Part-A

1) Transform the equation $x^2y''+xy'=x$ into a linear differential equation with constant coefficients.

Solution: Put $x=e^z$, $z=\log x$

$$x\frac{dy}{dx}=D'y \; ; \; x^2\frac{d^2y}{dx^2}=D'(D'-1)y$$

The given differential equation is $(x^2D^2+xD)y=x$

$$[D'(D'-1)+D']y=e^z \Rightarrow \boxed{D'^2y=e^z}$$

2. Find the particular Integral of $(D^2+1)y=\sin x$.

$$P.I=\frac{1}{D^2+1}\sin x=\frac{1}{-1+1}\sin x = x.\frac{1}{2D}\sin x=\frac{x}{2}\int \sin x\,dx=\frac{-x\cos x}{2}.$$

3. Is the position vector $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ irrotational? Justify.

$$\nabla\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x & y & z\end{vmatrix}=0 \quad \therefore \vec{r} \text{ is irrotational}$$

4. State Gauss Divergence theorem.

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S then, $\iint_S \vec{F}.\hat{n}\,ds=\iiint_V \nabla.\vec{F}\,dv$.

5. Verify whether the function $u=x^3-3xy^2+3x^2-3y^2+1$ is harmonic

Solution:

$$u=x^3-3xy^2+3x^2-3y^2+1$$

$$\frac{\partial u}{\partial x}=3x^2-3y^2+6x \quad \Big| \quad \frac{\partial u}{\partial y}=-6xy-6y$$

$$\frac{\partial^2 u}{\partial x^2}=6x+6 \quad \Big| \quad \frac{\partial^2 u}{\partial y^2}=-6x-6$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6x+6-6x-6=0 \quad \therefore u \text{ is harmonic}$$

6) Find the constants a, b, c if $f(z) = x+ay+i(bx+cy)$ is analytic.

(2)

Solution:-

$$f(z) = u+iv = (x+ay)+i(bx+cy)$$

$u = x+ay$ | $v = bx+cy$

$\frac{\partial u}{\partial x} = 1$ | $\frac{\partial v}{\partial x} = b$

$\frac{\partial u}{\partial y} = a$ | $\frac{\partial v}{\partial y} = c$

By Cauchy Riemann equations,

$u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$ can assumes any value but a is negative of b (or) b is negative of a

7. What is the value of the integral $\int_C \frac{3z^2+7z+1}{z+1} dz$ where C is $|z| = \frac{1}{2}$?

Solution:

The pole of $\frac{3z^2+7z+1}{z+1}$ is given by $z+1=0$.

i.e $z=-1$ which is a simple pole.

As the circle C is $|z| = 1/2$ with centre at the origin and radius $1/2$ and $z=-1$ lies outside the circle $|z| = 1/2$.

$$\therefore \int_C \frac{3z^2+7z+1}{z+1} dz = 0.$$

8. If $f(z) = \frac{-1}{z-1} - 2[1+(z-1)+(z-1)^2+\cdots]$ find the residue of $f(z)$ at $z=1$.

Solution:

Residue of $f(z)$ at $z=1$ $= \lim_{z\to 1} (z-1) f(z)$

$$= \lim_{z\to 1} (z-1)\left[\frac{-1}{z-1} - 2(1+(z-1)+(z-1)^2\right.$$

Residue $= -1$ which is the coefficient of $\frac{1}{z-1}$ in its Taylor's expansion.

9: Find the Laplace transform of unit step function.

Solution:

Unit step function denoted by $u(t-a)$ is defined by (3)

$$U(t-a)=\begin{cases}0, & t<a \\ 1, & t\geq a\end{cases} \quad \text{where } a\geq 0.$$

10. Find the inverse Laplace Transform of $\cot^{-1}s$.

Solution:

$$L^{-1}[\cot^{-1}s] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}s\right] = \frac{-1}{t} L^{-1}\left[\frac{-1}{s^2+1}\right] = \frac{1}{t}\sin t.$$

Part-B

11) a) (i) Solve the equation $(D^2+4D+3)y = e^{-x}\sin x$.

Solution:

The auxiliary equation is $m^2+4m+3=0 \Rightarrow m=-1,-3$.

The complementary function is $C.F = Ae^{-x}+Be^{-3x}$.

Particular Integral $P.I = \frac{1}{D^2+4D+3} e^{-x}\sin x = e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$

$$= e^{-x}\frac{1}{D^2+1-2D+4D-4+3}\sin x = e^{-x}\frac{1}{D^2+2D}\sin x = e^{-x}\cdot\frac{1}{-1+2D}\sin x$$

$$= e^{-x}\frac{1}{2D-1}\times\frac{2D+1}{2D+1}\sin x = e^{-x}\cdot\frac{2D+1}{4D^2-1}\sin x$$

$$= e^{-x}\cdot\frac{(2D+1)}{-5}\sin x$$

$$= \frac{e^{-x}}{-5}\left[2D\sin x+\sin x\right] = \frac{e^{-x}}{-5}\left[2\cos x+\sin x\right]$$

Therefore the solution is $y = C.F + P.I$

$$\text{i.e } y = Ae^{-x}+Be^{-3x} - \frac{e^{-x}}{5}(2\cos x+\sin x).$$

(ii) Solve the equation $(D^2+1)y = x\sin x$ by the method of variation of parameters.

Solution: The auxiliary equation is $m^2+1=0 \Rightarrow m=\pm i$

$C.F = A\cos x + B\sin x$

$f_1 = \cos x \quad | \quad f_2 = \sin x$

$f_1' = -\sin x \quad | \quad f_2' = \cos x$

$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1.$

$$P = -\int \frac{f_2 f(x)}{f_1 f_2' - f_1' f_2} dx = -\int \frac{\sin x \cdot x \sin x}{1} dx = -\int x \sin^2 x \, dx$$

$$= -\int x\left(\frac{1-\cos 2x}{2}\right) dx = -\left[\int \frac{x}{2} dx - \int \frac{x \cos 2x}{2} dx\right)$$

$$P = -\frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\int x \cos 2x \, dx = -\frac{x^2}{4} + \frac{1}{2}\left[\frac{x \sin 2x}{2} + \frac{\cos 2x}{2}\right]$$

$$Q = \int \frac{f_1 f(x)}{f_1 f_2' - f_2 f_1'} dx = \int x \sin x \cos x \, dx = \int x \frac{\sin 2x}{2} dx = \frac{x}{2}\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{8}$$

$$P.I = P f_1 + Q f_2 = \cos x\left[-\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{4}\right] + \sin x\left[-\frac{x \cos 2x}{4} + \frac{\sin 2x}{8}\right]$$

$$= \frac{\cos x}{8} + \frac{x \sin x}{4} - \frac{x^2 \cos x}{4}$$

$\therefore$ The solution is $y = C.F + P.I$

i.e. $y = A\cos x + B\sin x + \frac{\cos x}{8} + \frac{x \sin x}{4} - \frac{x^2 \cos x}{4}$.

b) i) Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$.

Solution:

put $x = e^z$ or $z = \log x$

$xD = D'$, $x^2D^2 = D'(D'-1)$.

$\therefore [D'(D'-1) - 2D' - 4]y = e^{2z} + 2z$

$[D'^2 - 3D' - 4]y = e^{2z} + 2z$

Auxiliary equation is $m^2 - 3m - 4 = 0$.

$\boxed{m = -1, 4}$

$\therefore C.F = Ae^{4x} + Be^{-x}$.

Particular Integral P.I $= \frac{1}{D'^2-3D'-4} e^{2z} + 2z$ (5)

$$= \frac{1}{D'^2-3D'-4} e^{2z} + \frac{1}{D'^2-3D'-4} 2z$$

$$= \frac{1}{4-6-4} e^{2z} + \frac{2}{-4}\left[1-\left(\frac{D'^2-3D'}{4}\right)\right]^{-1} z$$

$$= \frac{1}{-6} e^{2z} - \frac{1}{2}\left[1+\left(\frac{D'^2-3D'}{4}\right) z - \cdots\right] z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left[z + \frac{D'^2}{4} z - \frac{3}{4} Dz\right]$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left[z - 3/4\right]$$

$$= \frac{-x^2}{6} - \frac{1}{2}(\log x - 3/4)$$

$\therefore$ Complete solution is $y = C.F + P.I.$

$$y = Ae^{4z} + Be^{-z} - \frac{e^{2z}}{6} - \frac{1}{2}(z - 3/4)$$

i.e $y = Ax^4 + \frac{B}{x} - \frac{x^2}{6} - \frac{1}{2}\left(\log x - 3/4\right)$

(ii) Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$; $\frac{dy}{dt} + 3x + 2y = 0$.

solution :

The given equations are

$$(D+2)x + 3y = 2e^{2t} \quad \text{--- (1)}$$

$$3x + (D+2)y = 0 \quad \text{--- (2)}$$

(1) $\times 3 \Rightarrow 3(D+2)x + 9y = 6e^{2t}$

(2) $\times (D+2) \Rightarrow 3(D+2)x + (D+2)^2 y = 0$

$$\left[(D+2)^2 - 9\right] y = 6e^{2t}$$

$$\boxed{(D^2+4D-5)y = -6e^{2t}}$$

The auxiliary equation is $m^2 + 4m - 5 = 0 \Rightarrow m = 1, -5$.

$\therefore C.F = Ae^{t} + Be^{-5t}$.

Particular Integral $P.I = \frac{1}{D^2+4D-5}(-6e^{2t})$

$= -\frac{6}{7}e^{2t}$

$\therefore$ The solution is $y = C.F + P.I = Ae^{t} + Be^{-5t} - \frac{6}{7}e^{2t}$.

$$\frac{dy}{dt} = Ae^{t} - 5Be^{-5t} - \frac{12}{7}e^{2t}$$

Now $\frac{dy}{dt} + 3x + 2y = 0$.

$$x = -\frac{1}{3}\left[\frac{dy}{dt} + 2y\right]$$

$$x = -\frac{1}{3}\left[Ae^{t} - 5Be^{-5t} - \frac{12}{7}e^{2t} + 2Ae^{t} + 2Be^{-5t} - \frac{12}{7}e^{2t}\right]$$

$$\boxed{x = -Ae^{t} + Be^{-5t} + \frac{8}{7}e^{2t}}$$

12) a) i) Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational and find the scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Solution :

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{i}\,[-1+1] - \vec{j}\left[3z^2 - 3z^2\right] + \vec{k}\,(6x - 6x) = 0$$

$\therefore \vec{F}$ is irrotational.

To find ϕ:

$$\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}.$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}.$$

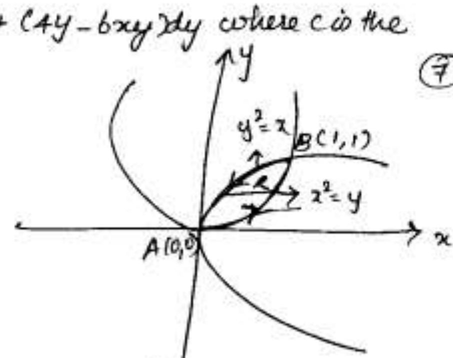
$\frac{\partial\phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = 3x^2y + xz^3 + f(y,z)$.

$\frac{\partial\phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2y - zy + f(x,z)$

$\frac{\partial\phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f(x,y)$

$\therefore \phi = 3x^2y + \underline{xz^3} - yz + C$

(ii) Verify Green's theorem for $\int_c (3x^2-8y^2)dx + (4y-6xy)dy$ where c is the boundary of the region by $x=y^2$, $y=x^2$. (7)



Green's theorem states that

$$\int_c pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

L.H.S $\int_C pdx+Qdy = \left(\int_{AB} + \int_{BA}\right)(pdx+Qdy)$.

$$\int_{AB} pdx+Qdy = \int (3x^2-8x^4)dx + (4x^2-6x^3)\,2xdx = \int_0^1 (3x^2-8x^4+8x^3-12x^4)dx$$

$y=x^2$

$dy=2xdx$

$$= \int_0^1 (3x^2+8x^3-20x^4)dx = \left(\frac{3x^3}{3}+\frac{8x^4}{4}-\frac{20x^5}{5}\right)_0^1 = 1+2-4 = -1.$$

$$\int_{BA} Pdx+Qdy = \int (3y^4-8y^2)\,2ydy + (4y-6y^3)dy$$

$x=y^2$

$dx=2ydy$

$$= \int_1^0 (6y^5-16y^3+4y-6y^3)dy = \int_1^0 (6y^5-22y^3+4y)dy$$

$$= \left(\frac{6y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2}\right)_1^0 = 0-1+\frac{11}{2}-2 = 5/2$$

$\therefore \int pdx+Qdy = -1+5/2 = 3/2$ — (1).

R.H.S $\iint_R (Q_x-P_y)dxdy$

$$u = 3x^2-8y^2 \quad\Big|\quad V = 4y-6xy$$

$$\frac{\partial u}{\partial y} = -16y \quad\Big|\quad \frac{\partial v}{\partial x} = -6y$$

$$Q_x-P_y = -6y-(-16y) = 10y.$$

$$\therefore \iint_R (Q_x - P_y)\,dxdy = \iint_R 10y\,dxdy = 10\int_0^1\int_{y^2}^{\sqrt{y}} y\,dxdy \qquad (8)$$

$$= 10\int_0^1 y\left(\int_{y^2}^{\sqrt{y}} dx\right)dy = 10\int_0^1 y\,(x)_{y^2}^{\sqrt{y}}\,dy = 10\int_0^1 y(\sqrt{y} - y^2)\,dy$$

$$= 10\int_0^1 (y^{3/2} - y^3)\,dy = 10\left(\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right)_0^1 = 10\left(\frac{2}{5} - \frac{1}{4}\right) = 10\left(\frac{8-5}{20}\right)$$

$$= 3/2 \quad \text{—} (2).$$

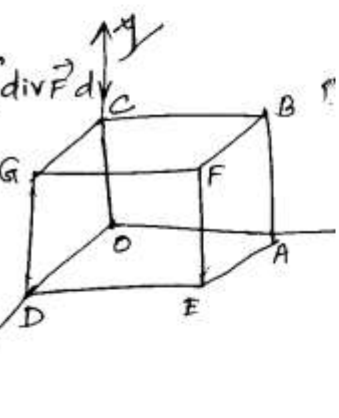
From (1) and (2),

$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dxdy$. Hence Green's theorem is verified.

b) Verify Gauss-Divergence theorem for the vector function $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$ over the cube bounded by $x=0, x=a$, $y=0, y=a$ and $z=0, z=a$.

Solution:

By Gauss Divergence theorem $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\vec{F}\,dv$



L.H.S $$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{x=a\\ \hat{n}=\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{\substack{y=a\\ \hat{n}=\vec{j}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{z=0\\ \hat{n}=-\vec{k}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{z=a\\ \hat{n}=\vec{k}}} \vec{F}\cdot\hat{n}\,ds.$$

$$= \int_0^a\int_0^a yz\,dydz + \int_0^a\int_0^a (a^3 - yz)\,dydz + \int_0^a\int_0^a 0\,dxdz + \int_0^a\int_0^a -2ax^2\,dxdz$$

$$+ \int_0^a\int_0^a -2\,dxdy + \int_0^a\int_0^a 2\,dxdy.$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^a z\,dz + \int_0^a \left(a^3y - \frac{y^2 z}{2}\right)_0^a dz - 2a\int_0^a \left(\frac{x^3}{3}\right)_0^a dz + 2\int_0^a -a\,dy + 2\int_0^a a\,dy.$$

$$= \frac{a^2}{2}\left(\frac{z^2}{2}\right)_0^a + \int_0^a \left(a^4 - \frac{a^2}{2}z\right)dz + \left(-\frac{2a^4}{3}\right)(z)_0^a - 2a(y)_0^a + 2a(y)_0^a$$

$$= \frac{a^4}{4} + \left(a^4 z - \frac{a^2 z^2}{4}\right)_0^a - \frac{2a^5}{3} - 2a^2 + 2a^2$$

$$= \frac{a^4}{4} + a^5 - \frac{a^4}{4} - \frac{2a^5}{3}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^5}{3} \quad \text{—(1)}.$$

R.H.S $\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz = \left(\frac{x^3}{3}\right)_0^a (y)_0^a (z)_0^a = \frac{a^5}{3}$ —(2)

$\therefore$ (1) = (2).

i.e $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$ $\therefore$ Gauss divergence theorem is verified.

---

a) i) Prove that every analytic function $w = u+iv$ can be expressed as a function z alone not as a function of $\bar{z}$.

Solution: Let $z = x+iy$; $\bar{z} = x-iy$.

$z+\bar{z} = x+iy+x-iy = 2x \Rightarrow x = \frac{z+\bar{z}}{2}$; $\frac{\partial x}{\partial z} = \frac{1}{2}$; $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$

$z-\bar{z} = x+iy-x+iy = 2iy \Rightarrow y = \frac{z-\bar{z}}{2i}$; $\frac{\partial y}{\partial z} = \frac{1}{2i}$; $\frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}\cdot\frac{+i}{2}$

$= -i/2$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u+iv) = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \quad \text{—(1)}$$

Now $\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right]$ —(2)

Since $f(z) = u+iv$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ ; \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also $\frac{\partial v}{\partial \bar{z}} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right]$ — (3). (10)

From (1), $\frac{\partial \omega}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + i\left[\frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)\right]$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)$$

$\frac{\partial \omega}{\partial \bar{z}} = 0 \Rightarrow \omega$ is independent of $\bar{z}$

$\Rightarrow \omega$ does not contain $\bar{z}$

ω can not expressed interms of $\bar{z}$ ie it can be expressed interms of z alone.

(ii) Find the bilinear transformation which maps the points $z = 0, 1, \infty$ into $\omega = i, 1, -i$ respectively.

Solution: $z_0 = 0,\ z_1 = 1,\ z_2 = \infty$; $\omega_0 = i,\ \omega_1 = 1,\ \omega_2 = -i.$

or

$z_1 = 0,\ z_2 = 1,\ z_3 = \infty$; $\omega_1 = i,\ \omega_2 = 1,\ \omega_3 = -i.$

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega_1-\omega_2)(\omega_3-\omega)} = \frac{(z-z_1)}{(z_1-z_2)} \cdot \frac{z_3(z_2/z_3 - 1)}{z_3(1 - z/z_3)}$$

$$\frac{(\omega-i)(1+i)}{(i-1)(-i-\omega)} = \frac{(z-0)(-1)}{(-1)(1)}$$

$$\frac{(\omega-i)}{\omega+i}\left(\frac{1+i}{1-i}\right) = z$$

$$\Rightarrow \frac{\omega-i}{\omega+i} = z\left(\frac{1-i}{1+i}\right)$$

$$= z(-i)$$

$$\omega - i = (\omega+i)(-iz)$$

$\omega - i = -i\omega z + z \Rightarrow \omega + i\omega z = z + i \Rightarrow \omega(1+iz) = z+i$

$\frac{1-i}{1+i} \times \frac{1+i}{1-i} = \frac{(1-}{1^2-}$

$= \frac{\cancel{1}+\cancel{i^2}-2i}{1-(-1)} =$

$=$

$$\therefore \boxed{w = \frac{z+i}{1+iz}}$$

b) i) Find the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$.

Solution: $w=u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\left(\frac{-y}{x^2+y^2}\right)$.

$u=\frac{x}{x^2+y^2}$ & $v=\frac{-y}{x^2+y^2}$

$x=\frac{u}{u^2+v^2}$ and $y=\frac{-v}{u^2+v^2}$.

$\therefore x^2-y^2=1 \Rightarrow \frac{u^2}{(u^2+v^2)^2}-\frac{v^2}{(u^2+v^2)^2}=1$

$\Rightarrow u^2-v^2=(u^2+v^2)^2$

$\Rightarrow u^2-v^2=u^4+v^4+2u^2v^2$

$u^4+v^4-u^2+v^2+2u^2v^2=0$

$u^2(u^2-1)+v^4+v^2+2u^2v^2=0$

$u^2(u+1)(u-1)+(v^4+v^2+2v^2u^2)=0.$

This is the image of $x^2-y^2=1$ in the w-plane.

c) Prove that the transformation $w=\frac{z}{1-z}$ maps the upper half of z-plane onto the upper half of the w-plane. What is the image of $|z|=1$ under this transformation?

Solution: $w=\frac{z}{1-z} \Rightarrow w(1-z)=z \Rightarrow w-wz=z \Rightarrow w=(1+w)z$

$z=\frac{w}{w+1}$

Putting $z=x+iy$, $w=u+iv$, we get $x+iy=\frac{u+iv}{u+iv+1}$

$x+iy=\frac{u+iv}{(u+1)+iv}\times\frac{(u+1)-iv}{(u+1)-iv}$

$=\frac{(u^2+v^2+u)+iv}{(u+1)^2+v^2}$

Equating the real and imaginary parts we get

$$x = \frac{u^2+v^2+u}{(u+1)^2+v^2} \; ; \; y = \frac{v}{(u+1)^2+v^2}$$

Now $y=0 \Rightarrow \frac{v}{(u+1)^2+v^2} = 0 \Rightarrow v = 0.$

$y>0 \Rightarrow \frac{v}{(u+1)^2+v^2} > 0 \Rightarrow v>0 \; [\because (u+1)^2+v^2>0 \text{ always}]$

Hence the upper half of the z plane $y \geq 0$ is mapped onto the upper half of the w plane $v>0$ under $w = \frac{z}{1-z}$.

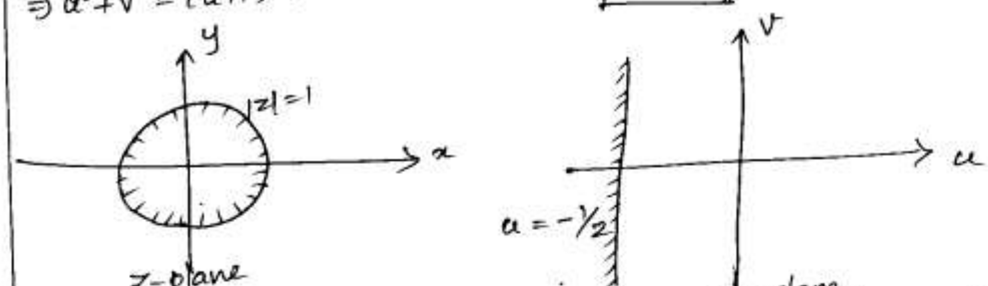
Now $|z|=1 \Rightarrow \left|\frac{w}{w+1}\right| = 1$

$\Rightarrow |w| = |w+1|$

$\Rightarrow |u+iv| = |u+iv+1|$

$\sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$

$\Rightarrow u^2+v^2 = (u+1)^2+v^2 \Rightarrow 2u+1=0 \Rightarrow \boxed{u = -\frac{1}{2}}$



$\therefore$ The image of the circle $|z|=1$ in z-plane is the straight line $u=-\frac{1}{2}$ in w-plane under $w = \frac{-z}{z-1}$ or $\frac{z}{1-z}$.

---

14) a) i) Find the Laurent's series of $f(z) = \frac{7z-2}{z(z+1)(z+2)}$ in $1<|z+1|<3$.

Solution:

Let $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$ (13)

Put $z=0$, $A=1$

$z=2$, $B=2$

$z=-1$, $C=-3$

$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

$1<|z+1|<3 \Rightarrow$ Let $u = z+1 \Rightarrow z = u-1$.

i.e $1<|u|<3$

$1<|u|$ and $|u|<3$

$\frac{1}{|u|}<1$ and $\frac{|u|}{3}<1$

$\left|\frac{1}{u}\right|<1$ and $\left|\frac{u}{3}\right|<1$

$$\therefore f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-u/3\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right] - \frac{2}{3}\left[1+u/3+(u/3)^2+\cdots\right] - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right] - \frac{3}{z+1}$$

$$f(z) = \frac{-3}{z+1} + \sum_{n=1}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n}.$$

(c) Using Cauchy's integral formula, evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where c is the $|z| = 3/2$.

Solution:-

The contour C is the circle $|z|=3/2$ with centre at the origin and radius $3/2$ and $z=0$ and $z=1$ lies inside C.

Using Partial fractions

$$\frac{4-3z}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$4-3z = A(z-1)(z-2) + B(z)(z-2) + C z(z-1)$

Put $z=0$, $4 = A(-1)(-2) \Rightarrow 4 = 2A \Rightarrow \boxed{A=2}$

put $z=1$, $1 = 0 + B(-1) + 0 \Rightarrow \boxed{B=-1}$

put $z=2$, $-2 = 0+0+C2(1) \Rightarrow -2 = 2C \Rightarrow \boxed{C=-1}$

$$\therefore \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} + \frac{-1}{z-1} + \frac{-1}{z-2}$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\int \frac{dz}{z-0} - \int \frac{dz}{z-1} - \int \frac{dz}{z-2}$$

$= 2[2\pi i f(0)] - 2\pi i f(1) - 0$ as $z=2$ lies outside c.

$= 4\pi i \cdot 1 - 2\pi i \cdot 1$

$$\int_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\pi i.$$

14) b)i) Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$ using contour integration.

Solution:

$$f(z) = \frac{z^2-z+2}{z^4+10z^2+9}.$$

The poles of $f(z)$ are given by $z^4+10z^2+9=0$.

i.e $(z^2+9)(z^2+1)=0$

i.e $z = \pm 3i, \pm i$

The two poles $z=i$ and $z=3i$ lies inside c and the other two poles $z=-i, -3i$ lies outside c.

Residue of $f(z)$ at $z=i$ $= \lim_{z\to i} (z-i) \frac{z^2-z+2}{(z-i)(z+i)(z^2+9)} = \lim_{z\to i} \frac{z^2-z+2}{(z+i)(z^2+9)}$

$= \frac{1-i}{16i}$

Residue of $f(z)$ at $z=3i$ $= \lim_{z\to 3i} (z-3i) \frac{z^2-z+2}{(z^2+1)(z+3i)(z-3i)}$

Residue of $f(z)$ at $z=3i = \underset{z\to 3i}{Lt}\ (z-3i)\dfrac{z^2-z+2}{(z^2+1)(z+3i)(z-3i)}$

$$= \underset{z\to 3i}{Lt}\ \frac{z^2-z+2}{(z^2+1)(z+3i)} = \frac{7+3i}{48i}$$

$\therefore$ Sum of the residues of $f(z)$ is $\Sigma R = \dfrac{1-i}{16i} + \dfrac{7+3i}{48i} = \dfrac{10}{48i}$

But $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz$

When $R\to\infty$, $\int_{\Gamma} f(z)dz = 0$.

$\therefore \int_C f(z)dz = \int_{-\infty}^{\infty} \dfrac{x^2-x+2}{x^4+10x^2+9}\,dx = 2\pi i\left(\dfrac{10}{48i}\right) = \dfrac{5\pi}{12}$.

Evaluate $\int_0^{2\pi} \dfrac{d\theta}{2+\cos\theta}$ using contour integration

Solution:

Let $I = \int_0^{2\pi} \dfrac{d\theta}{2+\cos\theta}$

Put $z = e^{i\theta}$ then $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z+1/z)$

Then $I = \int_C \dfrac{dz/iz}{2+\frac{1}{2}(z+1/z)}$, where C is the unit circle

$$= \int_C \frac{dz/iz}{4+\left(\frac{z^2+1}{z}\right)} = \frac{2}{i}\int_C \frac{dz}{4z+z^2+1} \quad \text{—— (1)}$$

$$= \frac{2}{i}\int_C f(z)dz \ (\text{say})$$

The poles of $f(z)$ are given by $z^2+4z+1=0 \Rightarrow z = \dfrac{-4\pm\sqrt{16-4}}{2} = \dfrac{-4\pm 2\sqrt{3}}{2}$

i.e $z = -2\pm\sqrt{3}$.

The poles are $z=-2+\sqrt{3}$, $z=-2-\sqrt{3}$.

The pole $z=-2+\sqrt{3}$ lies inside C.

$\left.\begin{array}{l}\text{Res } f(z) \text{ at} \\ z=-2+\sqrt{3}\end{array}\right\} = \underset{z\to -2+\sqrt{3}}{Lt}\ (z+2-\sqrt{3})\dfrac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$ (1b)

The pole $z = -2-\sqrt{3}$ lies outside c.

Res $f(z)$ at $z = -2-\sqrt{3} = 0$.

$\therefore$ By Residue theorem $\int_c f(z)dz = 2\pi i$ (Sum of the residue)

$$= 2\pi i\left(\frac{1}{2\sqrt{3}}\right) = \frac{\pi i}{\sqrt{3}}$$

$$① \Rightarrow I = \frac{2}{i}\left(\frac{\pi i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

$$\boxed{\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}}$$

15)a) i) Apply convolution theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$.

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \sin au \cdot \cos a(t-u)du$$ Since $\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2au - at)\right] du$$

$$= \frac{1}{2a}\int_0^t \sin at \, du + \frac{1}{2a}\int_0^t \sin(2au-at) \cdot du$$

$$= \frac{\sin at}{2a}\left(u\right)_0^t + \left(\frac{-\cos(2au-at)}{2a}\right)_0^t$$

$$= \left(\frac{t\sin at}{2a} - 0\right) + \left(\frac{-1}{2a}\right)\left(\frac{\cos(2at-at)}{1} - \cos(-at)\right)$$

$$= \frac{t\sin at}{2a} + \left(\frac{-1}{2a}\right)\left[\cos at - \cos at\right)$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a} //.$$

ii) Find the Laplace transform of the following triangular wave function given by $f(t) = \begin{cases} t, & 0 \le t \le \pi \\ 2\pi - t, & \pi \le t \le 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$. (17)

Solution:

$$L[f(t)] = \frac{1}{1-e^{-st}} \int_0^t e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \cdot t\,dt + \int_{\pi}^{2\pi} e^{-st}(2\pi - t)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\left\{ t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right) \right\}_0^{\pi} + \left\{ (2\pi - t)\frac{e^{-st}}{-s} + \left(\frac{e^{-st}}{s^2}\right) \right\}_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{-\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} + \frac{1}{s^2} + \frac{e^{-2\pi s}}{s^2} + \frac{\pi e^{-s\pi}}{s} - \frac{e^{-s\pi}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1+e^{-2\pi s} - 2e^{-s\pi}}{s^2} \right] = \frac{1}{1-e^{-2\pi s}} \frac{(1-e^{-s\pi})^2}{s^2}$$

$$= \frac{(1-e^{-s\pi})^{\cancel{2}}}{(\cancel{1-e^{-\pi s}})(1+e^{-\pi s})s^2} = \frac{1-e^{-s\pi}}{s^2(1+e^{-s\pi})} = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$$

15 (i) Verify initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution: $f(t) = 1 + e^{-t}(\sin t + \cos t)$ $\therefore F(s) = L[f(t)] = L[1 + e^{-t}\sin t + e^{-t}\cos t]$

$$\therefore F(s) = L[1] + L[e^{-t}\sin t] + L[e^{-t}\cos t] = \frac{1}{s} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\therefore F(s) = \frac{1}{s} + \frac{s+2}{s^2+2s+2} \Rightarrow sF(s) = s\left(\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right).$$

(i) Initial value theorem:-

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + e^0(0+1) = 1+1 = 2.$$

$$\underset{s\to\infty}{Lt}\; sF(s) = \underset{s\to\infty}{Lt}\; s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{\cancel{s^2}(1+2/s)}{\cancel{s^2}(1+2/s+2/s^2)}\right]$$

$$= 1 + \frac{1+0}{1+0+0}$$

$$= 1+1$$

$\underset{s\to\infty}{Lt}\; sF(s) = 2.$ Hence Initial value theorem is verified.

(ii) Final Value theorem: $\underset{t\to\infty}{Lt}\; f(t) = \underset{s\to 0}{Lt}\; sF(s)$

$$\underset{t\to\infty}{Lt}\; f(t) = \underset{t\to\infty}{Lt}\left[1 + e^{-t}(\sin t + \cos t)\right] = 1 + e^{-\infty}(\sin\infty + \cos\infty) = 1.$$

$$\underset{s\to 0}{Lt}\; sF(s) = \underset{s\to 0}{Lt}\; s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right] = \underset{s\to 0}{Lt}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1.$$

Hence Final Value theorem is verified.

---

ii) Using Laplace Transform solve the differential equation

$y'' - 3y' + 4y = 2e^{-t}$ with $y(0) = 1 = y'(0)$.

Solution: $y'' - 3y' - 4y = 2e^{-t}$

$$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L[e^{-t}]$$

$$s^2 L(y) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{1} - 3[sL(y) - \underbrace{y(0)}_{1}] - 4L[y] = 2\cdot\frac{1}{s+1}.$$

$$s^2 Y(s) - s - 1 - 3(sY(s) - 1) - 4Y(s) = \frac{2}{s+1}$$

$$(s^2-3s-4)Y(s)-s+2=\frac{2}{s+1}$$

$$(s^2-3s-4)Y(s)=\frac{2}{s+1}+s-2$$

$$\therefore Y(s)=\frac{2}{(s+1)(s^2-3s-4)}+\frac{s}{s^2-3s-4}-\frac{2}{s^2-3s-4}$$

$$=\frac{2+(s-2)(s+1)}{(s^2-3s-4)(s+1)}=\frac{\cancel{2}+s^2-s-\cancel{2}}{(s+1)^2(s-4)}$$

$$Y(s)=\frac{s^2-s}{(s+1)^2(s-4)} \quad \text{—①}$$

consider $\dfrac{s^2-s}{(s+1)^2(s-4)}=\dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s-4}$.

$$\therefore s^2-s=A(s+1)(s-4)+B(s-4)+C(s+1)^2$$

Put $s=-1$, $\quad 1+1=0+B(-5)+0 \quad \Rightarrow \boxed{B=\frac{-2}{5}}$

Put $s=4$, $\quad 16-4=0+0+25C \quad \Rightarrow \boxed{C=\frac{12}{25}}$

Put $s=0$, $\quad 0=A(1)(-4)+B(-4)+C(1)^2$

$$0=-4A-4B+C \Rightarrow 4A=C-4B \Rightarrow A=\frac{1}{4}(C-4B)$$

$$A=\frac{1}{4}\left[\frac{12}{25}+\frac{8}{5}\right]=\frac{1}{4}\left[\frac{12+40}{25}\right]=\frac{\overset{13}{\cancel{52}}}{\cancel{4}(25)}=\frac{13}{25} \quad \text{ie } \boxed{A=\frac{13}{25}}$$

$$\text{①} \Rightarrow Y(s)=\frac{13}{25}\cdot\frac{1}{s+1}-\frac{2/5}{(s+1)^2}+\frac{12/25}{s-4}$$

$$\therefore y(t)=L^{-1}\left[\frac{13}{25}\frac{1}{s+1}-\frac{2}{5}\cdot\frac{1}{(s+1)^2}+\frac{12}{25}\cdot\frac{1}{s-4}\right]$$

$$=\frac{13}{25}L^{-1}\left[\frac{1}{s+1}\right]-\frac{2}{5}L^{-1}\left[\frac{1}{(s+1)^2}\right]+\frac{12}{25}L^{-1}\left[\frac{1}{s-4}\right]$$

$$\boxed{y(t)=\frac{13}{25}e^{-t}-\frac{2}{5}e^{-t}\cdot t+\frac{12}{25}e^{4t}}$$

since $L^{-1}\left[\frac{1}{(s+1)^2}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2}\right]=e^{-t}\cdot t$.