Anna University

B.E / B.Tech. Degree Examination, May / June 2012

Second Semester

(R2008)

PART-A

1. Transform the equation $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x \quad \to (1)$ into a differential equation with constant coefficients.

Sln

Let $(2x+3) = e^z \Rightarrow z = \log(2x+3)$

$(2x+3)D = 2D'$

$(2x+3)^2 D^2 = 2^2 D'(D'-1) = 4D'(D'-1)$

$\therefore$ (1) becomes,

$[4D'(D'-1) - 2(2D') - 12]y = 6\left[\frac{e^z-3}{2}\right]$

$[4D'^2 - 8D' - 12]y = 3(e^z - 3)$

$[D'^2 - 2D' - 3]y = \frac{3}{4}e^z - \frac{9}{4}$

Auxillary equation is

$m^2 - 2m - 3 = 0 \Rightarrow m = 3, -1$

$\therefore$ The complementary function is

$C.F = Ae^{3z} + Be^{-z}$

$$P.I_1 = \frac{1}{D'^2 - 2D' - 3} \cdot \left(\frac{3}{4}e^z\right)$$

[Here $a=1$ put $D' = a = 1$]

$$P.I_1 = \frac{1}{1-2-3}\left(\frac{3}{4}\right)e^z = \frac{-3}{16}e^z$$

$$P.I_2 = \frac{1}{(D'^2 - 2D' - 3)}\left(\frac{9}{4}\right) = \frac{1}{(D'^2-2D'-3)}\left(\frac{9}{4}\right)e^{0z}$$

[Here $a=0$ put $D'=a=0$]

$$= \frac{9}{4} \cdot \frac{1}{0-0-3}$$

$P.I_2 = -9/12$

$\therefore$ The sln is $\quad y = C.F + P.I_1 + P.I_2$

$$y = Ae^{3z} + Be^{-z} - \frac{3}{16}e^{z} - \frac{9}{12}$$

$$y = A(e^z)^3 + B(e^z)^{-1} - \frac{3}{16}e^z - \frac{3}{4}$$

$$y = A(2x+3)^3 + \frac{B}{2x+3} - \frac{3}{16}(2x+3) - \frac{3}{4}$$

2) Find the particular Integral of $(D-1)^2 y = e^x \sin x$

Sln

$$P.I = \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x$$

$$= e^x \frac{1}{D^2} \sin x$$

$$= e^x \frac{1}{D}(-\cos x)$$

$$P.I = -e^x \sin x$$

$\boxed{\frac{1}{D} = \int \quad \frac{1}{D^2} = \iint}$

3) Find λ such that $\bar{F} = (3x-2y+z)\bar{i} + (4x+\lambda y-z)\bar{j} + (x-y+2z)\bar{k}$ is solenoidal

Sln. Given $\bar{F} = (3x-2y+z)\bar{i} + (4x+\lambda y-z)\bar{j} + (x-y+2z)\bar{k}$ is solenoidal.

ie) $\nabla \cdot \bar{F} = 0$

$$3 + \lambda + 2 = 0$$

$$\boxed{\lambda = -5}$$

4. State Gauss divergence theorem.

Statement:
The surface Integral of the normal Component of a vector function $\bar{F}$ taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of $\bar{F}$ taken throughout the volume V.

ie) $\iint\limits_S \bar{F}\cdot\vec{ds} = \iiint\limits_V \text{div}\,\bar{F}\,dv.$

~~5. State the basic difference between the limit of a function of a real Variable and that of a Complex Variable.~~

6. Prove that a bilinear transformation has atmost two fixed points.

The fixed points of the general bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $\frac{az+b}{cz+d} = z$

$$az+b = cz^2+dz \Rightarrow cz^2+dz-az-b=0 \Rightarrow cz^2+(d-a)z-b=0$$

This is a quadratic equation in z.
Hence a bilinear transformation has atmost two fixed points.

7. Define Singular point.

A point $z=a$ is said to be a Singular point (or) Singularity of $f(z)$ if $f(z)$ is not analytic at $z=a$

8. Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Sln:

$z=2$ is a simple pole.

$$\underset{z=2}{\text{Res}}\ f(z) = \lim_{z\to 2} (z-2)\frac{4}{z^3(z-2)} = \frac{1}{2}$$

9. State the first shifting theorem on Laplace transform

Statement:

If $L\{f(t)\} = F(s)$ then

(i) $L\{e^{-at} f(t)\} = F(s+a)$ (ii) $L\{e^{at} f(t)\} = F(s-a)$

10. Verify initial value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Sln:

Given $f(t) = 1 + e^{-t}(\sin t + \cos t)$

$$F(s) = L\{f(t)\} = L\{1 + e^{-t}(\sin t + \cos t)\} = L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$F(s) = \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} \Rightarrow s.F(s) = s\left(\frac{1}{s} + \frac{(s+2)}{(s+1)^2+1}\right)$$

IVT: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = [1 + e^0(0+1)] = 2.$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right)$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2+2s}{(s^2+2s)\left[1+\frac{2}{s^2+2s}\right]}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{1}{1+\frac{2}{s^2+2s}}\right]$$

$$= 1+1 = 2$$

$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 2$. Hence IVT verified.

PART – B

11) a) (i) Solve $(D^2+a^2)y = \sec ax$ using the method of Variation of parameters.

Given $(D^2+a^2)y = \sec ax$.

A.E $m^2+a^2=0 \Rightarrow m = \pm ai$

$C.F = A\cos ax + iB\sin ax$

Let $y_1 = \cos ax \quad y_2 = \sin ax$

$y_1' = -a\sin ax \; ; \; y_2' = a\cos ax$

$w = y_1 y_2' - y_2 y_1' = a[\cos^2 ax + \sin^2 ax] = a$

$P.I = uy_1 + vy_2$ where $u = -\int \frac{X y_2}{W} dx$

$$u = -\int \frac{\sec ax \cdot \sin ax}{a} dx = \frac{-1}{a}\int \frac{\sin ax}{\cos ax} dx = +\frac{1}{a^2}\log\cos ax$$

$$v = \int \frac{X y_1}{w} dx = \int \frac{\sec ax \cdot \cos ax}{a} dx = \frac{1}{a}[x] = \frac{x}{a}$$

$$\therefore P.I = \frac{\log\cos ax}{a^2}\cdot\cos ax + \frac{x}{a}\sin ax$$

$$\therefore y = C.F + P.I = A\cos x + B\sin x + \frac{1}{a^2}(\log\cos ax)\cos ax + \frac{x}{a}\sin ax$$

(ii) Solve: $(D^2-4D+3)y = e^x \cos 2x$

Sln: A.E is $m^2-4m+3=0 \Rightarrow m = 1, 3$.

$\therefore C.F = Ae^x + Be^{3x}$

$$P.I = \frac{1}{D^2-4D+3} e^x \cos 2x = e^x \frac{1}{(D+1)^2-4(D+1)+3} \cos 2x$$

$$= e^x \frac{1}{D^2+2D+1-4D-4+3} \cos 2x = e^x \frac{1}{D^2-2D} \cos 2x$$

Here $a=2$ put $D^2 = -a^2 = -2^2 = -4$.

$$P.I = e^x \frac{1}{-4-2D}\cos 2x = -e^x \frac{1}{(2D+4)}\cos 2x$$

$$P.I = -e^{x}\frac{(2D-4)}{4D^2-16}\cos 2x$$

$$= \frac{e^{x}}{32}(2D-4)\cos 2x$$

$$= \frac{e^{x}}{32}\left[-4\sin 2x - 4\cos 2x\right]$$

$$P.I = -\frac{e^{x}}{8}\left[\sin 2x + \cos 2x\right]$$

$\therefore$ The Sln is $y = C.F + P.I = Ae^{x} + Be^{3x} - \frac{e^{x}}{8}(\sin 2x + \cos 2x)$

11)b)(i) Solve the differential equation

$$(x^2D^2 - xD + 4)y = x^2\sin(\log x) \longrightarrow (1)$$

Sln: Let $x = e^{z}$ then $z = \log x$

$\therefore xD = D'$ & $x^2D^2 = D'(D'-1)$ where $D' = \frac{d}{dz}$

$\therefore$ (1) becomes, $(D'(D'-1) - D' + 4)y = e^{2z}\sin z$

$$(D'^2 - 2D' + 4)y = e^{2z}\sin z$$

A.E $m^2 - 2m + 4 = 0 \Rightarrow m = 1 \pm i\sqrt{3}$

$$\therefore C.F = e^{z}(A\cos\sqrt{3}z + B\sin\sqrt{3}z)$$

$$P.I = \frac{1}{D'^2 - 2D' + 4}(e^{2z}\sin z) = e^{2z}\frac{1}{(D'+2)^2 - 2(D'+2) + 4}\sin z$$

$$P.I = e^{2z}\left[\frac{1}{D'^2 + 2D' + 4}\right]\sin z = e^{2z}\left[\frac{1}{-1 + 2D' + 4}\right]\sin z$$

$$= e^{2z}\left[\frac{2D' - 3}{4D'^2 - 9}\right]\sin z = \frac{e^{2z}}{13}(2D' - 3)\sin z$$

$$= \frac{e^{2z}}{13}(3\sin z - 2\cos z) = \frac{x^2}{13}\left[3\sin(\log x) - 2\cos(\log x)\right]$$

$$\therefore y = C.F + P.I = x\left[A\cos\sqrt{3}\log x + B\sin(\sqrt{3}\log x)\right] + \frac{x^2}{13}\left[3\sin(\log x) - 2\cos(\log x)\right]$$

b) (ii) Solve the simultaneous differential equation

$$\frac{dx}{dt} + 2y = \sin 2t \;;\; \frac{dy}{dt} - 2x = \cos 2t$$

Sln $Dx + 2y = \sin 2t \rightarrow (1)$; $Dy - 2x = \cos 2t \longrightarrow (2)$

$(1) \times D \Rightarrow D^2x + 2Dy = -2\cos 2t$

$$-4x + 2Dy = 2\cos 2x$$
$$(+) \quad (-) \quad (-)$$

$$D^2x + 4x = -4\cos 2t$$

$$(D^2+4)x = -4\cos 2t$$

A.E $m^2+4=0 \Rightarrow m = \pm 2i$ $\therefore$ C.F $= A\cos 2t + B\sin 2t$

$$P.I = \frac{1}{D^2+4}(-4\cos 2t) = -4\frac{1}{0}\cos 2t$$

$$= \frac{x}{2D}(-4\cos 2t) = -2x\frac{1}{D}\cos 2t = -\frac{2x\sin 2t}{2}$$

$P.I = -x\sin 2t$

$x = A\cos 2t + B\sin 2t - x\sin 2t \longrightarrow (3)$

$\therefore Dx = -2A\sin 2t + 2B\cos 2t - \sin 2t - 2x\cos 2t$

$Dx = -2\sin 2t(1+A) + 2\cos 2t(B-x)$

$\therefore (1)$ becomes,

$-2\sin 2t(1+A) + 2\cos 2t(B-x) + 2y = \sin 2t$

$y = \sin 2t \cdot (1 + 2(1+A)) - 2\cos 2t\,(B-x) \Rightarrow (4)$

$y = \sin 2t\,(3+2A) + 2\cos 2t\,(x-B) \longrightarrow (4)$

(3) & (4) give the sln. of (1) & (2).

12)

a)(i) Show that $\bar{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2 z - y + 2z)\vec{k}$

is irrotational & hence find its scalar potential.

Sln:

$$\nabla \times \bar{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 + 2xz) & 2xy - z & 2x^2 z - y + 2z \end{vmatrix}$$

$\nabla \times \vec{F} = \vec{i}(0) - \vec{j}(0) + \bar{k}(0) = \vec{0} \Rightarrow \bar{F}$ is irrotational.

Let $\bar{F} = \nabla\phi$ where ϕ is the scalar potential of $\vec{F}$

$$\therefore \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\bar{j} + (2x^2 z - y + 2z)\vec{k}$$

$$\frac{\partial\phi}{\partial x} = y^2 + 2xz^2 \; ; \quad \frac{\partial\phi}{\partial y} = 2xy - z \; ; \quad \frac{\partial\phi}{\partial z} = 2x^2 z - y + 2z$$

Integrating,

$\phi = xy^2 + x^2 z^2 + f_1$; $\phi = x - yz + f_2$; $\phi = x^2 z^2 - yz + z^2$

$$\therefore \phi = xy^2 + x^2 z^2 - yz + z^2 + C$$

(ii) Verify Green's theorem in a plane for

$$\int_C \left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right],$$ where C is the

boundary of the region defined by $x = 0$, $y = 0$ and $x + y = 1$

Sln:

By Green's theorem we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Let $I = \int_C \left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right]$

Here $M = 3x^2 - 8y^2$ $\quad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$

$$I = \int_0^1 \int_0^{1-y} (-6y + 16y)\,dx\,dy \quad \int_0^1 10y\,[x]_0^{1-y}\,dy$$

$$I = 10\left(\frac{1}{6}\right) = \frac{5}{3}$$

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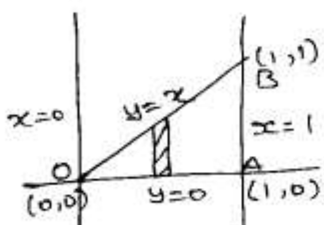
12) b) i) Using Stoke's theorem, evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = y^2\bar{i} + x^2\bar{j} - (x+z)\bar{k}$ and C is the boundary of the Δ^{le} with vertices at $(0,0,0)$ $(1,0,0)$ $(1,1,0)$

Sln: By stoke's theorem, $\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$



Given $\bar{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$

$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \bar{j} + 2(x-y)\vec{k}$

Since C is the boundary of the Δ^{le} whose Z-coordinates are zero, it is in xy-plane & hence $\hat{n} = \bar{k}$

$\therefore \text{Curl}\,\bar{F}\cdot\hat{n} = \left[\bar{j} + 2(x-y)\vec{k}\right]\cdot\vec{k} = 2(x-y)$ & $ds = dx\,dy$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_0^1\int_0^x 2(x-y)\,dy\,dx = \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right)dx = \frac{1}{3}$$

$\boxed{\text{In } OAB,\ x=0 \text{ to } x=1,\ y=0 \text{ to } y=x}$

b) (ii) Find the work done in moving a particle in the force field given by $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sln: $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\vec{k}$

C is the straight line from $(0,0,0)$ to $(2,1,3)$

Work done by $\bar{F} = \int_C \bar{F} \cdot d\vec{r}$

$= \int_C 3x^2\,dx + (2xz-y)\,dy + z\,dz$

Equation of the line joining the point $(0,0,0)$ to $(2,1,3)$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

$x = 2t \Rightarrow dx = 2dt$; $y = t \Rightarrow dy = dt$; $z = 3t \Rightarrow dz = 3dt$

$\therefore \bar{F} \cdot d\bar{r} = 24t^2 + 12t^2\,dt - t\,dt + 9t\,dt = (36t^2 + 8t)\,dt$

At $t=0$, $x = y = z = 0$; At $t=1$, $(x, y, z) = (2,1,3)$

$\therefore t$ varies from 0 to 1.

$\therefore$ Work done by $\bar{F} = \int_0^1 (36t^2 + 8t)\,dt = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$

$= \underline{\underline{16}}$

3) a) i) Prove that every analytic function $w = u+iv$ can be expressed as a function of z alone, not as a function ~~of u+iv~~ of $\bar{z}$

Proof

Let $z = x+iy$ Then $\bar{z} = x-iy$ $\therefore z+\bar{z} = 2x \Rightarrow x = \frac{z+\bar{z}}{2}$

$y = \frac{z-\bar{z}}{2i}$. Now, $\frac{\partial x}{\partial z} = \frac{1}{2}$; $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$; $\frac{\partial y}{\partial z} = \frac{1}{2i} = \frac{-i}{2}$

$\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i} = \frac{i}{2}$.

Now, $\frac{\partial w}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u+iv) = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \rightarrow (1)$

Now, $\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\left(\frac{i}{2}\right) = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right]$

Also $\frac{\partial v}{\partial \bar{z}} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right]$

Since $f(z) = u+iv$ is analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

From (1), $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + i\left[\frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)\right]$

$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right] = \frac{1}{2}(0) = 0$.

Hence $\frac{\partial w}{\partial \bar{z}} = 0 \Rightarrow w$ is independent of $\bar{z} \Rightarrow w$ does not Contain $\bar{z} \Rightarrow w$ can be expressed interms of z alone.

(ii) Find the bilinear transformation which maps the points $z = 0, 1, \infty$ into $w = i, 1, -i$ respectively.

Sln $z_1 = 0 \quad z_2 = 1 \quad z_3 = \infty$ & $w_1 = i$, $w_2 = 1$, $w_3 = -i$.

$$\frac{(w-w_1)}{(w_1-w_2)}\cdot\frac{(w_2-w_3)}{(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)}\frac{(z_2-z_3)}{(z_3-z)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)\cancel{z_3}\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)\cancel{z_3}\left(1-\frac{z}{z_3}\right)}.$$

$$\frac{(w-i)(1+i)}{(i-1)(-i-w)} = \frac{(z-0)(-1)}{(-1)(1)} \Rightarrow \frac{(w-i)(1+i)}{(w+i)(i-1)} = z$$

$$\frac{w-i}{w+i} = z\cdot\frac{(1-i)}{(1+i)} = \frac{z}{2}(1-i)^2 = \frac{z}{2}[1-2i-1] = -iz$$

$$w-i = iz(w+i) = iwz+z \Rightarrow w(1+iz) = z+i \Rightarrow w = \frac{z+i}{1+iz}$$

(3) b)(i) If $f(z)$ is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$$

Sln:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log\left(f(z)\,\overline{f(z)}\right)^{1/2}$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\,\frac{1}{2}\left[\log[f(z)] + \log[f(\bar{z})]\right]$$

$$= 2\frac{\partial}{\partial z}\left(\frac{\partial}{\partial\bar{z}}\right)\left[\log f(z) + \log f(\bar{z})\right]$$

$$= 2\frac{\partial}{\partial z}\left[0 + \frac{1}{f(\bar{z})}\cdot f'(\bar{z})\right] = 2(0) = 0.$$

(ii) Show that the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$ is the lemniscate $r^2=\cos 2\theta$.

Sln: $w=u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$; $u=\frac{x}{x^2+y^2}$ & $v=\frac{-y}{x^2+y^2}$

$x=\frac{u}{u^2+v^2}$ & $y=\frac{-v}{u^2+v^2}$ $\therefore x^2-y^2=1 \Rightarrow \frac{u^2}{(u^2+v^2)^2}-\frac{v^2}{(u^2+v^2)^2}=1$

$\therefore u^2-v^2=(u^2+v^2)^2 \Rightarrow u^2-v^2=u^4+v^4+2u^2v^2$

$\Rightarrow u^4+v^4-u^2+v^2+2u^2v^2=0 \Rightarrow u^2(u^2-1)+v^4+v^2+2u^2v^2=0$

$u^2(u+1)(u-1)+(v^4+v^2+2u^2v^2)=0 \Rightarrow u^2(u+1)(u-1)+(v^4+v^2+2u^2v^2)=0$

This is the image of $x^2-y^2=1$ in the w-plane.

14) a) i) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where C is $|z-2|=\frac{1}{2}$ by using cauchy's integral formula.

Sln: Let $I=\int_C \frac{z\,dz}{(z-1)(z-2)^2}$. $|z-2|=\frac{1}{2}$ is the circle centre $(2,0)$ & radius $\frac{1}{2}$.

By cauchy's integral formula, $f'(a)=\frac{1}{2\pi i}\int\frac{f(z)\,dz}{(z-a)^2}$

$$\therefore \int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2}dz = 2\pi i\, f'(2).$$

Now, $f(z)=\frac{z}{z-1} \Rightarrow f'(z)=\frac{-1}{(z-1)^2} \Rightarrow f'(2)=-1.$

$$\therefore \int \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

4) a) i) Evaluate $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for the region $|z|>3$ & $1<|z|<3$.

Sln $f(z) = \frac{1}{(z+1)(z+3)}$

$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \Rightarrow 1 = A(z+3) + B(z+1)$

when $z=-3$; $B = -\frac{1}{2}$ & $z=-1 \Rightarrow A = \frac{1}{2}$

$\therefore f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$

(i) The region is $|z|>3$.

$\therefore \frac{|z|}{3} > 1$ or $\frac{3}{|z|} < 1$

$\therefore f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$

$= \cancel{\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n}$

$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 \cdots\right]$

The expansion $\left(1+\frac{1}{z}\right)^{-1}$ is valid in the region $\left|\frac{1}{z}\right| < 1$

ie) $|z|>1$

The expansion $\left(1+\frac{3}{z}\right)^{-1}$ is valid in the region $\left|\frac{3}{z}\right| < 1 \Rightarrow |z| > 3$

The whole expansion of $f(z)$ is valid $1<|z|<3$

(ii) The

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4) b) i) Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ where C is the circle $|z-1|=2$ using Cauchy residue theorem

Sln. The singular points of $f(z) = \frac{z-1}{(z+1)^2(z-2)}$ are given by

$(z+1)^2(z-2) = 0$. ie) $z=-1$ & $z=2$.

$z=-1$ lies inside C & $z=2$ lies outside C where $C: |z-1$

Res of $f(z) = \frac{\left(\frac{z-1}{z-2}\right)}{(z+1)^2}$ at the double pole $z=-1$ is given

$\frac{\phi'(-1)}{1!}$ where $\phi(z) = \frac{z-1}{z-2}$ $\cancel{=\frac{-1}{}}$ $\Rightarrow \phi'(z) = \frac{-1}{(z-2)^2} \Rightarrow \phi'(-1) = -\frac{1}{9}$

Res of $f(z) = \frac{-1}{(-1-2)^2} = -\frac{1}{9}$.

$\therefore$ By Cauchy's Residue theorem,

$$\int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i \text{ Res}\left(f(z)\right) = -\frac{2\pi i}{9}.$$

(ii) Evaluate $\int_0^\infty \frac{\cos ax}{x^2+a^2}\,dx$ using Contour integration

Sln:

Let $\phi(z) = \frac{e^{iaz}}{z^2+a^2}$. Consider the contour C which is upper semicircle Γ bounded by the diameter $[-R, R]$.

Then

$$\int_C \phi(z)\,dz = \int_\Gamma \phi(z)\,dz + \int_{-R}^{R} \phi(z)\,dz = \underbrace{\int_\Gamma \phi(z)dz}_{(i)} + \int_{-R}^{R}\phi(x)\,dx$$

Now as $R \to \infty$ (i) $\int_{-R}^{R}\phi(x)\,dx = \int_{-\infty}^{\infty}\phi(x)\,dx$ (ii) $\int_\Gamma \phi(z)dz \to 0$.

(iii) C becomes very large [lies upper half plane]

$\therefore$ By (i), $\int_C \phi(z)\,dz = \int_{-\infty}^{\infty}\phi(x)\,dx \Rightarrow \int_C \frac{e^{iaz}}{z^2+a^2}\,dz = \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+a^2}\,dx \rightarrow (2)$

The poles of $\phi(z)$ are given by $z^2+a^2=0 \Rightarrow z = \pm ia$

Clearly $z = +ia, -ia$ are the poles of order 1 & $z=ai$ lies in the upper half plane

$$\therefore \left.\text{Res }\phi(z)\right|_{z=ai} = \lim_{z\to ai}(z-ai)\phi(z) = \lim_{z\to ai}(z-ai)\frac{e^{iaz}}{(z-ai)(z+ai)}$$

$$\underset{z=ai}{\text{Res}}\ \phi(z) = \frac{e^{i^2a^2}}{2ai} = \frac{e^{-a^2}}{2ai}.$$

By Cauchy's Residue theorem,

$$\int_C \phi(z)dz = 2\pi i \sum \text{Res }\phi(z) = 2\pi i \cdot \frac{e^{-a^2}}{2ai} = \frac{\pi e^{-a^2}}{a}$$

$$\therefore \text{By (2)}, \quad \int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+a^2}\,dx = \frac{\pi e^{-a^2}}{a}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos ax}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty}\frac{\sin ax}{x^2+a^2}\,dx = \frac{\pi e^{-a^2}}{a}$$

Equating real parts on bothsides, we get,

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-a^2}$$

5) a) i) Apply convolution theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sln. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right] = \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \sin au \cos a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2au - at)\right] du$$

$$= \frac{1}{2a}\left[u \sin at - \frac{\cos(2au-at)}{2a}\right]_0^t = \frac{t \sin at}{2a}$$

(ii) Find the Laplace transform of the following triangular wave function given by $f(t) = \begin{cases} t & : 0 \le t \le \pi \\ 2\pi - t & : \pi \le t \le 2\pi \end{cases}$ and $f(t+2\pi)=f(t)$

Sln.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} t e^{-st}\,dt + \int_{\pi}^{2\pi} e^{-st}(2\pi - t)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^{\pi} + \left[(2\pi - t)\left(\frac{e^{-st}}{s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\left[\left(\frac{\pi e^{-s\pi}}{-s} - \frac{e^{-\pi s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] + \left[\left(0 + \frac{e^{-2\pi s}}{s^2}\right) - \left(\frac{\pi e^{-s\pi}}{s} + \frac{e^{-s\pi}}{s^2}\right)\right]\right.$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{-\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} + \frac{e^{-2\pi s}}{s^2} + \frac{\pi e^{-s\pi}}{s} + \frac{e^{-s\pi}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{1+e^{-2\pi s} - 2e^{-s\pi}}{s^2}\right] = \frac{1}{1-e^{-2\pi s}}\frac{(1-e^{-\pi s})^2}{s^2}$$

$$= \frac{1}{(1+e^{-\pi s})\cancel{(1-e^{-\pi s})}}\frac{(1-e^{-\pi s})^{\cancel{2}}}{s^2} = \frac{1-e^{-\pi s}}{s^2(1+e^{-\pi s})}$$

$$L\{f(t)\} = \frac{1 - \dfrac{e^{-\pi s/2}}{e^{\pi s/2}}}{s^2\left(1+\dfrac{e^{-\pi s/2}}{e^{\pi s/2}}\right)} = \frac{e^{\pi s/2} - e^{-\pi s/2}}{s^2\left(e^{\pi s/2}+e^{-\pi s/2}\right)} = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$$

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15) b)
i) Find the Laplace transform of $\dfrac{e^{-at} - e^{-bt}}{t}$

Sln. $$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L(e^{-at} - e^{-bt})\,ds = \int_s^\infty\left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= \Big[\log(s+a) - \log(s+b)\Big]_s^\infty = \log\left[\frac{(s+a)}{(s+b)}\right]_s^\infty$$

$$= \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^\infty = \log 1 - \log\frac{s+a}{s+b} = \log\left(\frac{s+b}{s+a}\right)$$

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15) b) (ii) Evaluate $\displaystyle\int_0^\infty t e^{-2t}\cos t\,dt$ using Laplace transform.

Sln. Let $$I = \int_0^\infty t e^{-2t}\cos t\,dt$$

Consider,

$$L[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]$$

$$L(t\cos t) = \frac{1-s^2}{(s^2+1)^2}$$

put $s=2$ $\therefore \int_0^\infty e^{-2t}\, t\cos t\, dt = \frac{1-2^2}{(2^2+1)^2} = \frac{-3}{25}$

15) b) (ii) Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$ with $y(0)=1$ & $y'(0)=0$ using Laplace transform

Sln: Given $y'' - 3y' + 2y = e^{-t}$

Taking Laplace transform on bothsides,

$$L(y'') - 3L(y') + 2L(y) = L(e^{-t})$$

$$L(y'') - y'(0) + 3[sL(y) - y(0)] + 2L(y) = \frac{1}{s+1}$$

$$s^2L(y) - sy(0) - y'(0) + 3[sL(y) - 1] + 2L(y) = \frac{1}{s+1}$$

$$\therefore s^2L(y) - s - 0 + 3[sL(y) - 1]$$

$$(s^2+3s+2)L(y) = \frac{1}{s+1} + s + 3$$

$$(s+1)(s+2)L(y) = \frac{s^2+4s+3}{(s+1)}$$

$$L(y) = \frac{s^2+4s+3}{(s+1)^2(s+2)} \Rightarrow y = L^{-1}\left[\frac{s^2+4s+3}{(s+1)^2(s+2)}\right]$$

$$y = L^{-1}\left[\frac{s+3}{(s+1)(s+2)}\right] = L^{-1}\left[\frac{\frac{2}{1}}{s+1} + \frac{\frac{1}{-1}}{s-2}\right]$$

$$= 2L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s-2}\right] = 2e^{-t} - e^{2t}$$