November / December 2008

Part-A

2. A continuous random variable X has a density function given by $f(x) = k(1+x), 2 < x < 5$. Find $P(x < 4)$.

Solution:

Given that $f(x)$ is a pdf

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{2}^{5} k(1+x)dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left(\left[5 + \frac{5^2}{2}\right] - \left[2 + \frac{2^2}{2}\right]\right) = 1 \Rightarrow k\left(\frac{35}{2} - 4\right) = 1$$

$$\frac{27}{2}k = 1 \Rightarrow k = \frac{2}{27}$$

$$P(x < 4) = \int_{2}^{4} k(1+x)dx = \frac{2}{27}\int_{2}^{4}(1+x)dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left(\left[5 + \frac{5^2}{2}\right] - \left[2 + \frac{2^2}{2}\right]\right)$$

$$P(x < 4) = \frac{2}{27}\left(\left[4 + \frac{4^2}{2}\right] - \left[2 + \frac{2^2}{2}\right]\right) = \frac{2}{27}(12 - 4) = \frac{16}{27}$$

3. The number of monthly breakdowns of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (a) without a breakdown (b) with only one breakdown.

Solution:

Let X denote the number of breakdown of the computer in a month. X follows a Poisson distribution with mean $\lambda = 1.8$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2, \dots$$

$(a)P(X = 0) = \dfrac{e^{-\lambda}\lambda^0}{0!} = e^{-1.8}$

$(b)P(X = 1) = \dfrac{e^{-\lambda}\lambda^1}{1!} = 1.8\, e^{-1.8}$

4. Find the distribution function of the random variable $Y = g(x)$, in terms of the distribution function of X, if it is given that

$$g(x) = \begin{cases} x - c \ \ for\ x > c \\ 0 \ \ for\ |x| \leq c \\ x + c \ \ for\ x < -c \end{cases}$$

Solution:

If $Y < 0$, $F_Y(y) = P(Y \leq y) = P(X + c \leq y) = P(X \leq y - c)$

$F_Y(y) = F_X(y - c)$

If $Y \geq 0$, $F_Y(y) = P(Y \leq y) = P(X - c \leq y) = P(X \leq y + c)$

$F_Y(y) = F_X(y + c)$

5. Define independence of two random variables X and Y, both in the discrete case and in the continuous case.

Solution:

Let (X, Y) be a two dimensional discrete random variable, then X and Y are said to be independent if

$P_{ij} = P_{i.}\ P_{.j}$

Let (X, Y) be a two dimensional continuous random variable, then X and Y are said to be independent if

$f(x, y) = f_X(x) f_Y(y)$

6. Comment on the following:

"The random variables X and Y are independent iff $cov(x, y) = 0$".

Solution:

When X and Y are independent,

$$E(xy) = E(x).E(y)$$

$$cov(x, y) = E(xy) - E(x).E(y) = 0$$

But $cov(x, y) = 0$ even if $E(xy) \neq E(x).E(y)$

Therefore the converse is not true.

If X and Y are independent then $cov(x, y) = 0$, but if $cov(x, y) = 0$ then X and Y need not be independent.

7. If $X(s, t)$ is a random process, what is the nature of $X(s, t)$ when (a) s is fixed (b) t is fixed?

Solution:

If s is fixed, $X(s,t)$ is a single time function.

If t is fixed, $X(s,t)$ is a random variable.

8. What is stochastic matrix? When is it said to be regular?

Solution:

A square matrix, in which the sum of all the elements of each row is 1, is called stochastic matrix. A stochastic matrix P is said to be regular if all the entries of P^m are positive.

9. What do you mean by transient state and steady state queueing systems?

Solution:

If the characteristics of a queueing system are independent of time or equivalently if the behaviour of the system is independent of time, the system is said to be in steady state. Otherwise it is said to be in transient state.

10. If the people arrive to purchase cinema tickets at the average rate of six per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives two minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket. Can he expect to be seated for the start of the picture?

Solution:

$$\lambda = 6/minute$$

$$\mu = \frac{1}{7.5} per\ second = 8/minute$$

Average waiting time to purchase the ticket is given by

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} minute$$

Total time taken to purchase and to reach the correct seat

$$= \frac{1}{2} + 1\frac{1}{2} = 2\ minutes$$

Hence he can just be seated for the start of the picture.

Part-B

11.(a) (ii) A random variable X has the probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find

$$(i)\ P\left(X < \frac{1}{2}\right), (ii)\ P\left(\frac{1}{4} < X < \frac{1}{2}\right), (iii) P\left(X > \frac{3}{4} \,/\, X > \frac{1}{2}\right)$$

Solution:

$$(i)\ P\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} 2x\, dx = 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{1}{4}$$

$$(ii)\ P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$$(iii)\ P\left(X > \frac{3}{4} \,/\, X > \frac{1}{2}\right) = \frac{P\left(X > \frac{3}{4}\ and\ X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \frac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$$

$$P\left(X > \frac{3}{4}\right) = \int_{\frac{3}{4}}^{1} 2x\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{3}{4}}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

$$P\left(X > \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} 2x\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(X > \frac{3}{4} \,/\, X > \frac{1}{2}\right) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}$$

11.(b) (i) If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & otherwise \end{cases}$$

(1) Find 'a' (2) Find the cumulative distribution function of X.

Solution:

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1 \Rightarrow a\left(\frac{1}{2} + 2 - 1 + 9 - \frac{9}{2} - 6 + 2\right) = 1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$F(x) = \int_0^x f(x)\,dx = \int_0^x ax\,dx = a\frac{x^2}{2} = \frac{x^2}{4}, 0 \le x \le 1$$

$$F(x) = \int_0^1 ax\,dx + \int_1^x a\,dx = a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^x = \frac{1}{2}\left(\frac{1}{2} + x - 1\right) = \frac{1}{2}\left(x - \frac{1}{2}\right), 1 \le x \le 2$$

$$F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)dx = \frac{1}{2}\left(\frac{1}{2} + 2 - 1 + 3x - \frac{x^2}{2} - 6 + 2\right)$$

$$= \frac{1}{2}\left(3x - \frac{x^2}{2} - \frac{5}{2}\right), 2 \le x \le 3$$

$$F(x) = \begin{cases} \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{2}\left(x - \frac{1}{2}\right), & 1 \le x \le 2 \\ \frac{1}{2}\left(3x - \frac{x^2}{2} - \frac{5}{2}\right), & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

11. (b) (ii) If the moments of a random variable X are defined by $E(x^r) = 0.6, r = 1, 2, 3, \ldots.$ Show that $P(X = 0) = 0.4, P(X = 1) = 0.6, P(X = 2) = 0.$

Solution:

We know that

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} e^{tx} P(x) = P(0) + e^t P(1) + \sum_{r=2}^{\infty} e^{tx} P(x) \ldots (1)$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r) = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}(0.6) = 1 + 0.6(e^t - 1) = 0.4 + 0.6e^t \ldots (2)$$

From (1) and (2), we get

$$P(0) = 0.4, P(1) = 0.6, \sum_{r=2}^{\infty} e^{tx} P(x) = 0 \Rightarrow P(x) = 0, x \geq 2$$

12.(a) (i) A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are (1) exactly 3 defectives, (2) not more than 3 defectives.

Solution:

$$P = 0.05, n = 15, q = 1 - p = 0.95$$

It follows binomial distribution, the probability mass function is given by

$P(X = x) = nC_x P^x q^{n-x}, x = 0,1,2, \ldots$

$$i)\ P(X = 3) = 15C_3(0.05)^3(0.95)^{12} = 0.0307$$

$$ii)\ P(X \leq 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$$

$$= 15C_0(0.05)^0(0.95)^{15} + 15C_1(0.05)^1(0.95)^{14} + 15C_2(0.05)^2(0.95)^{13} + 15C_3(0.05)^3(0.95)^{12}$$

$$P(X \leq 3) = 0.994$$

12. (a) (ii) A die is cast until 6 appears. What is the probability that it must be cast more than 5 times?

Solution:

It follows geometric distribution, the probability mass function is given by

$$P(X = x) = q^x p, x = 0,1,2,3, \ldots$$

$p = \frac{1}{6}, q = 1 - p = \frac{5}{6}$

$P(X > 5) = 1 - P(X \geq 5)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)]$

$= 1 - \left[\left(\frac{5}{6}\right)^0 \frac{1}{6} + \left(\frac{5}{6}\right)^1 \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6}\right]$

$$= 1 - \frac{1}{6}\left[\left(\frac{5}{6}\right)^0 + \left(\frac{5}{6}\right)^1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4 + \left(\frac{5}{6}\right)^5\right] = 0.5021$$

12. (b) (i) If X is uniformly distributed over $(-\alpha, \alpha), \alpha > 0$. Find α so that

$$(\mathbf{1})\boldsymbol{P}[\boldsymbol{X} > 1] = \frac{\mathbf{1}}{\mathbf{3}} \quad (\mathbf{2})\boldsymbol{P}[|\boldsymbol{X}| < 1] = \boldsymbol{P}[|\boldsymbol{X}| > 1]$$

Solution:

If X is uniformly distributed in $(-\alpha, \alpha)$, then its probability density function is

$$f(x) = \frac{1}{2\alpha}, -\alpha < x < \alpha$$

(1) $P[X > 1] = \frac{1}{3}$

$$P[X > x] = \int_x^\infty f(x)dx$$

$$P[X > 1] = \int_1^\alpha \frac{1}{2\alpha}dx \Rightarrow \frac{1}{2\alpha}(x)_1^\alpha = \frac{1}{3} \Rightarrow \frac{1}{2\alpha}(\alpha - 1) = \frac{1}{3} \Rightarrow 3\alpha - 3 = 2\alpha \Rightarrow \alpha = 3$$

(2) $P[|X| < 1] = P[|X| > 1] \Rightarrow P[|X| < 1] = 1 - P[|X| \leq 1] \Rightarrow P[-1 < x < 1] = 1 - P[-1 \leq x \leq 1]$

$$\int_{-1}^{1} \frac{1}{2\alpha}dx = 1 - 11\frac{1}{2\alpha}dx \Rightarrow \frac{1}{\alpha}\int_{-1}^{1} dx = 1 \Rightarrow \frac{2}{\alpha} = 1 \Rightarrow \alpha = 2$$

12.(b) (ii) Assume that mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall?

Solution:

Given $\mu = 68.22, \sigma^2 = 10.8 \Rightarrow \sigma = 3.286$

$$P(X > 6\,feet) = P(X > 72\,inches)$$

$$Z = \frac{X - \mu}{\sigma} = \frac{72 - 68.22}{3.286} = 1.1503$$

$$P(X > 72) = P(Z > 1.1503) = 0.5 - P(0 < x < 1.1503) = 0.5 - 0.3749 = 0.1251$$

Therefore there are 125 (0.1251×1000) soldiers greater than 6 feet tall.

13. (a) (i) If the joint distribution function of X and Y is given by

$$\boldsymbol{F(x,y)} = \begin{cases} \boldsymbol{(1-e^{-x})(1-e^{-y})}, & \boldsymbol{x} > 0, y > 0 \\ \boldsymbol{0}, & \boldsymbol{otherwise} \end{cases}$$

(1) Find the marginal density function of X and Y.

(2) Are X and Y independent.

(3) $P(1 < X < 3, 1 < Y < 2)$

Solution:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 (1-e^{-x})(1-e^{-y})}{\partial x \partial y} = e^{-x}e^{-y}, x > 0, y > 0$$

$$f(x,y) = \begin{cases} e^{-x}e^{-y}, & x > 0, y > 0 \\ 0, & otherwise \end{cases}$$

The marginal density function of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} e^{-x}e^{-y}dy = e^{-x}(-e^{-y})_0^{\infty} = e^{-x}, x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{\infty} e^{-x}e^{-y}dx = e^{-y}(-e^{-x})_0^{\infty} = e^{-y}, y > 0$$

$$f_X(x)f_Y(y) = e^{-x}e^{-y}, x > 0, y > 0 = f(x,y)$$

$\therefore X$ and Y are independent.

$$P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3)P(1 < Y < 2) \quad [\because \ X \text{ and } Y \text{ are independent}]$$

$$P(1 < X < 3, 1 < Y < 2) = \int_1^3 e^{-x}dx \int_1^2 e^{-y}dy = (-e^{-x})_1^3(-e^{-y})_1^2 = (e^{-1} - e^{-3})(e^{-1} - e^{-2})$$

13. (a) (ii) Find the coefficient of correlation between industrial production and export using the following data:

Production (X):	**55**	**56**	**58**	**59**	**60**	**60**	**62**
Export (Y):	**35**	**38**	**37**	**39**	**44**	**43**	**44**

Solution:

X	Y	X^2	Y^2	XY
55	35	3025	1225	1925
56	38	3136	1444	2128
58	37	3364	1369	2146
59	39	3481	1521	2301
60	44	3600	1936	2640
60	43	3600	1849	2580
62	44	3844	1936	2728
410	280	24050	11280	16448

$$n = 7, \sum x_i = 410, \quad \sum y_i = 280, \quad \sum {x_i}^2 = 24050\,, \quad \sum {y_i}^2 = 11280,$$

$$\sum x_i y_i = 16448$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{410}{7} = 58.57$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{280}{7} = 40$$

$$\rho = \frac{\sum x_i y_i - n\bar{x}\,\bar{y}}{\sqrt{\sum {x_i}^2 - n\bar{x}^2}\sqrt{\sum {y_i}^2 - n\bar{y}^2}}$$

$$\rho = \frac{16448 - 7 \times 58.57 \times 40}{\sqrt{24050 - 7 \times (58.57)^2}\sqrt{11280 - 7 \times (40)^2}}$$

$$\rho = \frac{48.4}{\sqrt{36.89}\sqrt{80}}$$

$$\rho = 0.8919$$

13. (b) (i) The life time of a certain brand of an electric bulb may be considered a random variable with mean 1200 hours and standard deviation 250 hours. Find the probability, using central limit theorem, the average life time of 60 bulbs exceeds 1250 hours.

Solution:

Let X_i represent the life time of the bulb.

$E[X_i] = 1200$ and $Var(X_i) = 250^2$

Let $\bar{X}$ denote the mean life time of 60 bulbs. By Lindeberg-Levi form of central limit theorem

$$\bar{X}\ follows\ N\left(1200, \frac{250}{\sqrt{60}}\right)$$

$$P(\bar{X} > 1250) = P\left(\frac{\bar{X} - 1200}{\frac{250}{\sqrt{60}}} > \frac{1250 - 1200}{\frac{250}{\sqrt{60}}}\right) = P\left(Z > \frac{\sqrt{60}}{5}\right) = P(Z > 1.55) = 0.0606$$

13. (b) (ii) The two lines of regression are $8x - 10y + 66 = 0, 40x - 18y + 214 = 0$. The variance of X is 9.Find (1) The mean values of X and Y. (2) Correlation coefficient between X and Y.

Solution:

Since the lines of regression passes through the mean values $\bar{x}$ and $\bar{y}$. The point $(\bar{x}\,, \bar{y})$ must satisfy the two given regression lines

$$8\,\bar{x} - 10\,\bar{y} + 66 = 0 \ldots (1)$$
$$40\,\bar{x} - 18\,\bar{y} + 214 = 0 \ldots (2)$$

Solving equations (1) and (2) we get,

$$\bar{x} = 13, \bar{y} = 17$$

$$8x - 10y + 66 = 0 \Rightarrow y = \frac{1}{10}(8x + 66) \ldots (3)$$

$$40x - 18y + 214 = 0 \Rightarrow x = \frac{1}{40}(18y - 214) \ldots (4)$$

$$b_{yx} = \frac{8}{10} = \frac{4}{5}$$

$$b_{xy} = \frac{18}{40} = \frac{9}{20}$$

$$r_{xy} = \pm\sqrt{b_{yx}\, b_{xy}} = \pm\sqrt{\frac{4}{5}\,\frac{9}{20}} = \pm 0.6$$

Since both the regression coefficients are positive $r_{xy} = 0.6$

14. (a) (i) Given a random variable Ω with density $f(w)$ and another random variable ϕ uniformly distributed in $(-\pi, \pi)$ and independent of Ω and $X(t) = a\cos(\Omega t + \phi)$, prove that $\{X(t)\}$ is a WSS.

Solution:

Given $X(t) = a\cos(\Omega t + \phi)$

$$E[X(t)] = E[a\cos(\Omega t + \phi)] = a\,E[\cos\Omega t\ \sin\phi - \sin\Omega t\cos\phi]$$

$$= a\,E[\cos\Omega t\,]E[\sin\phi] - aE[\sin\Omega t]E[\cos\phi] \qquad (\because \phi \text{ and } \Omega \text{ are independent})$$

$$= a\,E[\cos\Omega t\,]\int_{-\pi}^{\pi}\frac{1}{2\pi}\sin\phi\, d\phi - aE[\sin\Omega t]\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos\phi\, d\phi = -\frac{2}{2\pi}aE[\sin\Omega t]\int_{0}^{\pi}\cos\phi\, d\phi$$

$$\begin{pmatrix} \because \sin\phi \text{ is an odd function } \displaystyle\int_{-\pi}^{\pi} \sin\phi \, d\phi = 0 \\ \cos\phi \text{ is an odd function } \displaystyle\int_{-\pi}^{\pi} \sin\phi \, d\phi = 2\int_{0}^{\pi} \cos\phi \, d\phi \end{pmatrix}$$

$$= -\frac{1}{\pi} aE[\sin \Omega t][\sin \ \phi]_0^{\pi} = 0$$

$$E[X(t)] = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E[a\cos(\Omega t + \phi)\, a\cos(\Omega(\mathrm{t}+\tau) + \phi)]$$

$$= a^2 E\left[\frac{\cos(\Omega t + \phi + \Omega(\mathrm{t}+\tau) + \phi) + \cos(\Omega t + \phi - \Omega(\mathrm{t}+\tau) - \phi)}{2}\right]$$

$$= \frac{a^2}{2} E[\cos(2\Omega t + 2\phi + \Omega\tau)] + \frac{a^2}{2} E[\cos \Omega\tau]$$

$$= 0 + \frac{a^2}{2}\int_{-\infty}^{\infty} f(w)\cos\Omega\tau \, d\mathrm{w} = \frac{a^2}{2}\cos\Omega\tau \int_{-\infty}^{\infty} f(w)\, d\mathrm{w} = \frac{a^2}{2}\cos\Omega\tau$$

$$R_{XX}(t, t+\tau) = \frac{a^2}{2}\cos\Omega\tau$$

$\therefore$ Mean of $X(t)$ is constant and auto correlation function is a function of τonly

Hence $\{X(t)\}$ is WSS.

14. (a)(ii) The transition probability matrix of a Markov chain $X_n, n = 1, 2, 3, \dots$ having 3 states 1, 2 and 3 is

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{0.1} & \mathbf{0.5} & \mathbf{0.4} \\ \mathbf{0.6} & \mathbf{0.2} & \mathbf{0.2} \\ \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.3} \end{pmatrix}$$

and the initial distribution is $(\mathbf{0.7}, \mathbf{0.2}, \mathbf{0.1})$

Find (1) $P(X_2 = 3)$ (2) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

Solution:

$$P^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

$$(1)\ P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3, X_0 = i) = \sum_{i=1}^{3} P(X_2 = 3/X_0 = i)\, P(X_0 = i)$$

$$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2) + P(X_2 = 3/X_0 = 3)P(X_0 = 3)$$

$$= P_{13}^{(2)} P(X_0 = 1) + P_{23}^{(2)} P(X_0 = 2) + P_{33}^{(2)} P(X_0 = 3)$$

$$P(X_2 = 3) = (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1) = .279$$

$(2)\ P(X_3 = 2, X_2 = 3, X_1 = 3\ , X_0 = 2)$

$= P(X_3 = 2/\ X_2 = 3)P(\ X_2 = 3/X_1 = 3\ )P(X_1 = 3\ /X_0 = 2)P(X_0 = 2)$

$= P_{32}^{(1)} P_{33}^{(1)} P_{23}^{(1)} P(X_0 = 2) = (0.4)(0.3)(0.2)(0.2)$

$$P(X_3 = 2, X_2 = 3, X_1 = 3\ , X_0 = 2) = 0.0048$$

14.(b)(i) A man goes to his office by car or catches the train every day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to go by car again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and went by car to work if and only if a 6 appeared. Find (1) probability that he went by train on the third day and (2) the probability he went by car to work in a long run.

Solution:

State space is $\{Train\ T,\ Car\ C\}$

The tpm is

$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$$P(going\ by\ car) = P(getting\ six\ in\ the\ toss\ of\ the\ dice) = \frac{1}{6}$$

$$P(going\ by\ train) = 1 - \frac{1}{6} = \frac{5}{6}$$

The initial state probability distribution is

$$P^{(1)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

$$(1) P^{(2)} = P^{(1)}P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

$$Probability\ that\ he\ went\ by\ train\ on\ the\ third\ day\ = \frac{11}{24}$$

(2) let $\pi = (\pi_1, \pi_2)$ be the stationary state distribution of the Markov chain.

We know that $\pi P = \pi$ and $\pi_1 + \pi_2 = 1 \dots (1)$

$$(\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$0\,\pi_1 + \frac{1}{2}\pi_2 = \pi_1 \Rightarrow \pi_2 = 2\pi_1 \dots (2)$$

$$1\,\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \Rightarrow \pi_2 = 2\pi_1 \dots (3)$$

Substituting equation (2) or (3) in (1), we get

$$\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 + 2\pi_1 = 1 \Rightarrow 3\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{3}$$

$$from\ (2), we\ get\ \pi_2 = \frac{2}{3}$$

$$The\ probability\ he\ went\ by\ car\ to\ work\ in\ a\ long\ run\ = \frac{2}{3}$$

14. (b)(ii) Write a short note on recurrent state, transient state and ergodic state.

Solution:

Recurrent state:

The state i is said to be recurrent if the return to state i is certain.

$$i.e., F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$$

Transient state:

The state i is said to be transient if the return to state i is uncertain.

$$i.e., F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} < 1$$

Ergodic state:

A non-null persistent and a periodic state is called an ergodic state.

15.(a)(i) A duplicating machine maintained for office is operated by an office assistant who earns Rs. 5 per hour. The time to complete each jobs varies according to an exponential distribution with mean 6 minutes. Assume the Poisson input with an average arrival rate of 5 jobs per hour. If an 8 hours day is used as a base determine

(1) The percentage idle time of the machine

(2) The average time a job in the system

(3) The average earning per day of the assistant.

Solution:

This is of type $(M/M/1):(\infty/FCFS)$

$\lambda = 5\ jobs\ /hour$

$\mu = \frac{1}{6}\ jobs\ per\ minute = 10\ jobs\ /\ hour$

(1) $P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{10} = 0.5$

(2) $W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu - \lambda)} = \frac{1}{(10 - 5)} = \frac{1}{5}\ hour$

(3) $L_s = \frac{\lambda}{(\mu - \lambda)} = \frac{5}{10 - 5} = 2.5\ jobs$

Average number of jobs in the system in an 8 hour day= $2.5 \times 8 = 20$

The average earning per day of the assistant $= 20 \times 5 = Rs. 100$

15.(a)(ii) A super market has two girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour,

(1) What is the probability that a customer has to wait for service?

(2) What is the expected percentage of idle time for each girl?

Solution:

This is of type $(M/M/C):(\infty/FCFS)$

$$C = 2, \lambda = 10\ customers/hour$$

$$\mu = \frac{1}{4}\ customers/minute = \frac{60}{4} = 15\ customers/hour$$

$$(1)\ P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^C}{C!\left(1-\frac{\lambda}{\mu c}\right)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{10}{15}\right)^n + \frac{\left(\frac{10}{15}\right)^2}{2!\left(1-\frac{10}{15(2)}\right)}\right]^{-1}$$

$$= \left[1 + \frac{10}{15} + \frac{\left(\frac{2}{3}\right)^2}{2!\left(\frac{2}{3}\right)}\right]^{-1} = \frac{1}{2} = 0.5$$

$$L_q = \frac{1}{C.C!}\frac{\left(\frac{\lambda}{\mu}\right)^{C+1} P_0}{\left(1-\frac{\lambda}{\mu c}\right)^2} = \frac{1}{2.2!}\frac{\left(\frac{10}{15}\right)^{2+1}(0.5)}{\left(1-\frac{10}{15(2)}\right)^2} = 0.0833\ customer$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.0833}{10} = 0.00833\ hour$$

$$(2)\ P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{15} = 0.33$$

The expected percentage of idle time for each girl is 33 %

15.(b)(i) Customers arrive at a one-man barber shop according to a Poisson process with a mean interarrival time of 12 minutes. Customers spend an average of 10 minutes in the barber's chair.

(1) What is the expected number of customers in the barber shop and in the queue?

(2) What is the probability that more than 3 customers are in the system?

Solution:

This is of type $(M/M/1):(\infty/FCFS)$

$$\lambda = \frac{1}{12}\ customers/minute$$

$\mu = \dfrac{1}{10}\ customers/minute$

(1) $L_s = \dfrac{\lambda}{\mu - \lambda} = \dfrac{\frac{1}{12}}{\frac{1}{10} - \frac{1}{12}} = \dfrac{10}{2} = 5\ customers$

$L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{\left(\frac{1}{12}\right)^2}{\frac{1}{10}\left(\frac{1}{10} - \frac{1}{12}\right)} = \dfrac{100}{24} = 4.17\ customers$

(2) $P[n > k] = \left(\dfrac{\lambda}{\mu}\right)^{k+1}$

$P[n > 3] = \left(\dfrac{\lambda}{\mu}\right)^{3+1} = \left(\dfrac{\frac{1}{12}}{\frac{1}{10}}\right)^4 = 0.4823$

15. (b)(ii) Derive the expected steady state system size for the single server queues with Poisson input and General service or Derive Pollaczek-Khintchine formula.

Solution:

Let n and n_1 be the number of customer in the system at times t and $t+T$, when two consecutive customers have just left the system after getting service.

Let $f(t), E(T)$ and $var(T)$ be the probability density function, mean and variance of T. Let k be the number of customers arriving in the system during the service time T.

Hence

$$n_1 = \begin{cases} k, & if\ n = 0 \\ n - 1 + k, & if\ n > 0 \end{cases}$$

Where $k = 0,1,2,3, \ldots$, is the number of arrivals during the service time. If

$$\delta = \begin{cases} 1\ if\ n = 0 \\ 0\ if\ n > 1 \end{cases}$$

Then $n_1 = n - 1 + \delta + k \qquad \ldots (1)$

$$E(n_1) = E(n - 1 + \delta + k) \Rightarrow E(n_1) = E(n) - 1 + E(\delta) + E(k) \qquad \ldots (2)$$

When the system has reached the steady state, the probability of the number of customers in the system will be constant. Hence

$$E(n_1) = E(n) \text{ and } E(n_1^2) = E(n^2) \qquad \dots (3)$$

Substituting (3) in (2), we get

$$-1 + E(\delta) + E(k) = 0 \Rightarrow E(\delta) = 1 - E(k) \qquad \dots (4)$$

$$n_1^2 = (n + k - 1 + \delta)^2$$

$$n_1^2 = n^2 + (k-1)^2 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta(k-1)$$

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta k - 2\delta \dots (5)$$

Since $\delta = \delta^2$ and n $\delta = 0$,we get

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta + 2n(k-1) + 2\delta k - 2\delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + 2\delta k - \delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)$$

$$E\big(2n(1-k)\big) = E\big(n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)\big)$$

$$2E(n)E(1-k) = E(n^2) - E(n_1^2) + E(k^2) - 2E(k) + 1 + E(\delta)E(2k-1)$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)E(2k-1) \qquad \text{(Using (3)\& (4))}$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)(2E(k)-1)$$

$$E(n) = \frac{\big(E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)(2E(k)-1)\big)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-1 + 1 - E(k)\big)(2E(k)-1)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-E(k)\big)(2E(k)-1)}{2\big(1-E(k)\big)} = \frac{E(k^2) + E(k) - 2E^2(k)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) - E^2(k) + E(k) - E^2(k)}{2\big(1-E(k)\big)} = \frac{Var\,(k) + E(k) - E^2(k)}{2\big(1-E(k)\big)} \dots (6)$$

Since the number k of arrivals follows Poisson process with parameter λ.

$$E(k/T) = \lambda T$$

$$E(k^2/T) = \lambda^2 T^2 + \lambda T$$

$$E(k) = \int_0^\infty E(k/T)\, f(t)dt = \lambda \int_0^\infty T\, f(t)dt = \lambda E(T) \dots (7)$$

$$E(k^2) = \int_0^\infty E(k^2/T)\, f(t)dt = \int_0^\infty (\lambda^2 T^2 + \lambda T)\, f(t)dt = \lambda^2 \int_0^\infty T^2\, f(t)dt + \lambda \int_0^\infty T\, f(t)dt$$

$$E(k^2) = \lambda^2 E(T^2) + \lambda E(T) \dots (8)$$

$$Var(k) = E(k^2) - E^2(k) = \lambda^2 E(T^2) + \lambda E(T) - \lambda^2 E^2(T) = \lambda^2\big(E(T^2) - E^2(T)\big) + \lambda E(T)$$

$$Var(k) = \lambda^2 Var(T) + \lambda E(T) \dots (9)$$

Substituting (7),(8) and (9) in (6), we get

$$E(n) = \frac{\lambda^2 Var(T) + \lambda E(T) + \lambda E(T) - \lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T) - 2\lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \frac{\lambda^2\big(Var(T) + E^2(T)\big) + 2\lambda E(T)\big(1 - \lambda E(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \lambda E(T) + \frac{\lambda^2\big(Var(T) + E^2(T)\big)}{2\big(1 - \lambda E(T)\big)}$$