Characteristic equation and Eigen value:

For any square matrix A, the equation $|A - \lambda I| = 0$ where λ is a scalar is called characteristic equation. Here the scalar λ is called Eigen value.

Eigen vector:

If A is a square matrix, a non-zero vector X is an Eigenvector of A if there is a scalar λ (lambda) such that

$$(A - \lambda I)X = 0$$

Properties of Eigen values and Eigen vectors:

1. The sum of Eigen values of a matrix is equal to the sum of diagonal elements of that matrix.
2. The product of Eigen values of a matrix is equal to the determinant of that matrix.
3. The Eigen values of upper and lower triangular matrices are its diagonal values.
4. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the Eigen values of a $n \times n$ matrix A then the Eigen values of the matrix
   (a) A^m are $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ where m is an integer.
   (b) $aA \pm bI$ are $a\lambda_1 \pm b, a\lambda_2 \pm b, \dots, a\lambda_n \pm b$ where a and b are real numbers.
5. If $X_1, X_2, \dots, X_n$ are the Eigen vectors of a $n \times n$ matrix A then the Eigen vectors of the matrix A^m, $aA \pm bI$ are $X_1, X_2, \dots, X_n$ where m is an integer, a and b are real numbers.
6. Eigen vectors are non zero vectors.
7. Eigen vectors are not unique.

Two Eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to 1 each. Find the Eigen values of A^{-1}.

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of matrix A.

Given $\lambda_1 = 1, \lambda_2 = 1$

We know that $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$

$$1 + 1 + \lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Eigen values of matrix A are 1,1,5.

Eigen values of matrix A^{-1} are $1, 1, \frac{1}{5}$.

The product of two Eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third Eigen value.

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of matrix A.

Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$$16\lambda_3 = 6(8) + 2(-4) + 2(4)$$

$$16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

Find the sum and product of the Eigen values of the matrix $\begin{pmatrix} 7 & 4 & 4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$

Solution:

Sum of Eigen values = $7 - 8 - 8 = -9$

Product of Eigen values= $7(63) - 4(-28) + 4(28) = 665$

One of the Eigen values of $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ is – 9, find the other two Eigen values.

Solution:

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of matrix A.

Given $\lambda_1 = -9$

We know that $\lambda_1 + \lambda_2 + \lambda_3 = 7 - 8 - 8 = -9 \Rightarrow -9 + \lambda_2 + \lambda_3 = -9 \Rightarrow \lambda_2 + \lambda_3 = 0 \ldots (1)$

$$\lambda_1 \lambda_2 \lambda_3 = |A| = \begin{vmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{vmatrix} = 7(63) - 4(-28) - 4(28) = 441$$

$$-9\lambda_2\lambda_3 = 441 \Rightarrow \lambda_2\lambda_3 = -49 \ldots (2)$$

Solving (1) and (2) we get

$$\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3 = 0 \Rightarrow \lambda^2 - 0\lambda - 49 = 0$$

$$\lambda^2 = 49 \Rightarrow \lambda = \pm 7$$

If the sum of two Eigen values and trace of 3×3 matrix A are equal, find the value of $|A|$.

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of matrix A.

Given $\lambda_1 + \lambda_2 = \text{Trace of } 3 \times 3 \text{ matrix } A$

We know that trace of 3×3 matrix $A = \lambda_1 + \lambda_2 + \lambda_3$

$$\lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 0$$

$$\lambda_1\lambda_2\lambda_3 = |A| \Rightarrow |A| = \lambda_1\lambda_2 0 = 0$$

$$\therefore |A| = 0$$

Find the Eigen values of A^2 and A^{-1}. Given the matrix $A = \begin{pmatrix} 3 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Solution:

The Eigen values of upper and lower triangular matrices are its diagonal values.

The Eigen values of A are $2, 3, 5$

The Eigen values of A^2 are $2^2, 3^2, 5^2$ i.e. $4, 9, 25$.

The Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$

Find the Eigen values and Eigen vectors of

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Solution: The characteristic equation is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = -2 + 1 + 0 = -1$

$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= -12 - 3 - 6 = -21$

$c = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(-12) - 2(-6) - 3(-3) = 24 + 12 + 9 = 45$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \ldots (1)$$

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

Here $\lambda = -3$ is one of the root of the (1). (1) reduces to $\lambda^2 - 2\lambda - 15 = 0 \Rightarrow \lambda = -3,5$

$\therefore The\ eigen\ values\ are\ \lambda = -3, -3, 5$

The Eigen vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponding to the Eigen value λ is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = -3$

$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$x + 2y - 3z = 0$

$2x + 4y - 6z = 0$

$-x - 2y + 3z = 0$

The three equations are similar so assume $z = 1, y = 0$ and substituting these values in first equation, we get

$$x + 2(0) - 3(1) = 0 \Rightarrow x = 3$$

$\therefore X_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Case 2: When $\lambda = -3$

The three equations are similar so assume $z = 0, y = 1$ and substituting these values in first equation, we get

$$x + 2(1) - 3(0) = 0 \Rightarrow x = -2$$

$\therefore X_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Case 3: When $\lambda = 5$

$$\begin{pmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-7x + 2y - 3z = 0$

$2x - 4y - 6z = 0$

$-x - 2y - 5z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}} \Rightarrow \frac{x}{-24} = \frac{-y}{48} = \frac{z}{24} \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$\therefore X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\therefore The\ eigen\ vectors\ are\ X_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Find the Eigen values and Eigen vectors of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution: The characteristic equation is given by

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = 1+5+1 = 7$

$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4-8+4 = 0$

$c = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4)-1(-2)+3(-14) = 4+2-42 = -36$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0 \ldots (1)$$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

Here $\lambda = -2$ is one of the root of the (1). (1) reduces to $\lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 3,6$

$\therefore The\ eigen\ values\ are\ \lambda = -2, 3, 6$

The Eigen vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponding to the Eigen value λ is given by $(A-\lambda I)\,X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = -2$

$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$3x + y + 3z = 0$

$x + 7y + z = 0$

$3x + y + 3z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}1 & 3\\7 & 1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}3 & 3\\1 & 1\end{vmatrix}} = \frac{z}{\begin{vmatrix}3 & 1\\1 & 7\end{vmatrix}} \Rightarrow \frac{x}{-20} = \frac{-y}{0} = \frac{z}{20} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$$\therefore X_1 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Case 2: When $\lambda = 3$

$$\begin{pmatrix}-2 & 1 & 3\\1 & 2 & 1\\3 & 1 & -2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$-2x + y + 3z = 0$

$x + 2y + z = 0$

$3x + y - 2z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}1 & 3\\2 & 1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}-2 & 3\\1 & 1\end{vmatrix}} = \frac{z}{\begin{vmatrix}-2 & 1\\1 & 2\end{vmatrix}} \Rightarrow \frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5} \Rightarrow \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$\therefore X_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Case 3: When $\lambda = 5$

$$\begin{pmatrix}-4 & 1 & 3\\1 & 0 & 1\\3 & 1 & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$-4x + y + 3z = 0$

$x + 0y + z = 0$

$3x + y - 4z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}1 & 3\\0 & 1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}-4 & 3\\1 & 1\end{vmatrix}} = \frac{z}{\begin{vmatrix}-4 & 1\\1 & 0\end{vmatrix}} \Rightarrow \frac{x}{1} = \frac{-y}{-7} = \frac{z}{-1} \Rightarrow \frac{x}{1} = \frac{y}{7} = \frac{z}{-1}$$

$\therefore X_3 = \begin{pmatrix} 1 \\ 7 \\ -1 \end{pmatrix}$

$$\therefore The\ eigen\ vectors\ are\ X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 7 \\ -1 \end{pmatrix}$$

Orthogonal matrix:

A square matrix A is said to be orthogonal if $AA^T = A^T A = I$ i.e. $A^{-1} = A^T$.

Diagonal matrix:

A square matrix A is said to be diagonal matrix if all the values of the matrix A is zero except diagonal elements.

Quadratic form:

The general form of quadratic form is $ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_2x_3 + 2fx_3x_1$

Matrix representation of Quadratic forms:

The general form of quadratic form is $ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_2x_3 + 2fx_3x_1$

The matrix of the above quadratic form is $\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}$

Rank:

The rank of the quadratic form is equal to the number of non zero Eigen values of the matrix of quadratic form.

Index:

The index of the quadratic form is equal to the number of positive Eigen values of the matrix of quadratic form.

Signature:

The index of the quadratic form is equal to the difference between the number of positive Eigen values and the number of negative Eigen values of the matrix of quadratic form.

Nature of the quadratic form:

Positive definite: If all the Eigen values of the matrix of quadratic form are positive.

Negative definite: If all the Eigen values of the matrix of quadratic form are negative.

Positive Semi definite: If all the Eigen values of the matrix of quadratic form are non negative.

Negative Semi definite: If all the Eigen values of the matrix of quadratic form are non positive.

Indefinite: If the Eigen values of the matrix of quadratic form are both non positive and non negative.

Nature of the quadratic form:

The general form of quadratic form is $ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_2x_3 + 2fx_3x_1$

The matrix of the above quadratic form is $\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}$

Let $D_1 = a, D_2 = \begin{vmatrix} a & d \\ d & b \end{vmatrix}, D_3 = \begin{vmatrix} a & d & f \\ d & b & e \\ f & e & c \end{vmatrix}$

Positive definite: If $D_1,\ D_2, D_3 > 0$.

Negative definite: If $D_1,\ D_2, D_3 < 0$.

Positive Semi definite: If $D_1,\ D_2, D_3 \geq 0$.

Negative Semi definite: If $D_1,\ D_2, D_3 \leq 0$.

Indefinite: If $D_1,\ D_2, D_3 \leq and\ \geq 0$

Find the nature of the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$

Solution: The matrix of the quadratic form is

$$A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$$

$$D_1 = 10 > 0, D_2 = \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 16 > 0, D_3 = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix} = 0$$

The nature of the quadratic form is Positive semi definite.

Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to canonical form by orthogonal transformation and also find rank, index, signature and nature.

Solution: The matrix of the quadratic form is

$$A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = 10 + 2 + 5 = 17$

$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 1 + 25 + 16$
$= 42$

$$c = |A| = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix} = 10(1) + 2(5) - 5(4) = 0$$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0 \Rightarrow \lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 42) = 0 \Rightarrow \lambda = 0\ or\ \lambda^2 - 17\lambda + 42 = 0 \ \Rightarrow \lambda = 0, 3, 14$

$\therefore The\ eigen\ values\ are\ \ \lambda = 0, 3, 14$

The Eigen vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponding to the Eigen value λ is given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = 0$

$$\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$10x - 2y - 5z = 0$

$-2x + 2y + 3z = 0$

$-5x + 3y + 5z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}-2 & -5\\ 2 & 3\end{vmatrix}} = \frac{-y}{\begin{vmatrix}10 & -5\\ -2 & 3\end{vmatrix}} = \frac{z}{\begin{vmatrix}10 & -2\\ -2 & 2\end{vmatrix}} \Rightarrow \frac{x}{4} = \frac{-y}{20} = \frac{z}{16} \Rightarrow \frac{x}{1} = \frac{-y}{5} = \frac{z}{4}$$

$$\therefore X_1 = \begin{pmatrix}1\\ -5\\ 4\end{pmatrix}$$

Case 2: When $\lambda = 3$

$$\begin{pmatrix}10-3 & -2 & -5\\ -2 & 2-3 & 3\\ -5 & 3 & 5-3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0$$

$7x - 2y - 5z = 0$

$-2x - y + 3z = 0$

$-5x + 3y + 2z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}-2 & -5\\ -1 & 3\end{vmatrix}} = \frac{-y}{\begin{vmatrix}7 & -5\\ -2 & 3\end{vmatrix}} = \frac{z}{\begin{vmatrix}7 & -2\\ -2 & -1\end{vmatrix}} \Rightarrow \frac{x}{-11} = \frac{-y}{11} = \frac{z}{-11} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore X_2 = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

Case 3: When $\lambda = 14$

$$\begin{pmatrix}10-14 & -2 & -5\\ -2 & 2-14 & 3\\ -5 & 3 & 5-14\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0$$

$-4x - 2y - 5z = 0$

$-2x - 12y + 3z = 0$

$-5x + 3y - 9z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix}-2 & -5\\ -12 & 3\end{vmatrix}} = \frac{-y}{\begin{vmatrix}-4 & -5\\ -2 & 3\end{vmatrix}} = \frac{z}{\begin{vmatrix}-4 & -2\\ -2 & -12\end{vmatrix}} \Rightarrow \frac{x}{-66} = \frac{-y}{-22} = \frac{z}{44} \Rightarrow \frac{x}{-3} = \frac{y}{1} = \frac{z}{2}$$

$$\therefore X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} 1 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 5 + 4 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 1 + 2 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -3 - 5 + 8 = 0$$

Therefore the Eigen vectors are pair wise orthogonal.

The modal matrix $M = \begin{pmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

The normalized matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{1^2 + (-5)^2 + 4^2}} & \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} & -\frac{3}{\sqrt{(-3)^2 + 1^2 + 2^2}} \\ -\frac{5}{\sqrt{1^2 + (-5)^2 + 4^2}} & \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} & \frac{1}{\sqrt{(-3)^2 + 1^2 + 2^2}} \\ \frac{4}{\sqrt{1^2 + (-5)^2 + 4^2}} & \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} & \frac{2}{\sqrt{(-3)^2 + 1^2 + 2^2}} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

The diagonal matrix $D = N^T A N$

$$D = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} \\ -\frac{42}{\sqrt{14}} & \frac{14}{\sqrt{14}} & \frac{28}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

The canonical form is $Q = Y^T D Y = (y_1 \quad y_2 \quad y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 3y_2^2 + 14y_3^2$

Rank =2, Index=2, Signature=2 and the nature is Positive semi definite.

Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 - 4x_2x_3 - 2x_3x_1 + 2x_1x_2$ to canonical form by orthogonal transformation and also find rank, index, signature and nature.

Solution: The matrix of the quadratic form is

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = 2 + 1 + 1 = 4$

$$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1$$
$$= -1$$

$$c = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = 2(-3) - 1(-1) - 1(-1) = -4$$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0 \ldots (1)$$

1	1	-4	-1	4
	0	1	-3	-4
	1	-3	-4	0

Here $\lambda = 1$ is one of the root of the (1). (1) reduces to $\lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda = -1,4$

$\therefore \textit{The eigen values are } \lambda = -1, 1, 4$

The Eigen vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponding to the Eigen value λ is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = -1$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$3x + y - z = 0$

$x + 2y - 2z = 0$

$-x - 2y + 2z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}} \Rightarrow \frac{x}{0} = \frac{-y}{-5} = \frac{z}{5} \Rightarrow \frac{x}{0} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 1$

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x + y - z = 0$

$x + 0y - 2z = 0$

$-x - 2y + 0z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} \Rightarrow \frac{x}{-2} = \frac{-y}{-1} = \frac{z}{-1} \Rightarrow \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$$\therefore X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case 3: When $\lambda = 4$

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x + y - z = 0$

$x - 3y - 2z = 0$

$-x - 2y - 3z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ -3 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix}} \Rightarrow \frac{x}{-5} = \frac{-y}{5} = \frac{z}{5} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

Therefore the Eigen vectors are pair wise orthogonal.

The modal matrix $M = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

The normalized matrix

$$N = \begin{pmatrix} \frac{0}{\sqrt{0^2+1^2+1^2}} & \frac{2}{\sqrt{2^2+(-1)^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+(-1)^2}} \\ \frac{1}{\sqrt{0^2+1^2+1^2}} & \frac{-1}{\sqrt{2^2+(-1)^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+(-1)^2}} \\ \frac{1}{\sqrt{0^2+1^2+1^2}} & \frac{1}{\sqrt{2^2+(-1)^2+1^2}} & \frac{-1}{\sqrt{1^2+1^2+(-1)^2}} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

The diagonal matrix $D = N^T A N$

$$D = \begin{pmatrix} \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The canonical form is $Q = Y^TDY = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -y_1^2 + y_2^2 + 4y_3^2$

Rank =3, Index=2, Signature=1 and the nature is indefinite.

Reduce the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 + 4x_3x_1 - 4x_1x_2$ to canonical form by orthogonal transformation and also find rank, index, signature and nature.

Solution: The matrix of the quadratic form is

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = 6 + 3 + 3 = 12$

$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14$
$= 36$

$$c = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(8) + 2(-4) + 2(-4) = 32$$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

2	1	-12	36	-32
	0	2	-20	32
	1	-10	16	0

Here $\lambda = 2$ is one of the root of the (1). (1) reduces to $\lambda^2 - 10\lambda + 16 = 0 \Rightarrow \lambda = 2,8$

$\therefore The\ eigen\ values\ are\ \lambda = 2,2,8$

The Eigen vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponding to the Eigen value λ is given by $(A - \lambda I)\,X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1: When $\lambda = 2$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

Here all the three equations are similar, so consider only one equation.

Solving the third equation by putting $x = 0, y = 1$ we get

$$2(0) - 1 + z = 0 \Rightarrow\ z = 1$$

$$\therefore X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case 2: When $\lambda = 2$

Solving the third equation by putting $x = 1, y = 0$ we get

$$2(1) - 0 + z = 0 \Rightarrow\ z = -2$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Case 3: When $\lambda = 8$

$$\begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x - 2y + 2z = 0$

$-2x - 5y - z = 0$

$2x - y - 5z = 0$

Solving first two equations we get

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}} \Rightarrow \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \Rightarrow \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$X_1^T X_2 = (0 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 0 + 0 - 2 = -2 \neq 0$$

$$X_2^T X_3 = (1 \quad 0 \quad -2)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2 + 0 - 2 = 0$$

$$X_3^T X_1 = (2 \quad -1 \quad 1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

Therefore the Eigen vectors are not pair wise orthogonal.

We have to make these Eigen vectors orthogonal

So Put either $X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $or\ X_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $(Since\ X_1^T X_2 \neq 0)$

Such that $X_1^T X_2 = 0\ and\ X_3^T X_1 = 0$

$Otherwise\ X_1^T X_2 = 0\ and\ X_2^T X_3 = 0$

$$Let\ X_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$X_1^T X_2 = 0 \Rightarrow (a \quad b \quad c)\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 0 \Rightarrow a + 0b - 2c = 0 \ldots (1)$$

$$X_3^T X_1 = 0 \Rightarrow (2 \quad -1 \quad 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow 2a - b + c = 0 \ldots (2)$$

Solving (1) and (2), we get

$$\frac{a}{\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}} \Rightarrow \frac{a}{-2} = \frac{-b}{5} = \frac{c}{-1} \Rightarrow \frac{x}{2} = \frac{y}{5} = \frac{z}{1}$$

$$\therefore X_1 = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

The modal matrix $M = \begin{pmatrix} 2 & 1 & 2 \\ 5 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$

The normalized matrix

$$N = \begin{pmatrix} \frac{2}{\sqrt{2^2+5^2+1^2}} & \frac{1}{\sqrt{1^2+0^2+(-2)^2}} & -\frac{2}{\sqrt{2^2+(-1)^2+1^2}} \\ \frac{5}{\sqrt{2^2+5^2+1^2}} & \frac{0}{\sqrt{1^2+0^2+(-2)^2}} & \frac{-1}{\sqrt{2^2+(-1)^2+1^2}} \\ \frac{1}{\sqrt{2^2+5^2+1^2}} & \frac{-2}{\sqrt{1^2+0^2+(-2)^2}} & \frac{1}{\sqrt{2^2+(-1)^2+1^2}} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & \frac{0}{\sqrt{5}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{30}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{0}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

The diagonal matrix $D = N^T AN$

$$D=\begin{pmatrix}\frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}}\\ \frac{1}{\sqrt{5}} & \frac{0}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}}\\ \frac{5}{\sqrt{30}} & \frac{0}{\sqrt{5}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{30}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$D=\begin{pmatrix}\frac{4}{\sqrt{30}} & \frac{10}{\sqrt{30}} & \frac{2}{\sqrt{30}}\\ \frac{2}{\sqrt{5}} & \frac{0}{\sqrt{5}} & \frac{-4}{\sqrt{5}}\\ \frac{16}{\sqrt{6}} & \frac{-8}{\sqrt{6}} & \frac{8}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}}\\ \frac{5}{\sqrt{30}} & \frac{0}{\sqrt{5}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{30}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 8\end{pmatrix}$$

The canonical form is $Q=Y^TDY=(y_1\quad y_2\quad y_3)\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 8\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=2y_1^2+2y_2^2+8y_3^2$

Rank =3, Index=3, Signature=3 and the nature is Positive definite.

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Verify Cayley Hamilton theorem and hence find A^{-1} *and* A^4 for

$$A=\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}$$

Solution:

The characteristic equation is given by

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$$

Here $a=sum\ of\ diagonal\ values=2+2+2=6$

$b=sum\ of\ minors\ of\ diagonal\ values=\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}+\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}=3+3+3$
$=9$

$$c=|A|=\begin{vmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{vmatrix}=2(3)+1(-1)+1(-1)=4$$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

By Cayley Hamilton's theorem,

$$A^3 - 6A^2 + 9A - 4I = 0 \ldots (1)$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Therefore Cayley Hamilton theorem is verified.

Multiply (1) by A^{-1} we get

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0 \Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$4A^{-1} = \begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

Multiply (1) by A we get

$$A(A^3 - 6A^2 + 9A - 4I) = 0 \Rightarrow A^4 - 6A^3 + 9A^2 - 4A = 0$$

$$A^4 = 6A^3 - 9A^2 + 4A$$
$$= 6\begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 9\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 4\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix}$$

Use Cayley Hamilton theorem to find the value of the matrix given by

$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Solution:

The characteristic equation is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Here $a = sum\ of\ diagonal\ values = 2 + 1 + 2 = 5$

$b = sum\ of\ minors\ of\ diagonal\ values = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 + 3 + 2 = 7$

$c = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2) - 1(0) + 1(-1) = 3$

Therefore the characteristic equation is $\lambda^3 - a\lambda^2 + b\lambda - c = 0$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley Hamilton's theorem,

$$A^3 - 5A^2 + 7A - 3I = 0 \ldots (1)$$

$$
\begin{array}{r|l}
 & A^5 + 8A + 35I \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 - 5A^7 + 7A^6 - 3A^5 \\
\hline
 & 8A^4 - 5A^3 + 8A^2 - 2A + I \\
 & 8A^4 - 40A^3 + 56A^2 - 24A \\
\hline
 & 35A^3 - 48A^2 + 22A + I \\
 & 35A^3 - 175A^2 + 245A - 105\,I \\
\hline
 & 127A^2 - 223A + 106I
\end{array}
$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + 8A + 35I) \ + \ 127A^2 - 223A + 106I$$

$$= 0(A^5 + 8A + 35I) + \ 127A^2 - 223A + 106I \ \left(from\ (1)\right)$$

$$= 127A^2 - 223A + 106I$$

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I = 127A^2 - 223A + 106I$$

$$= 127\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix} - 223\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + 106\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I = \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 158 & 285 & 295 \end{pmatrix}$$