B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2012
Fifth Semester
Computer Science and Engineering
MA2265 – DISCRETE MATHEMATICS
(Regulation 2008)

Part - A

1. Using truth table, show that the proposition $P \vee \sim (P \wedge Q)$ is a tautology.

Solution:

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$	$P \vee \sim (P \wedge Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since all the values in last column are true. $P \vee \sim (P \wedge Q)$ is a tautology.

2. Write the negation of the statement $(\exists x)(\forall y)p(x,y)$.

Solution:

$$\sim (\exists x)(\forall y)p(x,y) \Rightarrow (\forall x)(\exists y) \sim p(x,y)$$

3. Find the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 = 11$.

Solution:

If there are r unknowns and their sum is n, then the number of non-negative integer solution for the problem is $(n+r-1)C_{r-1}$

Here there are 3 unknowns and the sum is 7

∴ The number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 = 11$ is

$$(11+3-1)C_{3-1} = 13C_2 = 78$$

4. Find the recurrence relation for the Fibonacci sequence.

Solution:

$$f_n = f_{n-1} + f_{n-2}, n \geq 2 \; and \; f_0 = 0, f_1 = 1$$

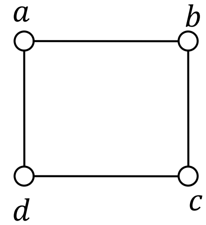
5. Define isomorphism of two graphs.

Ans:

Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exists one to one correspondence between the vertex sets preserves adjacency of the vertices.

6. Give an example of an Euler graph.

Ans:



7. Define a semi group.
Ans:
A nonempty set S together with the binary operation satisfying the following conditions
Closure: $\forall a, b \in S \Rightarrow a * b \in S$
Associative: $\forall a, b, c \in S,\ (a * b) * c = a * (b * c)$
then $(S,*)$ is called semi group.

8. If $'a'$ is a generator of a cyclic group G, then show that a^{-1} is also a generator of G.
Solution:

$$< a^{-1} >= \{(a^{-1})^n, n \in Z\}$$
$$= \{a^{-n}, n \in Z\}$$
$$= \{a^m, m \in Z\}$$
$$< a^{-1} > = < a >$$

9. In a Lattice $(L, \leq)$, prove that $a \wedge (a \vee b) = a$, for all $a, b \in G$.
Solution:
We know from the definition of GLB and LUB

$$a \wedge b \leq a \dots (1)$$
$$a \vee b \geq a \dots (2)$$
$$a \wedge (a \vee b) \leq\ a \dots (3)\quad [from\ (1)]$$
$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b)$$
$$a \wedge (a \vee b) = a \vee a \wedge b \geq a\ [from\ (2)]$$
$$\Rightarrow a \wedge (a \vee b) \geq a \dots (4)$$

From (3) and (4), we get

$$a \wedge (a \vee b) = a$$

10. Define a Boolean algebra.
Ans:
A complemented distributive lattice is called Boolean algebra.

Part - B

11. a)i) Prove that the following argument is valid: $p \to \sim q, r \to q, r \Rightarrow \sim p$.
Solution:

1.	$p \to \sim q$	Rule P
2.	$r \to q$	Rule P
3.	r	Rule P
4.	q	Rule T,2,3, Modus phones
5.	$\sim p$	Rule T,1,4, Modus tollens

ii) Determine the validity of the following argument:
If 7 is less than 4 then 7 is not a prime number, 7 is not less than 4. Therefore 7 is a prime number.
Solution:
Let L represents 7 is less than 4.
Let N represents 7 is a prime number
The inference is $L \to \sim N, \sim L \Rightarrow N$

1.	$L \to \sim N$	Rule P
2.	$\sim L$	Rule P

The argument is not valid, since $L \to \sim N, \sim L \nRightarrow N$

b) i) Verify the validity of the following argument. Every living thing is a plant or an animal. John's gold fish is alive and it is not a plant. All animals have hearts. Therefore John's gold fish has a heart.

Solution:

Let $L(x)$: x is a living thing

Let $P(x)$: x is a plant

Let $A(x)$: x is an animal

Let y represents John's gold fish

Let $H(x)$: x have heart

The inference is $\forall x\left(L(x) \to \left(P(x) \vee A(x)\right)\right), L(y) \wedge \sim P(y), \forall x(A(x) \to H(x)) \Rightarrow H(y)$

1.	$\forall x\left(L(x) \to \left(P(x) \vee A(x)\right)\right)$	Rule P
2.	$L(y) \wedge \sim P(y)$	Rule P
3.	$\forall x(A(x) \to H(x))$	Rule P
4.	$L(y) \to \left(P(y) \vee A(y)\right)$	Rule T,1,US
5.	$A(y) \to H(y)$	Rule T,3,US
6.	$L(y)$	Rule T,2, $p \wedge q \Rightarrow p$
7.	$\sim P(y)$	Rule T,2, $p \wedge q \Rightarrow q$
8.	$P(y) \vee A(y)$	Rule T,4,6, Modus phones
9.	$A(y)$	Rule T,7,8, disjunctive syllogism
10.	$H(y)$	Rule T,5,9, Modus phones

ii) Show that $(\forall x)\left(P(x) \to Q(x)\right), (\exists y)P(y) \Rightarrow (\exists x)Q(x)$.

Solution:

1.	$(\forall x)\left(P(x) \to Q(x)\right)$	Rule P
2.	$(\exists y)P(y)$	Rule P
3.	$P(a) \to Q(a)$	Rule T,1,US
4.	$P(a)$	Rule T,2,ES
5.	$Q(a)$	Rule T,3,4, Modus phones
6.	$(\exists x)Q(x)$	Rule T,5, EG

12.a) i) Prove by the principle of Mathematical induction, for $'n'$ a positive integer

$$\mathbf{1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

Solution:

$$Let\ P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \ldots (1)$$

$$P(1): 1^2 = \frac{1(1+1)(2+1)}{6}$$

$$1 = \frac{6}{6} \Rightarrow 1 = 1$$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true. Now we have to prove that $P(n+1)$ is true.

To prove:

$$P(n+1): 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad (from\ (1))$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$
$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$
$$= \frac{(n+1)[2n^2 + n + 6n + 6]}{6}$$
$$= \frac{(n+1)[2n^2 + 7n + 6]}{6}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$\therefore P(n+1)$ is true.
$\therefore$ By induction method,

$$P(n): 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \ is\ true\ for\ all\ positive\ integers.$$

ii) Find the number of distinct permutations that can be formed from all the letters of each word (1) RADAR (2) UNUSUAL.

Solution:

(1) The word RADAR contains 5 letters of which 2 A's and 2 R's are there.

$$The\ number\ of\ possible\ words\ = \frac{5!}{2!\,2!} = 30$$

Number of distinct permutation $= 30$.

(2) The word UNUSUAL contains 7 letters of which 3 U's are there.

$$The\ number\ of\ possible\ words\ = \frac{7!}{3!} = 840$$

Number of distinct permutation $= 840$.

b) Solve the recurrence relation, $S(n) = S(n-1) + 2S(n-2)$, with $S(0) = 3, S(1) = 1$, by finding its generating function.

Solution:

The given recurrence relation is $2a_{n-2} + a_{n-1} - a_n = 0$ with $a_0 = 3, a_1 = 1.$

$$Let\ G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \ldots(1)$$

where $G(x)$ is the generating function for the sequence $\{a_n\}$.

Given $2a_{n-2} + a_{n-1} - a_n = 0$

Multiplying by x_n and summing from 2 to ∞, we have

$$2\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} a_{n-1}x^n - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$2x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} + x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$2x^2(a_0 + a_1 x + a_2 x^2 + \cdots) + x(a_1 x + a_2 x^2 + \cdots) - (a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0$$
$$2x^2 G(x) + xG(x) - xa_0 - G(x) + a_0 + a_1 x = 0 \qquad [from\ (1)]$$
$$G(x)(2x^2 + x - 1) - 3x + 3 + x = 0$$
$$G(x)(2x^2 + x - 1) = 2x - 3$$

$$G(x)=\frac{2x-3}{(2x^2+x-1)}=\frac{2x-3}{-(1+x)(1-2x)}=\frac{3-2x}{(1+x)(1-2x)}$$

$$\frac{3-2x}{(1+x)(1-2x)}=\frac{A}{1+x}+\frac{B}{(1-2x)}$$

$3-2x=A(1-2x)+B(1+x)$ ... (2)

Put $x=\frac{1}{2}$ in (2)

$3-1=B\left(1+\frac{1}{2}\right)\Rightarrow\ \frac{3}{2}B=2\Rightarrow B=\frac{4}{3}$

Put $x=-1$ in (2)

$3+2=A(1+2)\Rightarrow 3A=5\Rightarrow A=\frac{5}{3}$

$$G(x)=\frac{\frac{5}{3}}{1+x}+\frac{\frac{4}{3}}{(1-2x)}$$

$$\sum_{n=0}^{\infty}a_nx^n=\frac{5}{3}\sum_{n=0}^{\infty}(-1)^n\,x^n+\frac{4}{3}\sum_{n=0}^{\infty}2^nx^n\qquad\left[\because\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\right]$$

a_n = Coefficient of x^n in $G(x)$

$a_n=\frac{5}{3}(-1)^n+\frac{4}{3}2^n$

13. a) Prove that a connected graph G is Eulerian if and only if all the vertices are on even degree.

Proof:

Suppose, G is an Euler graph. G contains an Eulerian circuit. While traversing through the circuit a vertex v is incident by two edges with one we entered and other exited. This is true, for all the vertices, because it is a circuit. Thus the degree of every vertex is even.

Conversely, suppose that all vertices of G are of even degree, we have to prove that G is an Euler graph. Construct a circuit starting at an arbitrary vertex v and going through the edge of G such that no edge id repeated. Because, each vertex is of even degree, we can exit from each end, every vertex we enter, the tracing can stop only at vertex v. Name the circuit as h. If h covers all edges of G, then G contains Euler circuit, and hence G is an Euler graph. If h does not cover all edges of G then remove all edges of h from G and obtain the remaining graph $G^{'}$. Since G and G' contains all the vertex of even degree. Every vertex in G' is also of even degree. Since G is connected, h will touch G' atleast one vertex $v^{'}$. Starting from v' we can again construct a new circuit h' in G'. This will terminate only at $v^{'}$, because every vertex in $G^{'}$ is of even degree. Now, this circuit h' combined with h forms a circuit starts and ends at v and has more edges than h, this process is repeated until we obtain a circuit covering all edges of G. Thus G is an Euler graph.

b) Show that graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2 .

Proof:

Suppose that such a partition exists. Consider two arbitrary vertices a and b of G such that $a\in V_1$ and $b\in V_2$. No path can exist between vertices a and b. Otherwise, there would be at least one edge whose one end vertex be in V_1 and the other in V_2. Hence if partition exists, G is not connected.

Conversely, let G be a disconnected graph.
Consider a vertex a in G. Let V_1be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a set V_2. No vertex in $V_1$is joined to any vertex in V_2 by an edge. Hence the partition.

14.a) Let $f : G \to G'$ be a homorphism of groups with Kernel K . Then prove that K is a normal subgroup of G and G / K is isomorphic to the image of f .
Proof:
Let f be a homomorphism from a group $(G,*)$ to a group (G',Δ), and let K be the kernel of f.

$$K = ker(f) = \{f(a) = e^{'} \backslash a \in G, e^{'} \in G'\}$$

To prove K is a subgroup of G:
We know that $f(e) = e^{'} \Rightarrow e \in K$
$\therefore K$ is a non-empty subset of G.
By the definition of homomorphism $f(a * b) = f(a)\,\Delta\, f(b), \forall a, b \in G$
Let $a, b \in K \Rightarrow f(a) = e^{'}\ and\ g(b) = e^{'}$
Now $f(a * b^{-1}) = f(a)\,\Delta\, f(b^{-1}) = f(a)\,\Delta\,\left(f(b)\right)^{-1} = e^{'}\,\Delta\,(e^{'})^{-1}$

$$= e^{'}\,\Delta\, e^{'} = e'$$

$$\therefore a * b^{-1} \in K$$

$\therefore K$ is a subgroup of G
To prove K is a normal subgroup of G:
For any $a \in G\ and\ k \in K$,

$$f(a^{-1} * k * a) = f(a^{-1})\,\Delta\, f(k)\Delta f(a) = f(a^{-1})\Delta\, f(k)\,\Delta\, f(a)$$

$$= f(a^{-1})\Delta\, e^{'}\,\Delta\, f(a) = f(a^{-1})\,\Delta\, f(a) = f(a^{-1} * a) = f(e) = e'$$

$$a^{-1} * k * a \in K$$

$\therefore K$ is a normal subgroup of G.
Let us define a mapping $h: G/K \to G'$ from the group $(G/K, \otimes)$ to the group $(G^{'}, \Delta)$ such that $h(aK) = f(a)$ ... (1)
To prove that h is well defined:
For any $a, b \in G$,

$$\therefore aK = bK$$

$$a * b^{-1} \in K$$

$$f(a * b^{-1}) = e^{'}\ [since\ k\ is\ kernel\ of\ homomorphism\ from\ G\ to\ G'\ ]$$

$$f(a)\Delta\, f(b^{-1}) = e^{'}\ \ [since\ f\ is\ homomorphism\ from\ G\ to\ G'\ ]$$

$$f(a)\Delta\left(f(b)\right)^{-1} = e^{'}\ \left[\because \left(f(b)\right)^{-1} = f(b^{-1})\ \right]$$

$$f(a)\Delta\left(f(b)\right)^{-1}\Delta\ f(b) = e^{'}\Delta\ f(b)$$

$$f(a)\Delta\, e^{'} = f(b) \Rightarrow f(a) = f(b)$$

$$h(aK) = h(bK)$$

$$aK = bK \Rightarrow h(aK) = h(bK)$$

$\therefore h$ is well defined.
To prove that h is homomorphism:

$$h(aK \otimes bK) = h\big((a * b)K\big)$$

$$= f(a * b)[from(1)]$$

$$= f(a)\,\Delta\, f(b)\ \ [since\ f\ is\ homomorphism\ from\ G\ to\ G'\ ]$$

$$= h(aK)\,\Delta\, h(bK)\ [from(1)]$$

$\therefore h$ is homomorphism
To prove that h is one to one:
For any $a, b \in G$,

$$h(aK) = h(bK)$$
$$f(a) = f(b)$$
$$f(a)\Delta\left(f(b)\right)^{-1} = f(b)\Delta\left(f(b)\right)^{-1}$$
$$f(a)\Delta f(b^{-1}) = e' \quad \left[\left(f(b)\right)^{-1} = f(b^{-1}) \ \& \ f(b)\Delta\left(f(b)\right)^{-1} = e'\right]$$
$$f(a * b^{-1}) = e' \ [since\ g\ is\ homomorphism\ from\ G\ to\ H\ ]$$
$$a * b^{-1} \in K \Rightarrow a \in Kb$$
$$\therefore aK = bK$$

$\therefore$ h is one to one
To prove that h is on to:
Let y be any element of G'.

$$f(a) = y$$
$$\therefore h(aK) = f(a) = y.$$

$\therefore$ $\forall y \in G'$ there is an pre-image aK in G/K.
$\therefore$ h is onto.
$\therefore$ $h: G/K \rightarrow G'$ is isomorphic.

b) State and Prove Lagrange's theorem.
Statement:
The order of a subgroup of a finite group is a divisor of the order of the group.
Proof:
Let aH and bH be two left cosets of the subgroup $\{H,*\}$ in the group $\{G,*\}$.
Let the two cosets aH and bH be not disjoint.
Then let c be an element common to aH and bH i.e., $c \in aH \cap bH$

$$\because c \in aH, c = a * h_1, for\ some\ h_1 \in H \ldots (1)$$
$$\because c \in bH, c = b * h_2, for\ some\ h_2 \in H \ldots (2)$$

From (1) and (2), we have

$$a * h_1 = b * h_2$$
$$a = b * h_2 * h_1^{-1} \ldots (3)$$

Let x be an element in aH
$x = a * h_3, for\ some\ h_3 \in H$
$= b * h_2 * h_1^{-1} * h_3, using\ (3)$
Since H is a subgroup, $h_2 * h_1^{-1} * h_3 \in H$
Hence, (3) means $x \in bH$
Thus, any element in aH is also an element in bH. $\therefore$ $aH \subseteq bH$
Similarly, we can prove that $bH \subseteq aH$
Hence $aH = bH$
Thus, if aH and bH are disjoint, they are identical.
The two cosets aH and bH are disjoint or identical. …(4)
Now every element $a \in G$ belongs to one and only one left coset of H in G,
For,
$a = ae \in aH, since\ e \in H \Rightarrow a \in aH$
$a \notin bH$, since aH and bH are disjoint i.e., a belongs to one and only left coset of H in G i.e., aH … (5)
From (4) and (5), we see that the set of left cosets of H in G form the partition of G. Now let the order of H be m.
Let $H = \{h_1, h_2, \ldots, h_m\}, where\ h_i's$ are distinct
Then $aH = \{ah_1, ah_2, \ldots, ah_m\}$

The elements of aH are also distinct, for, $ah_i = ah_j \Rightarrow h_i = h_j$, which is not true.
Thus H and aH have the same number of elements, namely m.
In fact every coset of H in G has exactly m elements.
Now let the order of the group $\{G,*\}$ be n, i.e., there are n elements in G
Let the number of distinct left cosets of H in G be p.
$\therefore$ The total number of elements of all the left cosets $= pm$ = the total number of elements of G. i.e., $n = pm$
i.e., m, the order of H is adivisor of n, the order of G.

15. a) Show that the direct product of any two distributive lattices is a distributive lattice.
Solution:
Let $(L,*,\oplus)$ and $(S,\wedge,\vee)$ $be\ two\ distributive\ lattices\ and\ let$ $(L \times S,.,+)$ $be\ the\ direct\ product\ of\ two\ lattices.$
For any $(a_1,b_1),(a_2,b_2)$ and $(a_3,b_3) \in L \times S$

$$(a_1,b_1).\big((a_2,b_2)+(a_3,b_3)\big) = (a_1,b_1).(a_2 \oplus a_3, b_2 \vee b_3)$$
$$= \big(a_1 * (a_2 \oplus a_3), b_1 \wedge (b_2 \vee b_3)\big)$$
$$= \big((a_1 * a_2) \oplus (a_1 * a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)\big)$$
$$= (a_1,b_1).(a_2,b_2) + (a_1,b_1).(a_3,b_3)$$

$\therefore$The direct product of any two distributive lattices is a distributive lattice.

b) Let B be a finite Boolean algebra and let A be the set of all atoms of B. Then prove that the Boolean algebra B is isomorphic to the Boolean Algebra $\rho(A)$, where $\rho(A)$ is the power set of .
Proof:
Let B be any finite Boolean algebra.
We will use induction on |B|.
We assume that the result is true when $|B| < n$.
Let B be the Boolean algebra with $|B| = n$.
Let $'a'$ be any element of B such that $0 < a < 1$.
Let B_1 be the Boolean algebra $[0,a]$ and B_2 be the Boolean Algebra $[0,a']$.
We know that $B \approx B_1 \times B_2$

$$\therefore 1 \notin B_1 \Rightarrow |B_1| < |B| = n$$
$$Similarly, 1 \notin B_2 \Rightarrow |B_2| < |B| = n$$

$\therefore$ By induction assumption, there exist a finite set X and Y such that
$B_1 \approx (\rho(X),\cup,\cap)$ and $B_2 \approx (\rho(Y),\cup,\cap)$
We know that Boolean algebra
$(\rho(X \cup Y),\cup,\cap) \approx (\rho(X),\cup,\cap) \times (\rho(Y),\cup,\cap)$
Let $Z = X \cup Y$ then

$$(\rho(Z),\cup,\cap) \approx (\rho(X),\cup,\cap) \times (\rho(Y),\cup,\cap) \approx B_1 \times B_2 \approx B$$

$\therefore B = (\rho(Z),\cup,\cap)$ for a suitable finite set Z.
Now, the smallest number of elements in Boolean algebra is 2 and any Boolean Algebra with two elements contain only 0 and 1.
If Z =Singleton set
$(\rho(Z),\cup,\cap) \approx$ the Boolean algebra with 2 elements
$\therefore$ By induction hypothesis, the result is true for any Boolean algebra.