B.E/B.Tech DEGREE EXAMINATION, NOVEMBER/DECEMBER 2012. (1)

MATHEMATICS - II MA2161.

(Regulation 2008)

PART A – (10 × 2 = 20 marks).

1. Find the Wronskian of y_1, y_2 of $y'' - 2y' + y = e^x \log x$.

The complementary function is the solution of $y'' - 2y' + y = 0$.

$C.F = (C_1 x + C_2) e^x$

$f_1 = y_1 = xe^x$, y_2 (or f_2) $= e^x$ and $X = e^x \log x$.

$y_1' = xe^x + e^x$, $y_2' = e^x$

$\therefore$ Wronskian $W = f_1 f_2' - f_1' f_2 = xe^{2x} - (xe^x + e^x)e^x = -e^{2x}$.

2. Find the particular integral of $(D^2 - 4D + 4)y = 2^x$.

Given $(D^2 - 4D + 4)y = 2^x$

$= e^{\log 2^x} = e^{x \log 2}$

Particular Integral $= \dfrac{1}{D^2 - 4D + 4} 2^x$

$$P.I = \frac{e^x}{(\log 2)^2 - 4\log 2 + 4}$$

3. Prove that $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ is irrotational.

$$\nabla \times \vec{F} = \text{curl}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\partial/\partial x(xy) - \partial/\partial z(yz)\right] + \vec{k}\left[\partial/\partial x(xz) - \partial/\partial y(yz)\right]$$

$$= \vec{i}[x - x] - \vec{j}[y - y] + \vec{k}[z - z]$$

$\nabla \times \vec{F} = \vec{0}$

$\therefore \vec{F}$ is irrotational.

(2)

4. State Gauss divergence theorem.

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

i.e., $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$.

5. Show that the function $f(z)=\bar{z}$ is nowhere differentiable.

Given $f(z)=\bar{z}$ — since $z=x+iy$, $\bar{z}=x-iy$

$u+iv = x-iy$ — $f(z)=u+iv$.

$v=-y$

$u=x$, $\frac{\partial u}{\partial x}=1$, $\frac{\partial u}{\partial y}=0$ | $\frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=-1$.

At all point (x,y), $u_x=1$ and $v_y=-1$. Hence C-R equations not satisfied anywhere. Hence $f(z)=\bar{z}$ is nowhere differentiable

6. Find the map of the circle $|z|=3$ under the transformation $w=2z$

Given $|z|=3 \Rightarrow x^2+y^2=9$ —(1).

Given $w=2z$

$u+iv=2(x+iy)$ $\therefore u=2x$, $v=2y \Rightarrow x=\frac{u}{2}$, $y=\frac{v}{2}$. —(2)

Substituting (2) in (1), we get

$(u/2)^2+(v/2)^2=9$

$u^2+v^2=36$ which is a circle in w-plane whose centre is at (0,0) and radius is $\sqrt{36}=6$.

Hence $|z|=3$ is transformed into $|w|=6$.

7. Evaluate $\int_c \frac{z\,dz}{(z-1)(z-2)}$, where c is the circle $|z|=1/2$.

$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2}$

$1=A(z-2)+B(z-1)$. Putting $z=1$, $A=-1$

Putting $z=2$, $B=1$.

$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$

$\therefore \int_C \frac{z}{(z-1)(z-2)} dz = -\int_C \frac{z}{z-1} dz + \int_C \frac{z\,dz}{z-2}$

Cauchy's integral formula is $\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$, z_0 lies on and inside c.

$\therefore \int_C \frac{z}{(z-1)(z-2)} dz = -0+0$ Since In $\int_C \frac{z}{z-1} dz$, $z_0=1$ lies outside $|z|=\frac{1}{2}$

$= 0.$

In $\int_C \frac{z}{z-2} dz$, $z_0=2$ also lies outside $|z|=\frac{1}{2}$.

8. If $f(z) = \frac{-1}{z-1} - 2\left[1+(z-1)+(z-1)^2+\ldots\right]$, find the residue of $f(z)$ at $z=1$.

The coeff of $\frac{1}{z-a}$ is the Laurent's expansion of $f(z)$ about 'a' is called the residue of $f(z)$ at $z=a$.

$\therefore$ Residue of $f(z)$ is coeff of $\frac{1}{z-1}$ i.e $\underline{\underline{-1}}$.

9. Is the linearity property applicable to $L\left[\frac{1-\cos t}{t}\right]$? Reason out.

Yes. $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]\, ds = \int_s^\infty \{L[1] - L[\cos t]\}\, ds$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds = \left[\log s - \tfrac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty = \left[\log \frac{1}{\sqrt{1+1/s^2}}\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{\sqrt{1+1/s^2}}\right) = 0 - \log \frac{1}{\sqrt{\frac{s^2+1}{s^2}}}$$

$$= -\log \frac{s}{\sqrt{s^2+1}}$$

$$L\left[\frac{1-\cos t}{t}\right] = \log \frac{\sqrt{s^2+1}}{s}$$

(4)

10. Find the inverse laplace transform of $\frac{1}{(s+1)(s+2)}$.

$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s+2}\right]$

Using Convolution theorem
$L^{-1}[F(s)G(s)] = f(t) * g(t)$,
$f(t) = L^{-1}[F(s)]$
$g(t) = L^{-1}[G(s)]$.
$f(t) * g(t) = \int_0^u f(u) g(t-u) du$

$= e^{-t} * e^{-2t}$

$= \int_0^t e^{-u} e^{-2(t-u)} du$

$= e^{-2t} \int_0^t e^{-u(1-2)} du = e^{-2t} \int_0^t e^{u} du = e^{-2t}\left(\frac{e^u}{1}\right)_0^t$

$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-2t}[e^t - 1] = e^{-t} - e^{-2t}$

PART - B (5 × 16 = 80 marks).

11(a) (i) Solve the equation $(D^2+5D+4)y = e^{-x}\sin 2x$

The auxiliary equation is $m^2+5m+4=0$
$m = -1, -4.$

$\therefore C.F = Ae^{-x} + Be^{-4x}.$

$P.I = \frac{1}{D^2+5D+4} e^{-x}\sin 2x = e^{-x}\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x$

$= e^{-x}\frac{1}{D^2+1-2D+5D-5+4}\sin 2x = e^{-x}\frac{1}{D^2+3D+\cancel{5}-\cancel{5}}\sin 2x$

$= e^{-x}\frac{1}{D^2+3D}\sin 2x = e^{-x}\frac{1}{-4+3D}\sin 2x = e^{-x}\frac{1}{3D-4}\sin 2x$

$= e^{-x}\frac{3D+4}{9D^2-16}\sin 2x = e^{-x}\cdot\frac{3D+4}{-36-16}\sin 2x = -\frac{e^{-x}}{52}[3D\sin 2x + 4\sin 2x]$

$P.I = \frac{e^{-x}}{-52}[6\cos 2x + 4\sin 2x]$

$\therefore y = Ae^{-x} + Be^{-4x} - \frac{e^{-x}}{52}(6\cos 2x + 4\sin 2x).$

a (ii) Solve the equation $\frac{d^2y}{dx^2}+y=\operatorname{cosec} x$ by the method of variation of parameters. (5)

Given $(D^2+1)y = \operatorname{cosec} x$.

Auxiliary equation is $m^2+1=0$ ie $m=\pm i$

$\therefore$ Complementary function $C.F = c_1\cos x + c_2 \sin x$

Particular integral $P.I = Pf_1 + Qf_2$ where $f_1=\cos x$, $f_2=\sin x$

$f_1' = -\sin x$, $f_2' = \cos x$

Now $W = f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$. $X = \operatorname{cosec} x$

$$P = -\int \frac{f_2 X}{W}dx = -\int \frac{\sin x \operatorname{cosec} x}{1}dx = -\int \sin x \cdot \frac{1}{\sin x}dx = \int dx = -x.$$

$$Q = \int \frac{f_1 X}{W}dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}dx = \int \frac{\cos x}{\sin x}dx = \log(\sin x)$$

$\therefore PI = Pf_1 + Qf_2 = -x\cos x + \log(\sin x)\cdot \sin x$

The complete solution is $y = C.F + P.I$ $\therefore y = c_1\cos x + c_2 \sin x - x\cos x + \sin x \log \sin x$.

11. b) (i) Solve $\frac{dx}{dt}+y=e^t$; $x-\frac{dy}{dt}=t$.

Given $Dx+y=e^t$ — (1) & $x-Dy=t$ — (2)

Multiplying (1) by D, $D^2x + Dy = D(e^t)$

$= e^t$

$x - Dy = t$

$(D^2+1)x = e^t + t$ — (3)

Equation (3) is a linear differential equation with constant coefficients.

$\therefore x = C.F + P.I$

To find C.F consider $(D^2+1)x=0$

The Auxiliary eqn is $m^2+1=0 \Rightarrow m=\pm i$ $\therefore C.F = c_1\cos t + c_2\sin t$

To find P.I

$$P.I = \frac{1}{D^2+1}(e^t+t) = \frac{1}{D^2+1}e^t + \frac{1}{D^2+1}(t)$$

$D=a=1$

$$= \frac{1}{2}e^t + (1+D^2)^{-1}t = \frac{e^t}{2} + (1-D^2+D^4-\cdots)t$$

$D(t)=1$

$D^2(t)=0$

$D^4(t)=0$

$$P.I = \frac{e^t}{2} + (t-0) = \frac{e^t}{2}+t$$

$\therefore$ solution of (3) is $\boxed{x = c_1\cos t + c_2\sin t + \frac{e^t}{2}+t}$

From equation (1), $y = e^t - Dx \Rightarrow y = e^t - \frac{d}{dt}\left(c_1\cos t + c_2 \sin t + \frac{e^t}{2} + t\right)$

$\therefore y = e^t + c_1 \sin t + c_2 \cos t + \frac{e^t}{2} + 1$

(b)(ii) Solve the equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}$.

Given $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 12 \log x$ — ①.

or $(x^2D^2 + xD)y = 12 \log x$ — ①.

Putting $x = e^z$ or $z = \log x$ we get $xD = D'$ & $x^2D^2 = D'(D'-1)$ — ②

Substituting ② in ①, we get $[D'(D'-1) + D']y = 12z$

i.e $D'^2 y = 12z$

$\frac{d^2y}{dz^2} = 12z \Rightarrow \frac{d}{dz}\left(\frac{dy}{dz}\right) = 12z \Rightarrow d\left(\frac{dy}{dz}\right) = 12z\, dz$ — ③

Integrating on bothsides of ③,

$\frac{dy}{dz} = 12\left(\frac{z^2}{2}\right) + c$ i.e $\frac{dy}{dz} = 6z^2 + c$ — ④ $\Rightarrow dy = (6z^2 + c)\, dz$ — ④

Again integrating ④, we get $y = 6\frac{z^3}{3} + cz + c_1$

$\therefore y = 2z^3 + cz + c_1$

$\therefore y = 2(\log x)^3 + c(\log x) + c_1$

12. (a) (i) Show that $\vec{F} = (2xy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - 2xz)\vec{k}$ is irrotational and find its scalar potential.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy - z^2 & x^2 + 2yz & y^2 - 2xz \end{vmatrix} = \vec{i}\left[\partial/\partial x\,(x^2 + 2yz) - \partial/\partial y\,(2xy - z^2)\right]$$

$$-\vec{j}\left[\partial/\partial x\,(y^2 - 2xz) - \partial/\partial z\,(2xy - z^2)\right]$$

$$+\vec{k}\left[\partial/\partial x\,(x^2 + 2yz) - \partial/\partial y\,(2xy - z^2)\right]$$

$\nabla \times \vec{F} = \vec{i}[2x - 2x] - \vec{j}[-2z + 2z] + \vec{k}[2x - 2x] = \vec{0}$

$\therefore \vec{F}$ is irrotational

To find Scalar Potential :

Since curl $\vec{F} = 0$, $\therefore \vec{F} = \nabla\phi$

$\vec{F} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$. ⑦

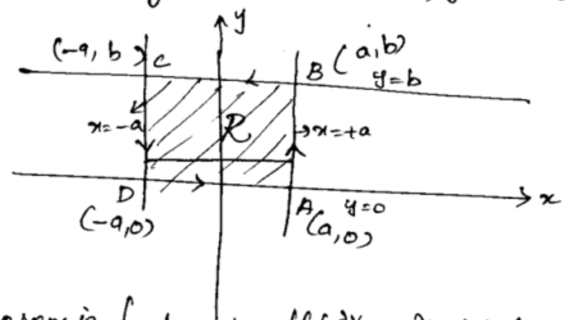
$\therefore \frac{\partial\phi}{\partial x} = 2xy - z^2 \;;\; \frac{\partial\phi}{\partial y} = x^2+2yz \;;\; \frac{\partial\phi}{\partial z} = y^2 - 2zx$.

$\int\frac{\partial\phi}{\partial x}dx = \int(2xy - z^2)dx \;;\; \phi = \int(x^2+2yz)dy \;;\; \phi = \int(y^2-2zx)dz$

$\phi = 2\frac{x^2}{2}y - z^2x + C_1 \;;\; \phi = x^2y + \frac{2y^2}{2}z + C_2 \;;\; \phi = y^2z - \frac{2z^2}{2}x + C_3$

$\therefore$ The scalar potential, $\boxed{\phi = x^2y - z^2x + y^2z + C}$

(ii) Verify Green's theorem for $\vec{V} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.



Limits R:

x: $-a$ to a

y: 0 to b

Green's theorem is $\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$

Given $\vec{V} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$\vec{V}\cdot\vec{dr} = (x^2+y^2)dx - 2xy\,dy$

$\therefore\; u = x^2+y^2 \;\&\; v = -2xy$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = -2y$

To evaluate $\int_C u\,dx + v\,dy$

$\int_C u\,dx + v\,dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} (u\,dx + v\,dy)$

(i) $\int_{AB} u\,dx + v\,dy = \int_{AB}(x^2+y^2)dx - 2xy\,dy = \int_0^b -2ay\,dy = -2a\left(\frac{y^2}{2}\right)_0^b = -a\left(\frac{b^2}{1} - 0\right)$

$x = a$, $dx = 0$ $\qquad \therefore I_1 = -\frac{ab^2}{1} \qquad\qquad = -\frac{ab^2}{1}$

(ii) $\int_{BC} (x^2+y^2)dx - 2xy\,dy = \int_a^{-a} (x^2+b^2)dx = \left(\frac{x^3}{3} + b^2x\right)_a^{-a} = \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3}+ab^2\right)$

(8)

Along BC

$y = b$

$dy = 0$

$I_2 = -\frac{2a^3}{3} - 2ab^2$

(iii) $I_3 = \int_{CD} (x^2+y^2)dx - 2xy\,dy = \int_b^0 0 - 2(-a)y\,dy = \int_b^0 2ay\,dy = \cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_b^0$

Along CD

$x = -a$

$\Rightarrow dx = 0$

$I_3 = 0 - ab^2 = -ab^2$

$I_4 = \int_{DA} (x^2+y^2)dx - 2xy\,dy = \int_{-a}^{a} x^2 dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} - \left(-\frac{a^3}{3}\right) = \frac{2a^3}{3}.$

Along DA, $y=0 \Rightarrow dy = 0$

$\therefore \int_C u\,dx + v\,dy = -\frac{ab^2}{\cancel{4}} + \left(-\cancel{\frac{2a^3}{3}} - 2ab^2\right) - ab^2 + \cancel{\frac{2a^3}{3}} = -4ab^2$ — (I)

To find $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$:

$\iint_R (V_x - U_y)dx\,dy = \iint_R [-2y - (+2y)]\,dx\,dy = \int_0^b\int_{-a}^{a} -4y\,dx\,dy$

$= -4\int_0^b y\,(x)_{-a}^{a}\,dy = -4\int_0^b y\cdot 2a\,dy = -\cancel{8}^{4}a\left(\frac{y^2}{\cancel{2}}\right)_0^b = -4ab^2$ — (II)

From (I) and (II), Green's theorem is verified.

(b) Verify Gauss's divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

Gauss's divergence theorem is $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\nabla\cdot\vec{F} = 4z - y$

$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz = \int_0^1\int_0^1 (4zx - yx)_0^1\,dy\,dz$

$= \int_0^1\int_0^1 (4z-y)\,dy\,dz = \int_0^1 \left[4z\cdot y - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right)dz$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1 (4z - \tfrac{1}{2})dz = \left(\frac{4z^2}{2} - \frac{1}{2}z\right)_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}.$$ (9)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{S_1\\x=1}} + \iint_{\substack{S_2\\x=0}} + \iint_{\substack{S_3\\y=1}} + \iint_{\substack{S_4\\y=0}} + \iint_{\substack{S_5\\z=1}} + \iint_{\substack{S_6\\z=0}} (\vec{F}\cdot\hat{n}\,ds)$$

$ds = dydz$

$$I_1 = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{i}\,dydz = \iint_{x=1} 4xz\,dydz = \int_0^1\int_0^1 4z\,dydz = 2.$$

$$I_2 = \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\\text{face}}} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i})dydz = \int_0^1\int_0^1 -4xz\,dydz = -4\int_0^1\int_0^1 0\,dydz = 0.$$

$$I_3 = \iint_{\substack{S_3\\y=1}} \vec{F}\cdot\hat{n}\,ds = \iint_{y=1} -y^2\,dxdz = -\iint_{y=1} dxdz = (\text{area of } y=1 \text{ face}) = -1.$$

$$I_4 = \iint_{\substack{S_4\\y=0}} [4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{j})\,dxdz = \iint_{y=0} 0\,dxdz = 0.$$

$$I_5 = \iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{z=1} yz\,dxdy = \int_0^1\int_0^1 y^2/2\,dy\,dx = 1/2.$$

$$I_6 = \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{z=0} -yz\,dxdy = \iint_{\substack{z=0\\\text{face}}} 0\,dxdy = 0.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 2 + 0 - 1 + 0 + 1/2 + 0 = 3/2.$$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\cdot ds$ Hence Gauss's divergence theorem is verified.

13.(a)(i) Find the bilinear transformation that maps the points $z = \infty, i, 0$ onto $w = 0, i, \infty$ respectively.

Substituting given values in Bilinear transformation $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

we get $\left(\frac{w-0}{0-i}\right)\left(\frac{\frac{w_2}{w_3}-1}{1-w/w_3}\right) = \left(\frac{i-0}{0-z}\right)\left(\frac{\frac{z}{z_1}-1}{1-z_2/z_1}\right)$

$$\frac{w}{i} = \frac{i}{z}$$

$$\boxed{w = -1/z}$$

(ii) Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$. (10)

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$\sin^2 2z = 1 - \cos^2 2z$
$= (1+\cos 2z)(1-\cos 2z)$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1+\cos 2z)(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\left[2\cos 2z - 2 - 2\cos 2z\right]}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-1}{\left(\frac{1-\cos 2z}{2}\right)} = \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,[2\sinh 2y]}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$\phi_2(z,0) = 0$.

By Milne's Thomson method, $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$f(z) = \int -\operatorname{cosec}^2 z\,dz - 0 = \cot z + C$.

(b) (i) Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = 1/z$.

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$x+iy = \frac{1}{Re^{i\phi}}$ i.e $x+iy = \frac{1}{R}e^{-i\phi}$
$= \frac{1}{R}(\cos\phi - i\sin\phi)$

$\therefore x = \frac{1}{R}\cos\phi$, $y = \frac{-1}{R}\sin\phi$

Given $x^2 - y^2 = 1 \Rightarrow \left(\frac{1}{R}\cos\phi\right)^2 - \left(\frac{-1}{R}\sin\phi\right)^2 = 1 \Rightarrow \frac{1}{R^2}(\cos^2\phi - \sin^2\phi) = 1$

i.e $\frac{\cos 2\phi}{R^2} = 1$ ie $\boxed{\cos 2\phi = R^2}$ which is lemniscate.

(b) (ii) Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of z-plane on to the upper half of w-plane. What is the image of $|z|=1$ under this transformation. (11)

(i) $w = \dfrac{z}{1-z}$

$$u+iv = \frac{x+iy}{1-(x+iy)} = \frac{x+iy}{1-x-iy} = \frac{x+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy} = \frac{x-x^2-y^2+iy}{(1-x)^2+y^2}$$

$$\therefore u = \frac{x-x^2-y^2}{(x-1)^2+y^2}, \quad v = \frac{y}{(1-x)^2+y^2}.$$

Upper half of the z-plane is $y>0$, when $y>0$ we have $v>0$ as $(1-x)^2+y^2>0$. Thus the upper half of the z-plane is mapped onto the upper half of the w-plane.

(ii) To find the image of $|z|=1$

Given: $|z|=1$ i.e., $|x+iy|=1 \Rightarrow x^2+y^2=1$

$$\therefore u = \frac{x-(x^2+y^2)}{(1-x)^2+y^2}, \quad v = \frac{y}{(1-x)^2+y^2}$$

$$= \frac{x-(x^2+y^2)}{1+x^2-2x+y^2}, \quad v = \frac{y}{1+x^2-2x+y^2}$$

$$= \frac{x-1}{2-2x}, \quad v = \frac{y}{2-2x}$$

$$= \frac{x-1}{-2(x-1)}, \quad v = \frac{y}{-2(x-1)}$$

$$u = -1/2, \quad v = \frac{-y}{2(x-1)}$$

The region $|z|<1$ transforms into $u > -1/2$.

14. (a) (i) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1+i|=2$, using Cauchy's integral formula.

Given $|z+1+i|=2$ ie $|z-[-(1+i)]|=2$ is a circle whose centre is $-(1+i)$ and radius is 2.

i.e., centre is $(-1,-1)$ and radius is 2.

$z^2+2z+5 = [z-(-1+2i)][z-(-1-2i)]$

$-1+2i$ i.e $(-1,2)$ lies outside C, $-1-2i$ ie $(-1,-2)$ lies inside C.

$$\therefore \int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{\left(\frac{z+4}{z-(-1+2i)}\right)}{z-(-1-2i)}dz$$

Here $f(z) = \dfrac{z+4}{z-(-1+2i)}$ is analytic inside c.

Hence by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

$$\therefore \int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i\,(-1-2i) = 2\pi i\left[\frac{-1-2i+4}{(-1-2i)-(-1+2i)}\right]$$

$$= 2\pi i\left(\frac{3-2i}{-1-2i+1-2i}\right) = 2\pi i\left(\frac{3-2i}{-4i}\right)$$

$$\int_C \frac{z+4}{z^2+2z+5}dz = -\frac{\pi}{2}(3-2i)$$

(ii) Find the residues of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)^2}$ at its isolated singularities using Laurent's series expansions. Also state the valid region.

Given $f(z) = \dfrac{z^2}{(z-1)^2(z+2)^2}$

Here $z=1$ is a pole of order 2.

$z=-2$ is a pole of orde 2.

$$\operatorname{Res}\left(f(z)\right)_{z=z_0} = \underset{z\to z_0}{Lt}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right].$$

Here $m=2$.

$$\therefore \operatorname{Res}\left(f(z)\right)_{z=1} = \frac{1}{1!}\underset{z\to 1}{Lt}\frac{d}{dz}\left[\cancel{(z-1)^2}\cdot\frac{z^2}{\cancel{(z-1)^2}(z+2)^2}\right] = \underset{z\to 1}{Lt}\frac{d}{dz}\left[\frac{z^2}{(z+2)^2}\right]$$

$$= \left[\frac{(z+2)^2\cdot 2z - z^2\cdot 2(z+2)}{(z+2)^4}\right]_{z=1} = \frac{9\times 2-2(3)}{3^4} = \frac{6-2}{3^3}$$

$$R_1 = \frac{4}{27}$$

$$R_2 = \operatorname{Res}\left(f(z)\right)_{z=-2} = \frac{1}{1!}\underset{z\to -2}{Lt}\frac{d}{dz}\left[\cancel{(z+2)^2}\frac{z^2}{(z-1)^2\cancel{(z+2)^2}}\right] = \underset{z\to -2}{Lt}\frac{d}{dz}\left[\frac{z^2}{(z-1)^2}\right]$$

$$\operatorname{Res}_{z=-2}(f(z)) = \lim_{z\to -2} \frac{d}{dz}\left[\frac{z^2}{(z-1)^2}\right] = \lim_{z\to -2}\left[\frac{(z-1)^2 \cdot 2z - z^2 2.(z-1).1}{(z-1)^4}\right]$$ (13)

$$R_2 = \frac{9\times 2(-2) - (4)(2)(-3)}{(-3)^4} = \frac{-12+8}{27} = \frac{-4}{27}.$$

(b) Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta$, $a>b>0$.

Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz d\theta \quad \therefore d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z^2+1}{2z}$ & $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{1}{a+b\frac{1}{2}\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{2\cancel{z}}{2az+bz^2+b} \cdot \frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i}\int_C \frac{1}{bz^2+2az+b} dz. = \frac{2}{ib}\int_C \frac{1}{z^2+\frac{2a}{b}z+1} dz$$

Let $f(z) = \frac{1}{z^2+\frac{2a}{b}z+1}$, $z^2 + \frac{2a}{b}z+1 = 0$

$$\therefore z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

i.e $z = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $z = \frac{-a-\sqrt{a^2-b^2}}{b}$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}.$

$\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$ Given $a>b>0$.

$\therefore z = \alpha$ lies inside c and $z=\beta$ lies outside c.

$$\operatorname{Res}_{z=\alpha}(f(z)) = \lim_{z\to\infty} (z-\alpha) f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta}$$

$$= \frac{b}{2\sqrt{a^2-b^2}}.$$

Hence by cauchy's residue theorem, $\int_C f(z)dz = 2\pi i$ [sum of residues of $f(z)$]

$$\int_C f(z)dz = 2\pi i\left(\frac{b}{2\sqrt{a^2-b^2}}\right) = \frac{b\pi i}{\sqrt{a^2-b^2}}$$ (14)

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{\cancel{ib}}\left(\frac{\cancel{b}\pi \cancel{i}}{\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

15.(a)(i) Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$ using convolution theorem.

$$L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{s}{s^2+4}\right]$$

$$= \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2u \cos 2(t-u)du = \int_0^t \cos 2u \cos(2t-2u)du.$$

$$= \frac{1}{2}\int_0^t \left[\cos(2u+2t-2u) + \cos(2u-2t+2u)\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos 2t + \cos(4u-2t)\right]du$$

$$= \frac{1}{2}\left[(\cos 2t)u + \frac{\sin(4u-2t)}{4}\right]_{u=0}^{u=t}$$

$$= \frac{1}{2}\left[\left(t\cos 2t + \frac{\sin 2t}{4}\right) - \left(0 - \frac{\sin 2t}{4}\right)\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \frac{1}{4}\left[\sin 2t + 2t\cos 2t\right]$$

(ii) Find the Laplace transform of the Half wave Rectifier

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases} \quad \text{and } f\left(t+\frac{2\pi}{\omega}\right) = f(t) \text{ for all } t.$$

Given function $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$.

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)dt. = \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-\frac{s\pi}{\omega}}\cdot\omega + \omega}{s^2+\omega^2}\right] = \frac{\omega\left(1+e^{-\frac{s\pi}{\omega}}\right)}{\left(1-e^{-\frac{s\pi}{\omega}}\right)\left(1+e^{-s\pi/\omega}\right)(s^2+\omega^2)}$$

$$L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}$$

(15)

(b) (i) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$.

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds = \int_s^\infty\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right] \qquad \text{since } \log 1 = 0.$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(ii) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x=0$ and $\frac{dx}{dt} = 5$ for $t=0$ using Laplace transform method.

Given $x'' - 3x' + 2x = 2$

$L[x''] - 3L[x'] + 2L[x] = L[2]$.

$s^2L(x) - s\,x(0) - x'(0) - 3[sL(x) - x(0)] + 2L[x] = 2/s$.

Given $x(0) = 0$ and $x'(0) = 5$.

$s^2L(x) - 5 - 3[sL(x) - 0] + 2L[x] = 2/s$

$L(x)[s^2 - 3s + 2] = \frac{2}{s} + 5$

$$L(x) = \frac{2}{s(s^2-3s+2)} + \frac{5}{s^2-3s+2}. \quad \text{(or)} \quad L[x] = \frac{5s+2}{s(s^2-3s+2)}$$

$$x = L^{-1}\left[\frac{2}{s(s^2-3s+2)}\right] + L^{-1}\left[\frac{5}{s^2-3s+2}\right] \quad \text{(or)} \quad x = L^{-1}\left[\frac{5s+2}{s(s^2-3s+2)}\right]$$

16

consider $\frac{5s+2}{s(s^2-3s+2)} = \frac{5s+2}{s(s-1)(s-2)}$

$\therefore \frac{5s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$\therefore 5s+2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$

put $s=0$, $2 = A(-1)(-2) + B(0) + C(0)$

$2 = 2A \quad \therefore \boxed{A=1}$

put $s=1$, $7 = A(0) + B(-1) + 0$

$\therefore \boxed{B=-7}$

put $s=2$, $12 = A(0) + B(0) + C(2)(1)$

$12 = 2C \quad \therefore \boxed{C=6}$

$\therefore x = L^{-1}\left[\frac{1}{s} + \frac{-7}{s-1} + \frac{6}{s-2}\right]$

$x = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{-7}{s-1}\right] + L^{-1}\left[\frac{6}{s-2}\right]$

$\therefore x(t) = 1 - 7e^{t} + 6e^{2t}$