of the regression lines becomes $y = \bar{y}$ and $x = \bar{x}$. The two lines are perpendicular.

4.Let the joint pdf of the random variable (X, Y) is given by $f(x, y) = 4xye^{-(x^2+y^2)}, x > 0, y > 0$. Are X and Y independent? Why or why not?

Solution:

$$f(x, y) = 4xye^{-(x^2+y^2)}, x > 0, y > 0$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{\infty} 4xye^{-(x^2+y^2)}dy = 2xe^{-x^2}, x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^{\infty} 4xye^{-(x^2+y^2)}dx = 2ye^{-y^2}, y > 0$$

$$f_X(x)f_Y(y) = 4xye^{-(x^2+y^2)} = f(x, y)$$

$\therefore$ X and Y independent.

5. Examine whether Poisson process $\{X(t)\}$ is stationary or not.

Ans:

$$P[X(t) = x] = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, x = 0,1,2, \dots$$

$$E\big(X(t)\big) = \lambda t$$

Mean of the Poisson process is a function of t.

Poisson process $\{X(t)\}$ is not stationary.

6. When is a Markov chain, called homogeneous?

Solution:

If the transition probability does not depend on the step then the Markov chain is called a homogeneous Markov chain.

7. Arrivals at a telephone booth are considered to be Poisson with an average time of 12 minutes between one arrival and the next. The length of the phone call is assumed to be distributed exponentially with mean 4 minutes. Find the average number of persons waiting in the system.

Solution:

$$\lambda = \frac{1}{12} customer/min$$

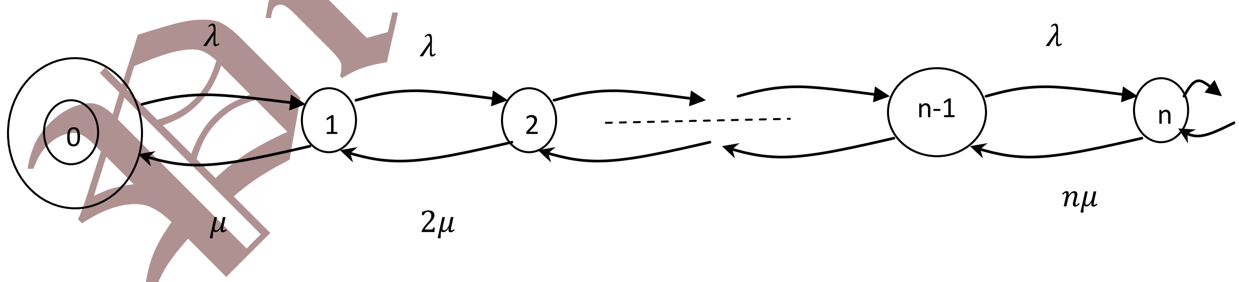
$$\mu = \frac{1}{4} customer/min$$

The average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = \frac{1}{2} \ customer \ per \ minute.$$

8. Draw the transition rate diagram of an $M/M/c$ queueing model.

Solution:



9. What do you mean by bottleneck of a network?

Ans:

The bottleneck of a network is the node with maximum traffic intensity.

10. Consider a service facility with two sequential stations with respective service rates of 3/min and 4/min. The arrival rate is 2/min. What is the average waiting time of the system, if the system could be approximated by a two stage Tandom queue?

Solution:

$$\lambda = 2\ /min, \mu_1 = 3\ /min, \mu_2 = 4/min$$

The average waiting time in the system is

$$W_s = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{3-2} + \frac{1}{4-2} = 1 + \frac{1}{2} = \frac{3}{2}\ minutes.$$

Part-B

11. (a) (i)The distribution function of a random variable X is given by $F(x) = 1 - (1+x)e^{-x}, x > 0$. Find the density function, mean and variance of X.

Solution:

$$F(x) = 1 - (1+x)e^{-x}, x > 0$$

$$f(x) = F'(x) = e^{-x} - e^{-x} + xe^{-x} = xe^{-x}, x > 0$$

$$E(x) = \int_{-\infty}^{\infty} xf(x)\,dx$$

$$= \int_0^{\infty} xxe^{-x}\,dx = \int_0^{\infty} x^2e^{-x}\,dx = \left(x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})\right)_0^{\infty}$$

$$E(x) = 2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} xx^2e^{-x}\,dx = \int_0^{\infty} x^3e^{-x}\,dx = \left(x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right)_0^{\infty}$$

$$E(x^2) = 6$$

$$var(x) = E(X^2) - (E(x))^2 = 6 - 4 = 2$$

11.(a)(ii) A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of head occurring is p. Find the value of p so that the probability that an odd number of tosses required is equal to 0.6. Can you find the value of p so that the probability is 0.5 that an odd number of tosses are required?

Solution:

Let X denote the number of tosses required to get the first head. Then X follows a geometric distribution given by $P(X = r) = q^{r-1}p, r = 1,2, \dots$

$$P(X = 1\ or\ 3\ or\ 5\ or \dots) = P(X = 2r-1) = \sum_{r=1}^{\infty} q^{2r-1-1}p = p\sum_{r=1}^{\infty} q^{2r-2}$$

$$= p(1 + q^2 + q^4 + q^6 + \cdots) = p(1 + q^2 + (q^2)^2 + (q^2)^3 + \cdots)$$

$$= p(1-q^2)^{-1} = \frac{p}{1-q^2} = \frac{p}{(1-q)(1+q)} = \frac{1}{(1+q)} \quad [Since\ 1 - q = p]$$

$$P(X = 2r-1) = \frac{1}{(1+q)} = \frac{1}{2-p} = 0.6$$

$$1.2 - 0.6p = 1 \Rightarrow 0.6\,p = 0.2 \Rightarrow p = \frac{1}{3}$$

$$when\ \frac{1}{2-p} = 0.5$$

$$1 - 0.5p = 1 \Rightarrow 0.5\,p = 0 \Rightarrow p = 0$$

$$\Rightarrow P(X = 2r-1) = \sum_{r=1}^{\infty} q^{2r-1-1} p = p \sum_{r=1}^{\infty} q^{2r-2} = 0$$

which is a contradiction to our assumption $P(X = 2r-1) = 0.5$
Therefore the value of p cannot be found out for $P(X = 2r-1) = 0.5$.

11.(b)(i) If $\boldsymbol{X}$ is a random variable with a continuous distribution function $\boldsymbol{F(x)}$, prove that $\boldsymbol{Y = F(x)}$ has a uniform distribution in $(\boldsymbol{0, 1})$. Further if

$$\boldsymbol{f(x) = \begin{cases} \frac{1}{2}(x-1), 1 \le x \le 3 \\ 0, \qquad otherwise \end{cases}}$$

Find the range of $\boldsymbol{Y}$corresponding to the range $\boldsymbol{1.1 \le X \le 2.9}$

Solution:

The distribution function of Y is given by

$$G_Y(y) = P(Y \le y) = \mathrm{P(F(X) \le y)}$$

$$= P\left(X \le F^{-1}(y)\right) = F\left(F^{-1}(y)\right) = y \qquad [Since\ F(x) = P(X \le x)]$$

$$G_Y(y) = y$$

Therefore the density function of Y is given by $g_Y(y) = G_Y'(y) = 1$

Also the range of Y is $0 \le y \le 1$. Since the range of $F(x)$ is $(0,1)$.

Therefore Y follows a uniform distribution in $(0,1)$.

$$Given\ f(x) = \begin{cases} \frac{1}{2}(x-1), 1 \le x \le 3 \\ 0, \qquad otherwise \end{cases}$$

$$Y = F(x) = \int_1^x \frac{1}{2}(x-1)dx = \frac{1}{2}\left[\frac{(x-1)^2}{2}\right]_1^x = \frac{1}{4}(x-1)^2$$

When $1.1 \le X \le 2.9, \frac{1}{4}(1.1-1)^2 \le Y \le \frac{1}{4}(2.9-1)^2 \Rightarrow 0.0025 \le Y \le 0.9025$

11.(b)(ii) The time (in hours) required to repair a machine is exponentially distributed with parameter $\boldsymbol{\lambda = \frac{1}{2}}$.

What is the probability that the repair time exceeds 2 hours?

What is the conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?

Solution:

$$f(x) = \lambda e^{-\lambda x}, x > 0$$

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}}, x > 0$$

$$P(X > 2) = \int_{2}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = \frac{1}{2}\left[-\frac{e^{-\frac{x}{2}}}{\frac{1}{2}}\right]_{2}^{\infty} = e^{-1}$$

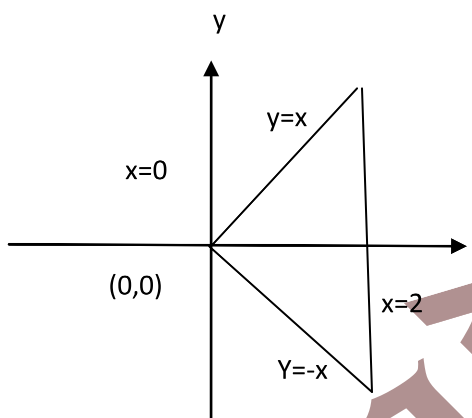
$$P(X \geq 10/X > 9) = P(X > 1)\ [Memoryless\, property]$$

$$= \int_{1}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = \frac{1}{2}\left[-\frac{e^{-\frac{x}{2}}}{\frac{1}{2}}\right]_{1}^{\infty} = e^{-\frac{1}{2}}$$

12.(a)(i) Given $f(x,y) = cx(x-y), 0 < x < 2, -x < y < x$ and 0 elsewhere. Evaluate c and find $f_X(x)$ and $f_Y(y)$.

Solution:

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$



$$\int_{0}^{2}\int_{-x}^{x} cx(x-y)dydx = 1$$

$$c\int_{0}^{2}\left[x^2 y - \frac{xy^2}{2}\right]_{-x}^{x} dx = 1$$

$$c\int_{0}^{2}\left[x^3 - \frac{x^3}{2} - \left(-x^3 - \frac{x^3}{2}\right)\right] dx = 1$$

$$c\int_{0}^{2} 2x^3 dx = 1 \Rightarrow 2c\left[\frac{x^4}{4}\right]_{0}^{2} = 1$$

$$c = \frac{1}{8}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-x}^{x} \frac{1}{8} x(x-y)dy$$

$$= \frac{2}{8}\int_{0}^{x} x^2 dy = \frac{1}{4}x^2 (y)_0^x = \frac{x^3}{4}$$

$$f_X(x) = \frac{x^3}{4}, 0 < x < 2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_{-y}^{2} \frac{1}{8} x(x-y)dx, -2 < y < 0 \\ \int_{y}^{2} \frac{1}{8} x(x-y)dx, 0 < y < 2 \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48}y^3, -2 < y < 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48}y^3, 0 < y < 2 \end{cases}$$

12.(a)(ii) Compute the correlation coefficient between X and Y using the following data:

$\boldsymbol{X}$:	**1**	**3**	**5**	**7**	**8**	**10**
$\boldsymbol{Y}$:	**8**	**12**	**15**	**17**	**18**	**20**

Solution:

X	Y	X^2	Y^2	XY
1	8	1	64	8
3	12	9	144	36
5	15	25	225	75
7	17	49	289	119
8	18	64	324	144
10	20	100	400	200
34	90	248	1446	582

$$n = 6, \sum x_i = 34, \quad \sum y_i = 90, \quad \sum x_i{}^2 = 248, \quad \sum y_i{}^2 = 1446,$$

$$\sum x_i y_i = 582$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{34}{6} = 5.67$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{90}{6} = 15$$

$$\rho = \frac{\sum x_i y_i - n\bar{x}\,\bar{y}}{\sqrt{\sum x_i{}^2 - n\bar{x}^2}\sqrt{\sum y_i{}^2 - n\bar{y}^2}}$$

$$\rho = \frac{582 - 6 \times 5.67 \times 15}{\sqrt{248 - 6 \times (5.67)^2}\sqrt{1446 - 6 \times (15)^2}}$$

$$\rho = \frac{71.7}{\sqrt{55.11}\sqrt{96}}$$

$$\rho = 0.9856$$

12.(b)(i) For two random variable X and Y with the same mean, the two regression equations are $\boldsymbol{y = ax + b}$ and $\boldsymbol{x = cy + d}$. Find the common mean, ratio of the standard deviations and also show that

$$\frac{\boldsymbol{d}}{\boldsymbol{b}} = \frac{\boldsymbol{1 - a}}{\boldsymbol{1 - c}}$$

Solution:

If μ is the common mean, the point (μ, μ) lies on $y = ax + b$ and $x = cy + d$

$$\mu = a\mu + b \Rightarrow \mu = \frac{b}{1 - a} \ldots (1)$$

$$\mu = c\mu + d \Rightarrow \mu = \frac{d}{1-c} \dots (2)$$

From (1) and (2) we get

$$\frac{b}{1-a} = \frac{d}{1-c} \Rightarrow \frac{d}{b} = \frac{1-a}{1-c}$$

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\left(\frac{b_{yx}}{b_{xy}}\right)} = \sqrt{\frac{a}{c}}$$

12.(b)(ii) If $X_1, X_2, \dots, X_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit theorem to estimate $P(120 \le S_n \le 160)$, where $S_n = X_1 + X_2 + \dots + X_n$ and $n = 75$.

Solution:

$E(X_i) = \lambda = 2$ and $var(X_i) = \lambda = 2$

By CLT, S_n follows $N(n\mu, \sigma\sqrt{n})$

S_n follows $N(150, \sqrt{150})$

$$P(120 \le S_n \le 160) = P\left(-\frac{30}{\sqrt{150}} \le \frac{S_n - 150}{\sqrt{150}} \le \frac{10}{\sqrt{150}}\right)$$

$$= P(-2.45 \le Z \le 0.85) = 0.4927 + 0.2939 = 0.7866$$

13.(a)(i) If customers arrive at a counter in accordance with a Poisson process with mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is more than 1 minute, between 1 and 2 minutes, and four minutes or less.

Solution:

The interval T between 2 consecutive arrivals follows exponential distribution with parameter $\lambda = 2$.

$$f(x) = \lambda e^{-\lambda t}, t > 0$$

$$f(x) = 2e^{-2t}, t > 0$$

$$P(T > 1) = \int_1^\infty 2e^{-2t} dt = 2\left(\frac{e^{-2t}}{-2}\right)_1^\infty = e^{-2}$$

$$P(1 < T < 2) = \int_1^2 2e^{-2t} dt = 2\left(\frac{e^{-2t}}{-2}\right)_1^2 = e^{-2} - e^{-4}$$

$$P(T \le 4) = \int_0^4 2e^{-2t} dt = 2\left(\frac{e^{-2t}}{-2}\right)_0^4 = 1 - e^{-8}$$

13.(a)(ii) An engineer analyzing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals follows a highly distorted signals, with no recognizable signal between, whereas 20 out of 23 recognizable signals follow recognizable signals, with no highly distorted signal between. Given that only highly distorted signals are not recognizable, Find the tpm and fraction of signals that are highly distorted.

Solution:

The state space of the Markov chain is

(recognizable, highly distorted)

The tpm is given by

$$P = \begin{pmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{pmatrix}$$

$$\pi P = \pi \ and \ \pi_1 + \pi_2 = 1 \ldots (1)$$

$$(\pi_1 \quad \pi_2) \begin{bmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{bmatrix} = (\pi_1 \quad \pi_2)$$

$$\frac{20}{23}\pi_1 + \frac{14}{15}\pi_2 = \pi_1 \Rightarrow \frac{3}{23}\pi_1 = \frac{14}{15}\pi_2$$

$$\frac{3}{23}\pi_1 + \frac{1}{15}\pi_2 = \pi_2 \Rightarrow \frac{3}{23}\pi_1 = \frac{14}{15}\pi_2 \Rightarrow \pi_1 = \frac{322}{45}\pi_2$$

$$\pi_1 + \pi_2 = 1 \Rightarrow \frac{322}{45}\pi_2 + \pi_2 = 1 \Rightarrow \frac{367}{45}\pi_2 = 1$$

$$\pi_2 = \frac{45}{367} = 0.123$$

$$from \ (1), we \ get \ \pi_1 = \frac{322}{367} = 0.877$$

Fraction of signals that are highly distorted is 87.7%

14.(a) If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket,

(i) Can he expect to be seated for the start of the picture?

(ii) What is the probability that he will be seated for the start of the picture?

(iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture?

Solution:

This is of type $(M/M/1):(\infty/FCFS)$.

$$\lambda = 6 \ tickets \ per \ minute$$

$$\mu = 7.5 \ seconds \ per \ ticket = \frac{60}{7.5} = 8 \ tickets \ per \ minute$$

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} \ minute$$

Expected time required to purchase and to reach the seat $= W_s + 1.5 = 2 \ minutes$.

Therefore he can just be seated for the start of the picture.

$$P(W > t) = e^{-(\mu - \lambda)t}$$

$$P\left(W \le \frac{1}{2}\right) = 1 - P\left(W > \frac{1}{2}\right) = 1 - e^{-\frac{(\mu-\lambda)}{2}} = 1 - e^{-1} = 0.63$$

$$P(W \le t) = 0.99$$

$$P(W > t) = 1 - P(W \le t) = 0.01$$

$$e^{-(\mu-\lambda)t} = 0.01 \Rightarrow -2t = \log 0.01 = -2.3 \Rightarrow t = 1.15\ minutes$$

Therefore it takes 1.15 minutes to purchase the ticket and 1.5 minutes to go to the seat.

Therefore the person must arrive at least 2.65 minutes early so as to be 99% sure of seeing the start of the picture.

14.(b) There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If the letters arrive for being typed at the rate of 15 letters per hour.

(i) What fraction of the time all typists will be busy?

(ii) What is the average number of letters waiting to be typed?

(iii) What is the average time a letter has to spend for waiting and for being typed?

Solution:

This is of type $(M/M/C):(\infty/FCFS)$

$$C = 3, \lambda = 15\ letters\ per\ hour$$

$$\mu = 6\ letters\ per\ minute$$

$$P(All\ typists\ are\ busy) = P(N > C) = \frac{\left(\frac{\lambda}{\mu}\right)^C P_0}{C!\left(1 - \frac{\lambda}{\mu c}\right)}$$

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^C}{C!\left(1 - \frac{\lambda}{\mu c}\right)}\right]^{-1}$$

$$P_0 = \left[1 + \frac{15}{6} + \frac{1}{2}\left(\frac{15}{6}\right)^2 + \frac{\left(\frac{15}{6}\right)^3}{3!\left(1 - \frac{15}{3(6)}\right)}\right]^{-1} = (22.25)^{-1} = 0.0449$$

$$P(N > C) = \frac{\left(\frac{15}{6}\right)^3}{3!\left(1 - \frac{15}{3(6)}\right)}(0.0449) = 0.7016$$

The fraction of time all the typists will be busy is 0.7016

$$L_q = \frac{1}{C.C!}\frac{\left(\frac{\lambda}{\mu}\right)^{C+1} P_0}{\left(1 - \frac{\lambda}{\mu c}\right)^2}$$

$$= \frac{1}{3.3!}\frac{\left(\frac{15}{6}\right)^{3+1}(0.0449)}{\left(1 - \frac{15}{3(6)}\right)^2} = 3.5078$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{15}\left[3.5078 + \frac{15}{6}\right] = 0.4005\ hours$$

15. (a) Derive the expected steady state system size for the single server queues with Poisson input and General service or Derive Pollaczek-Khintchine formula.

Solution:

Let n and n_1 be the number of customer in the system at times t and $t + T$, when two consecutive customers have just left the system after getting service.

Let $f(t), E(T)$ and $var(T)$ be the probability density function, mean and variance of T. Let k be the number of customers arriving in the system during the service time T.

Hence

$$n_1 = \begin{cases} k, & if\ n = 0 \\ n - 1 + k, & if\ n > 0 \end{cases}$$

Where $k = 0,1,2,3, \ldots,$ is the number of arrivals during the service time. If

$$\delta = \begin{cases} 1\ \ if\ n = 0 \\ 0\ if\ n > 1 \end{cases}$$

Then $n_1 = n - 1 + \delta + k$ ... (1)

$$E(n_1) = E(n - 1 + \delta + k) \Rightarrow E(n_1) = E(n) - 1 + E(\delta) + E(k) \qquad \ldots(2)$$

When the system has reached the steady state, the probability of the number of customers in the system will be constant. Hence

$$E(n_1) = E(n) \; and \; E(n_1^2) = E(n^2) \qquad \ldots(3)$$

Substituting (3) in (2), we get

$$-1 + E(\delta) + E(k) = 0 \Rightarrow E(\delta) = 1 - E(k) \qquad \ldots(4)$$

$$n_1^2 = (n + k - 1 + \delta)^2$$

$$n_1^2 = n^2 + (k-1)^2 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta(k-1)$$

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta k - 2\delta \ldots(5)$$

Since $\delta = \delta^2$ and n $\delta = 0$,we get

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta + 2n(k-1) + 2\delta k - 2\delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + 2\delta k - \delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)$$

$$E\big(2n(1-k)\big) = E\big(n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)\big)$$

$$2E(n)E(1-k) = E(n^2) - E(n_1^2) + E(k^2) - 2E(k) + 1 + E(\delta)E(2k-1)$$

$$2E(n)\big(1 - E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1 - E(k)\big)E(2k-1) \qquad \text{(Using (3)\& (4))}$$

$$2E(n)\big(1 - E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1 - E(k)\big)\big(2E(k) - 1\big)$$

$$E(n) = \frac{\big(E(k^2) - 2E(k) + 1 + (1 - E(k))(2E(k) - 1)\big)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-1 + 1 - E(k)\big)\big(2E(k) - 1\big)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-E(k)\big)\big(2E(k) - 1\big)}{2\big(1 - E(k)\big)} = \frac{E(k^2) + E(k) - 2E^2(k)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) - E^2(k) + E(k) - E^2(k)}{2\big(1 - E(k)\big)} = \frac{Var\,(k) + E(k) - E^2(k)}{2\big(1 - E(k)\big)} \ldots(6)$$

Since the number k of arrivals follows Poisson process with parameter λ.

$$E(k/T) = \lambda T$$

$$E(k^2/T) = \lambda^2 T^2 + \lambda T$$

$$E(k) = \int_0^\infty E(k/T)\, f(t)dt = \lambda \int_0^\infty T\, f(t)dt = \lambda E(T) \dots (7)$$

$$E(k^2) = \int_0^\infty E(k^2/T)\, f(t)dt = \int_0^\infty (\lambda^2 T^2 + \lambda T)\, f(t)dt = \lambda^2 \int_0^\infty T^2\, f(t)dt + \lambda \int_0^\infty T\, f(t)dt$$

$$E(k^2) = \lambda^2 E(T^2) + \lambda E(T) \dots (8)$$

$$Var(k) = E(k^2) - E^2(k) = \lambda^2 E(T^2) + \lambda E(T) - \lambda^2 E^2(T) = \lambda^2\big(E(T^2) - E^2(T)\big) + \lambda E(T)$$

$$Var(k) = \lambda^2 Var(T) + \lambda E(T) \dots (9)$$

Substituting (7),(8) and (9) in (6), we get

$$E(n) = \frac{\lambda^2 Var(T) + \lambda E(T) + \lambda E(T) - \lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T) - 2\lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \frac{\lambda^2\big(Var(T) + E^2(T)\big) + 2\lambda E(T)\big(1 - \lambda E(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \lambda E(T) + \frac{\lambda^2\big(Var(T) + E^2(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

15. (b) Write short notes on the following

(i) Queue networks

(ii) series Queues

(iii) Open networks

(iv) Closed networks.

Solution:

Queue networks: Queueing network is a part of organized systems. Network of service facilities where customers receive service at some or all of the facilities.

A network of queues is a group k nodes where each represents a service facility with C_i servers at node $i(i = 1,2, \dots, k)$. Customers may enter the system at one node, and after completion of service at one node may move to another node for further service and may leave the system from some other node. They may return to previously visited nodes, skip some nodes and may stay in the system for ever.

Series Queues: A special type of open queueing network called series queue.

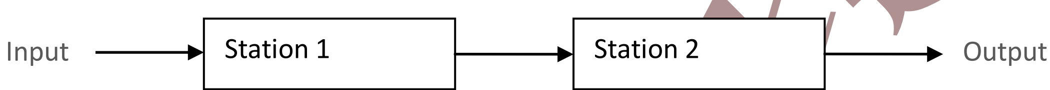
In open network there are a series of service facilities which each customer should visit in the given order before leaving the system. The nodes form a series system flow always in a single direction from node to node. Customers enter from outside only at node 1 and depart only from node k.

Example: Registration process in university, in clinic physical examination procedure.

There are two types of series queue.

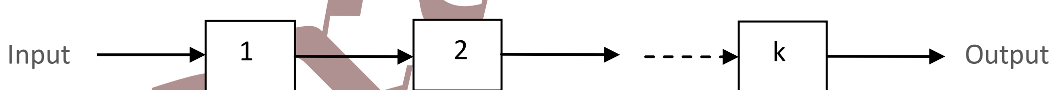
Series queues with blocking:

A queueing system with two stations in series. One server if each station and no queue allowed to form at either station. A customer entering for service has to go through station 1 and then station 2. No queues are allowed in front of station 1 and station 2.

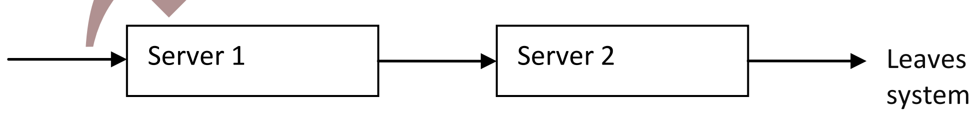


Series Queues with infinite capacity:

A System with k stations in series. Arrivals at station 1 are generated from an infinite population, according to a Poisson distribution with mean arrival rate λ. Serviced units will move successively from one station to the next until they depart from last station. Service time distribution of each station I is exponential with mean rate $\mu_i, i = 1,2, \dots, k$. There is no queue limit at any station.



Open networks : Consider a two server system in which customers arrive at a Poisson rate λ at server 1. After being served by server 1, they join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server serves only one customer at a time with server I taking exponential time with rate μ_i such a system is called a sequential system.



Consider a system of k servers customers arrive from outside the system to server i. $i = 1,2, \dots, k$ in accordance with independent Poisson process with rate r_i, they join the queue at i until their turn comes one a customer is served by server i, he then joins the queue in front of the server j, with probability P_{ij} or leaves the system with probability P_{i0}. Hence if we let λ_j be the total arrival rate of customers to server j, then

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$$

which is called flow balance equation.

Closed networks: A queueing network of k nodes is called a closed Jackson network if new customers never enter into and the existing customer never depart from the system i.e $r_i = 0$ and $P_{i0} = 0$ for all i.

The flow balance equation of this model becomes

$$\lambda_j = \sum_{i=1}^{k} \lambda_i P_{ij}$$

Jackson's open network concept can be extended when the nodes are multi server nodes.

In this case the network behaves as if each node is an independent $M/M/S$ model.