SL ①

AU – chennai, May / June 2009

MA1254 – Random Processes

(Regulation 2004).

Part – A.

1 In a coin tossing exp. if the coin shows head, one die is thrown and the result is recorded. But if the coin shows tail, two dice are thrown and their sum is recorded. what is the probability that the recorded number will be 2?

Out of syllabus.

2 Find the value of c given the pdf of a R.v $x$ as

$$f(x) = \begin{cases} \frac{c}{x^3}, & \text{if } 1 < x < \infty \\ 0, & \text{o.w} \end{cases}$$

Soln: $\int_{\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_1^{\infty} \frac{c}{x^3} dx = 1 \Rightarrow \left(\frac{-c}{2x^2}\right)_1^{\infty} = 1 \Rightarrow \boxed{c = 2}$

3 Weibull distribution.

Out of syllabus.

4 A R.v $x$ has pdf $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$. Find the density fn of $\frac{1}{X}$.

Soln $Y = \frac{1}{X}$. W.K.T $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$

$$f_Y(y) = \frac{1}{y^2} e^{-1/y}.$$

5 State central limit theorem.

If $X_1, X_2, \ldots X_n$ be a seq of independent identically distributes RVs with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, $i = 1, 2, \ldots$ and if $S_n = X_1 + X_2 + \ldots + X_n$, then under certain general conditions, $S_n$ follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n$ tends to infinity

6) If the joint pdf of $(X, Y)$ is given by $f(x,y) = 2-x-y$ in $0 \le x \le y \le 1$, find $E(X)$.

Soln:

$$f(x) = \int_{-\infty}^{\infty} f_X(x,y)\,dy = \int_x^1 (2-x-y)\,dy = \frac{3}{2}(1-2x+x^2), \quad 0<x<1$$

$$E(X) = \int_0^1 x \, \frac{3}{2}(1-2x+x^2)\,dx = \frac{1}{8}.$$

7) Prove that the sum of two independent poisson processes is a poisson process.

Let $X(t) = X_1(t) + X_2(t)$

$$P(X(t)=n) = \sum_{r=0}^{n} P(X_1(t)=r)\, P(X_2(t)=n-r)$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \, \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= \frac{e^{-t(\lambda_1+\lambda_2)}(\lambda_1 t+\lambda_2 t)^n}{n!}$$

$\therefore X_1(t) + X_2(t)$ is a poisson process with parameter $(\lambda_1+\lambda_2)t$.

8) Define sine wave process.

A sine wave process is represented as $X(t) = A\sin(\omega t+\theta)$ where the amplitude $A$, frequency $\omega$ and phase $\theta$.

9) Define WSS. Give example.

A R.P $X(t)$ with finite first & second order moments is called WSS if its mean is constant and the auto autocorrelation depends only on the time difference

(ie) $E[x(t)] = \mu$ + $E[x(t)\, x(t-\tau)] = R(\tau)$.

Ex:

The R.P $x(t) = A\cos(\omega_0 t + \theta)$ is wss if $A$ and $\omega_0$ are constants and $\theta$ is uniformly distributed R.V in $(0, 2\pi)$.

10) The power spectral density fn of a zero mean wss process $\{x(t)\}$ is given by $S(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{o.w} \end{cases}$ find $R(\tau)$.

Soln:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (\cos\omega\tau + i\sin\omega\tau)\, d\omega.$$

$$= \frac{1}{\pi}\int_{0}^{\omega_0} \cos\omega\tau\, d\omega = \frac{\sin\omega_0\tau}{\pi\tau}$$

Part-B

11) a)

(i) Out of syllabus.

a)

(ii) Out of 2000 families with 4 children each, how many would you expect to have

(1) atleast 1 boy (2) 2 boys (3) 1 or 2 girls (4) no girls.

Soln:

$n = 4$, $p = 1/2$, $q = 1/2$

(i) $P(\text{atleast 1 boy}) = P(X \geq 1) = 1 - P(X=0) = 1 - 4C_0 (1/2)^0 (1/2)^4$

$= 15/16$.

Number of families having atleast 1 boy $= 2000 \times 15/16$

$= 1875$

2) $P(\text{2 boys}) = P(x=2) = 4C_2 (1/2)^2 (1/2)^2 = 3/8$

No of families having 2 boys $= 2000 (3/8) = 750$.

3) $P(\text{1 or 2 girls}) = P(\text{3 or 2 boys})$

$= P(x=2) + P(x=3) = 4C_2 (1/2)^2 (1/2)^2 + 4C_3 (1/2)^3 ($

$= 5/8$.

No of families having 1 or 2 girls $= 2000 (5/8) = 1250$

4) $P(\text{No girls}) = P(\text{All boys}) = P(x=4) = 1/16$

No of families have no girls $= 2000 (1/16) = 125$.

(or)

b)

(i) A R.V $x$ has the following dist

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $p(x)$ | $0.1$ | $k$ | $0.2$ | $2k$ | $0.3$ | $3k$ |

(i) Find $k$

(ii) Evaluate $P(-2 < x < 3)$

(iii) Find the CDF of $x$.

Soln: $\Sigma p(x) = 1$

(i) $\boxed{k = 1/15}$

(ii) $P(-2 < x < 3) = P(x=-1) + P(x=0) + P(x=1) + P(x=2)$

$= 7/10$.

(iii) CDF $F(x) = 0$ when $x < -2$

$= \frac{1}{10}$, when $-2 \le x \le -1$

(3)

$= \frac{1}{6}$ when $-1 \le x < 0$

$= \frac{11}{30}$, when $0 \le x < 1$

$= \frac{1}{2}$, when $1 \le x < 2$

$= \frac{4}{5}$, when $2 \le x < 3$

$= 1$, when $3 \le x$.

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11 b)

(ii) A continuous R.V has the pdf $f(x) = kx^4$, $-1 < x < 0$

Find the value of k and $P[x > -1/2 \,|\, x < -1/4]$.

Soln: $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \boxed{k = 5}$

$$P(x > -1/2 \,|\, x < -1/4) = \frac{P[x > -1/2 \cap x < -1/4]}{P[x < -1/4]} = \frac{P[-1/2 < x < -1/4]}{P[x < -1/4]}$$

$$= \frac{\int_{-1/2}^{-1/4} 5x^4\,dx}{\int_{-1}^{-1/4} 5x^4\,dx} = \frac{\frac{31}{4^5}}{\frac{1023}{4^5}} = \frac{1}{33}.$$

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12) a)

(i) Define Geometric distribution, obtain its MGF and hence compute the first four moments.

Soln:

$P(X = r) = q^r p$, $r = 0, 1, 2, \ldots \infty$

$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} q^x p = p \sum_{x=0}^{\infty} (qe^t)^x$

$$\boxed{M_x(t) = \frac{p}{1 - qe^t}}$$

$$\mu_1' = M_x'(0) = q/p$$

$$\mu_2' = M_x''(0) = pq\left[\frac{p+2q}{p^3}\right]$$

$$\mu_2 = \mu_2' - \mu_1'^2 = q/p^2 .$$

12) a) (ii) For a normal distribution with mean 2 and variance 9. find the value of $x_1$, of the variable such that the probability of the variable lying the interval $(2, x_1)$ is 0.4115.

Soln: $\mu = 2, \quad \sigma^2 = 9.$

$$P(2 < x < x_1) = 0.4115$$

$$P\left(\frac{2-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{x_1-\mu}{\sigma}\right) = 0.4115$$

$$P\left(0 < z < \frac{x_1-2}{3}\right) = 0.4115 \quad \text{——— (1)}$$

From table $P(0 < z < 1.35) = 0.4115$ ——— (2)

From (1) & (2) $\frac{x_1 - 2}{3} = 1.35 \Rightarrow \boxed{x_1 = 6.05}$

(or)

12) b (i) A random variable $X$ has a uniform distribution over the interval $(-3, 3)$. Compute (a) $P(x=2)$ (b) $P(|x-2| < 2)$ (c) Find $k$ such that $P(x > k) = 1/3$.

Soln:

$$f(x) = \begin{cases} \frac{1}{6}, & (-3, 3) \\ 0, & \text{o.w.} \end{cases}$$

(4)

(i) $P(x=2)=0$

(ii) $P(|x-2|<2) = p(-2<x-2<2) = p(0<x<4)$

$$= \int_0^4 f(x)\,dx = \int_0^3 \frac{1}{6}\,dx = \frac{1}{2}$$

(iii) $P(x>k) = 1/3 \Rightarrow \int_k^3 f(x)\,dx = \frac{1}{3} \Rightarrow \int_k^3 \frac{1}{6}\,dx = 1/3.$

$$\frac{1}{6}(3-k) = \frac{1}{3} \Rightarrow \boxed{k=1}$$

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12)

b) (ii) Define Gamma distribution. Prove that the sum of the independent Gamma Variate is a Gamma Variate.

Soln:

A continuous Ran R.V is said to follow the Gamma dist/ with parameter $\lambda$, if the prob. density fn is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda, x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

If $x_1, x_2, \ldots x_k$ are independent gamma variate with parameter $\lambda_1, \lambda_2, \ldots \lambda_k$ respectively then $X_1+X_2+\ldots X_n$ is also a gamma variate with parameters $\lambda_1+\lambda_2+\ldots+\lambda_k$.

Since $x_i$ follows gamma distribution with parameter $\lambda$.

$$\text{w.k.t} \quad M_{x_i}(t) = (1-t)^{-\lambda_i}, \quad i=1,2,\ldots k.$$

$$M_{x_1+x_2+\ldots+x_k}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \ldots M_{x_k}(t)$$

$$= (1-t)^{-\lambda_1}(1-t)^{-\lambda_2} \ldots (1-t)^{-\lambda_k}$$

$$= (1-t)^{-(\lambda_1+\lambda_2+\ldots\lambda_k)}$$

The above R.H.S is the MGF of a gamma variate with parameter $\lambda_1+\lambda_2+\ldots+\lambda_k$.

(iii) In a book of 520 pages, 390 typographical error occur. Assuming poisson's law for the number of error per page, find the probability that a random sample of 5 pages will contain no error.

Ans: The avg. no. of typographical errors per page

$$\lambda = \frac{390}{520} = 0.75$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(\text{No error in a page}) = P(X=0) = \frac{e^{-0.75}(0.75)^0}{0} = e^{-0.75}$$

$$P(5 \text{ page contain no error}) = [P(X=0)]^5 = e^{-3.75}.$$

13) a)

(i) Find the marginal and conditional densities if $f(x,y) = K(x^3y + xy^3),\ 0 \le x \le 2,\ 0 \le y \le 2$.

Soln:

$$\iint f(x,y)\,dx\,dy = 1.$$

$$\int_0^2\int_0^2 K(x^3y + xy^3)\,dx\,dy = 1. \Rightarrow \boxed{K = \frac{1}{16}}$$

$$f_X(x) = \frac{1}{16}\int_0^2 (x^3y + xy^3)\,dy = \frac{1}{8}(x^3 + 2x),\ 0 \le x \le 2.$$

$$f_Y(y) = \frac{1}{16}\int_0^2 (x^3y + xy^3)\,dx = \frac{1}{8}(2y + y^3),\ 0 \le y \le 2.$$

$$\therefore f(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{16}xy(x^2+y^2)}{\frac{1}{8}(2y+y^3)} = \frac{x(x^2+y^2)}{2(2+y^2)},\ 0 \le x \le 2,\ 0 \le y \le 2.$$

$$f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{y(x^2+y^2)}{2(x^2+2)},\ 0 \le x \le 2,\ 0 \le y \le 2$$

(5)

13) a

(ii) The joint distribution of $(X, Y)$ where $X$ and $Y$ are discrete is given in the following table.

| $X$ \ $Y$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0.1 | 0.04 | 0.06 |
| 1 | 0.2 | 0.08 | 0.12 |
| 2 | 0.2 | 0.08 | 0.12 |

verify whether $X, Y$ are independent.

Soln:

If $P_{i*} \cdot P_{*j} = P_{ij} \ \forall i \ \& \ j$ then $X$ and $Y$ are independent

| $X$ \ $Y$ | 0 | 1 | 2 | $P_{i*}$ |
|---|---|---|---|---|
| 0 | 0.1 | 0.04 | 0.06 | $P_{0*} = 0.2$ |
| 1 | 0.2 | 0.08 | 0.12 | $P_{1*} = 0.4$ |
| 2 | 0.2 | 0.08 | 0.12 | $P_{2*} = 0.4$ |
| $P_{*j}$ | $P_{*0} = 0.5$ | $P_{*1} = 0.2$ | $P_{*2} = 0.3$ | 1 |

Now $P_{0*} \cdot P_{*0} = 0.1 = P_{00}$

$P_{0*} \cdot P_{*1} = 0.04 = P_{01}$

$P_{0*} \cdot P_{*2} = 0.06 = P_{02}$

$P_{1*} \cdot P_{*0} = 0.2 = P_{10}$

$P_{1*} \cdot P_{*1} = 0.08 = P_{11}$

$P_{1*} \cdot P_{*2} = 0.12 = P_{12}$

$P_{2*} \cdot P_{*0} = 0.2 = P_{20}$

$P_{2*} \cdot P_{*0} = 0.08 = P_{21}$

$P_{2*} \cdot P_{*2} = 0.12 = P_{22}$

Hence the R.V $X$ & $Y$ are independent.

(or)

13) b)

(i) The joint pdf of a 2D R.V $(X, Y)$ is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0<x<1,\ 0<y<2 \\ 0, & \text{elsewhere.} \end{cases}$$

(1) Find $P(X > 1/2)$
(2) $P(Y < X)$
(3) $P(Y < 1/2 \mid X < 1/2)$.

Soln:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{2x}{3}, \quad 0<x<1.$$

(i) $$P(X > 1/2) = \int_{1/2}^{1} f_X(x)\,dx = \int_{1/2}^{1}\left(2x^2 + \frac{2x}{3}\right) dx = \frac{5}{6}$$

(ii) $$P(Y<X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\,dx = \frac{7}{24}.$$

(iii) $$P(Y < 1/2 \mid X < 1/2) = \frac{P[X<1/2 \cap Y<1/2]}{P[X<1/2]} = \frac{\int_0^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy}{1 - P(X > 1/2)}$$

$$= \frac{5/192}{1/6} = \frac{5}{32}.$$

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13) b

(ii) For 10 observations in price $x$ and supply $y$ the following data were obtained

$\Sigma x = 130$, $\Sigma y = 220$, $\Sigma x^2 = 2288$, $\Sigma y^2 = 5506$ & $\Sigma xy = 3467$

Obtain the line of regression of $y$ on $x$ and estimate the supply when the price is 16 units.

Ans:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{130}{10} = 13$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{220}{10} = 22,$$

(6)

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = 59.8 \Rightarrow \sigma_x = 7.73$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2 = 66.6 \Rightarrow \sigma_y = 8.16$$

$$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y} = 60.7 ,$$

$$r_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{60.7}{(7.73)(8.16)} = 0.962 .$$

The regression line of y on x is

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y = 1.02x + 8.9 .$$

The estimate supply when the price is 16 units.

$$y(16) = (1.02)(16) + 8.9 = 25.22 ,$$

14) a

(i) Classify the R.p and explain with an example.

Soln:

A R.p is a collection of RVs $\{X(s,t)\}$ that are fns of a Real variable, namely time $t$ where $s \in S$ and $t \in T$.

Depending on the continuous or discrete nature of the state space $S$ and parameter set $T$, a random process can be classified into four types.

i) If both $T$ and $S$ are discrete, the R.p is called a discrete random seq.

Ex: If $X_n$ represents the outcome of the $n^{th}$ toss of a fair die then $\{X_n, n \geq 1\}$ is a discrete random seq, since $T = \{1, 2, 3 \ldots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

2) If T is discrete and S is continuous, the R.p is called a continuous random seq.

Ex: If $x_n$ represents the temp at the end of the $n^{th}$ hour of a day, then $\{x_n, 1 \le n \le 24\}$ is a continuous random seq.

3) If T is continuous and S is discrete, the R.p is called a discrete random process.

Ex: If $x(t)$ represents the no. of telephone calls received in the interval $(0, t)$ then $\{x(t)\}$ is a discrete random process, $\because S = \{0, 1, 2 \ldots\}$

4) If both T and S are continuous, the R.p is called a continuous random process

If $\{x(t)\}$ represents the max. temp at a place in $(0, t)$ then $\{x(t)\}$ is continuous R.P.

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14) a)

(ii) Given a R.v Y with charl. fn $\varphi(\omega)$ and a random process $X(t) = \cos(\lambda t + Y)$ show that $\{x(t)\}$ is stationary in the wide sense if $\varphi(1) = 0$ and $\varphi(2) = 0$.

Soln:

$$\varphi(\omega) = E[e^{i\omega y}]$$

$$E[x(t)] = E[\cos(\lambda t + y)] = \cos\lambda t \, E(\cos y) - \sin\lambda t \, E(\sin y) \quad (1)$$

Given $\varphi(1) = 0 \Rightarrow E(e^{iy}) = 0 \Rightarrow E(\cos y + i\sin y) = 0$

$\Rightarrow E(\cos y) = 0, \; E(\sin y) = 0$,

$\therefore (1) \Rightarrow \quad E[x(t)] = 0$

which is a constant.

(7)

Auto correlation $R_{xx}(t_1,t_2) = E[x(t_1)\, x(t_2)]$

$$= E[\cos(\lambda t_1 + y)\cos(\lambda t_2 + y)]$$

$$= E[(\cos\lambda t_1 \cos y - \sin\lambda t_1 \sin y)(\cos\lambda t_2 \cos y - \sin\lambda t_2 \sin y)]$$

$$= \frac{1}{2}\cos\lambda t_1 \cos\lambda t_2 [E(1) + E(\cos 2y)]$$

$$-\frac{1}{2}(\sin\lambda t_1 \cos\lambda t_2 + \cos\lambda t_1 \sin\lambda t_2) E(\sin 2y)$$

$$+\frac{1}{2}\sin\lambda t_1 \sin\lambda t_2 [E(1) - E(\cos 2y)] \quad \text{--- (2)}$$

$E[e^{2iy}] = 0$.

$E[\cos 2y] + i\,E(\sin 2y) = 0$.

$\Rightarrow E(\cos 2y) = 0, \; E(\sin 2y) = 0$

(2) $\Rightarrow \quad R_{xx}(t_1,t_2) = \frac{1}{2}\cos\lambda(t_1 - t_2) = \cos\lambda\tau. \; = f(\tau)$

$\therefore \{x(t)\}$ is a WSS process.

(or).

14) b)
(i) There boys $x, y, z$ are throwing a ball to each other. $x$ always throws the ball to $y$ and $y$ always throws the ball to $z$, but $z$ is just as likely throw the ball to $y$ as to $x$. Show that the process is Markovian. Find the transition probl. matrix and classify the states.

Soln :

The TPM is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = P$

states of $X_n$ depends only on states of $X_{n-1}$, but not on the states of $X_{n-2}, X_{n-3}, \ldots$

$\therefore X_n$ is a Markov chain

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}, \quad P^{(3)} = P^2 \cdot P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$P^4 = P^3 \cdot P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}, \quad P^{(5)} = \begin{pmatrix} 1/4 & 1/4 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix}.$$

Classification of states $X, Y, Z$.

$P_{11}^{(3)} > 0$, $P_{11}^{(5)} > 0$, $P_{11}^{(6)} > 0 \ldots$

Period of $X = GCD(3, 5, 6) = 1$,

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0 \ldots$

Period of $Y = GCD(2, 3, 4 \ldots) = 1$,

$P_{33}^{(2)} > 0$, $P_{33}^{(3)} > 0$, $P_{33}^{(4)} = 0$.

period of $Z = GCD(2, 3, 4 \ldots) = 1$,

All the states $X, Y, Z$ have period 1.

$\therefore$ They are a period.

Also the chain is irreducible. Since there are only three states. The chain is finite and irreducible.

$\therefore$ All the states are non-null persistent.

Since all the states are a periodic and non-null persistent They are ergodic.

14) b

(ii) A machine goes out of order whenever a component fails. The failure of this part follows a poisson process with a mean rate of 1 per week. Find the prob/ that 2 weeks have elapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the prob that the machine will not be out of order in the next 10 weeks.

Soln:

Mean = 1 week,

$P[X(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $P[X(2)=0] = \frac{e^{-2}}{0!} = 0.1353$

$P[X(10) \le 5) = P(X(10)=0) + P[X(10)=1) + P[X(10)=2) + P[X(10)=3] + P[X(10)=4) + P[X(10)=5]$

$= e^{-10}\left[\frac{(10)^0}{0!} + \frac{(10)}{1!} + \frac{(10)^2}{2!} + \frac{(10)^3}{3!} + \frac{(10)^4}{4!} + \frac{(10)^5}{5!}\right]$

$= 0.068$

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15) a)

(i) Consider two R.P's $x(t) = 3\cos(\omega t + \theta)$ and $y(t) = 2\cos(\omega t + \varphi)$ where $\varphi = \theta - \frac{\pi}{2}$ and $\theta$ is uniformly distributed R.v over $(0, 2\pi)$ verify whether $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0) R_{yy}(0)}$.

Soln:

$f(\theta) = \frac{1}{2\pi}$, $0 < \theta < 2\pi$

$R_{xx}(\tau) = \frac{9}{2}\cos\omega\tau \Rightarrow R_{xx}(0) = \frac{9}{2}$

$R_{yy}(\tau) = \frac{4}{2}\cos\omega\tau \Rightarrow R_{yy}(0) = 2$.

Now $\sqrt{R_{xx}(0) R_{yy}(0)} = \sqrt{\frac{9}{2} \cdot 2} = 3$ — ①

$R_{xy}(\tau) = E[x(t)\, y(t-\tau)]$

$= E[3\cos(\omega t+\theta)\; 2\cos \omega((t-\tau)+\theta-\pi/2)]$

$= \frac{3}{2\pi}\int_0^{2\pi} \cos(2\omega t - \omega\tau + 2\theta - \pi/2)\, d\theta + 3\cos(\omega\tau + \pi/2)$

$= 3\cos(\omega\tau + \pi/2)$

$|R_{xy}(\tau)| = |3\cos(\omega\tau+\pi/2)| \leq 3$ for any value of $\tau$ — ②

From ① & ②

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)\, R_{yy}(0)}$.

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5) a)

(ii) Define power spectral density and cross spectral density of a R.P. state their properties.

<u>Soln</u> Power spectral density:

If $x(t)$ is a stationary process with autocorrelation fn $R(\tau)$, then the Fourier transform of $R(\tau)$ is called the power spectral density fn of $x(t)$ and denoted by $S_{xx}(\omega)$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

Properties:

1) The value of the spectral density fn at zero frequency is equal to the total area under the graph of the auto correlation fn $S(0) = \int_{-\infty}^{\infty} R(\tau)\, d\tau$.

2) The mean square value of a wide-sense stationary process is equal to the total area under the graph of the spectral density $E[x^2(t)] = R(0) = \int_{-\infty}^{\infty} S(f)\, df$.

(9)

3) The spectral density fn of a real R.P is an even function.

Cross-Power spectral density:

If $\{x(t)\}$ and $\{y(t)\}$ are two jointly stationary R.P with cross-correlation fn $R_{xy}(\tau)$ then the Fourier Transform of $R_{xy}(\tau)$ is called the cross-power spectral density of $x(t)$ and $y(t)$ and denoted by $S_{xy}(\omega)$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-i\omega\tau}\, d\tau ,$$

15) b)

(i) If the PSD of a WSS is given by $S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|), & |\omega| \le a \\ 0, & |\omega| > a \end{cases}$

Find the ACF of the process.

Soln:

$$R(\tau) = \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\, e^{i\omega\tau}\, d\omega = \frac{b}{2a\pi}\int_{-a}^{a}(a-|\omega|)(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} - \frac{\cos\omega\tau}{\tau^2}\right]_0^a = \frac{b}{\pi a\tau^2}\left(2\sin^2\frac{a\tau}{2}\right)$$

$$R(\tau) = \frac{ab}{2\pi}\left(\frac{\sin\frac{a\tau}{2}}{\frac{a\tau}{2}}\right)^2 .$$

15) b)

(ii) For a linear system with random input $x(t)$. The impulse response $h(t)$ and $y(t)$. Obtain the power spectrum $S_{yy}(\omega)$ and cross power spectrum $S_{xy}(\omega)$,

Soln:

If $x(t)$ is a WSS process and if

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du \text{ then}$$

W.K.T $R_{yy}(\tau) = R_{xx}(\tau) * k(\tau)$ where $k(\tau) = h(t) * h(-t)$

Taking Fourier transform on both sides

$$S_{yy}(\omega) = F[k(\tau)]\, S_{xx}(\omega) \quad \text{—— (1)}$$

let $H(\omega) = F[h(t)] = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$ —— (2)

$$H^*(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} h(-s)\, e^{-i\omega s}\, ds \quad (\text{put } s = -t)$$

$$= F(h(-t))$$

Now $k(t) = h(t) * h(-t)$

$$F[k(t)] = F[h(t)]\, F(h(-t))$$

$$= H(\omega)\, H^*(\omega)$$

$$= |H(\omega)|^2$$

Sub in (1)

$$S_{yy}(\omega) = |H(\omega)|^2\, S_{xx}(\omega)$$