B.E | B.Tech Degree Examination, May | June 2010

R2008, First semester

Mathematics I

PART - A

1) If 1 and 2 are the eigen values of a 2×2 matrix A, what are the eigen values of A^2 and A^{-1}.

Sln:

Eigen values of A^2 are $1^2, 2^2$ ie) 1, 4.

Eigen values of A^{-1} are $\frac{1}{1}, \frac{1}{2}$ ie) $1, \frac{1}{2}$.

2) State Cayley - Hamilton theorem.

"Every square matrix satisfies its own characteristic equation.

3) Find the centre and radius of the sphere $2(x^2+y^2+z^2)+6x-6y+8z+9=0$.

Sln:

Given sphere is

$2(x^2+y^2+z^2)+6x-6y+8z+9=0$.

$\div$ by 2 on bothsides we get,

$x^2+y^2+z^2+3x-3y+4z+\frac{9}{2}=0$.

Centre is $\left(-\frac{1}{2}\text{coeff. of } x, -\frac{1}{2}\text{coeff. of } y, -\frac{1}{2}\text{coeff. of } z\right)$

$= \left(-\frac{1}{2}(3), -\frac{1}{2}(-3), -\frac{1}{2}(4)\right) = \left(\frac{-3}{2}, \frac{3}{2}, -2\right)$

Radius $= \sqrt{\left(\frac{-3}{2}\right)^2+\left(\frac{3}{2}\right)^2+(-2)^2-\frac{9}{2}}$.

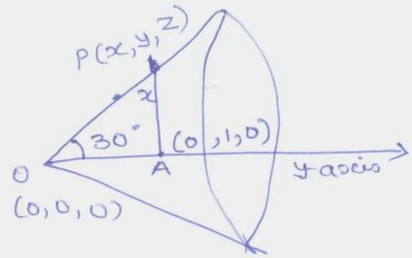
$= \sqrt{\cancel{\frac{9}{4}}+\cancel{\frac{9}{4}}+4-\cancel{\frac{9}{2}}}$

Radius $= \underline{\underline{2}}$.

4. Find the equation of the right circular Cone whose Vertex is the Origin, axis is the y-axis and semi Vertical angle is 30°

Sln:

$\angle POA = 30°$



From Δ^{le} OAP,

$$\cos 30° = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{y}{\sqrt{x^2+y^2+z^2}}.$$

$$\frac{3}{4} = \frac{y^2}{x^2+y^2+z^2}.$$

$$3(x^2+y^2+z^2) = 4y^2 \Rightarrow 3x^2+3y^2+3z^2-4y^2=0.$$

$$\Rightarrow 3x^2 - y^2 + 3z^2 = 0.$$

5. Find the radius of curvature $y = e^x$ at the point where it cuts the y-axis

Given $y = e^x$ ...(1); To find radius of curvature ρ

If it cuts the y-axis, we have to put $x = 0$ in (1).

$y = e^0 = 1$.

$\therefore$ The point at which the curve cuts the y axis is $(0,1)$.

$$\frac{dy}{dx} = e^x \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = e^0 = 1.$$

$$\frac{d^2y}{dx^2} = e^x \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{(0,1)} = e^0 = 1.$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{2^{3/2}}{1} = 2\sqrt{2}$$

6. Find the envelope of the family of straight lines $y = mx + \frac{1}{m}$, where m is a parameter

Sln:

Given $y = mx + \frac{1}{m} \longrightarrow (1)$.

Diff w. r. to 'm'

$$0 = (1)x - \frac{1}{m^2}$$

$$\frac{1}{m^2} = x.$$

$$m^2 = \frac{1}{x} \Rightarrow m = \frac{1}{\sqrt{x}}$$

$\therefore$ (1) becomes

$$y = \frac{1}{\sqrt{x}} x + \frac{1}{\frac{1}{\sqrt{x}}} = \sqrt{x} + \sqrt{x} = 2\sqrt{x}$$

$$y^2 = 4x \Rightarrow \underline{\underline{y^2 - 4x = 0}}$$

7. Given $u(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$ find the value of $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$.

Sln $u(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$.

$$u(tx, ty) = t^2 x^2 \tan^{-1}\left(\frac{\cancel{t}y}{\cancel{t}x}\right) = t^2 u(x,y).$$

$\Rightarrow u$ is a homogeneous fun/.. of degree 2.

$\therefore$ By Euler's ~~Extension~~ extension theorem,

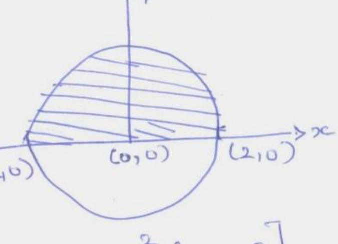
$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 2(2-1)u = 2u.$$

8. Write the sufficient condition for $f(x,y)$ to have a maximum value at (a,b).

$$\frac{\partial^2 f}{\partial x^2} < 0 \text{ or } \frac{\partial^2 f}{\partial y^2} < 0 \text{ and}$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \text{ at } (a,b)$$

9. Evaluate $\iint_R dx\,dy$ where R is the shaded region in the figure



Sln:

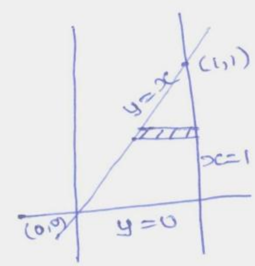
$\iint_R dx\,dx =$ Area of the shaded region

$= \frac{1}{2} \pi 2^2 \quad [\text{Area} = \pi r^2 \text{ \& } r = 2]$

$\iint_R dx\,dx = 2\pi$ sq. units

10. Change the order of integration for the double integral $\int_0^1 \int_0^x f(x,y)\,dx\,dy$.

$$\int_0^1 \int_0^x f(x,y)\,dx\,dy = \int_0^1 \int_0^x f(x,y)\,dy\,dx$$



$$= \int_0^1 \int_y^1 f(x,y)\,dx\,dy.$$

(considering horizontal strip).

11) a) (i) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sln.

We know, the characteristic equation,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 2+3+2 = 7$.

$S_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4+3+4 = 11$.

$S_3 = |A| = 2(4) - 2(2-1) + 1(2-3) = 8 - 2 - 1 = 5$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

$\lambda = 1 \Rightarrow 1 - 7 + 11 - 5 = 0$.

Using Synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1 \,\&\, 5$.

$\therefore$ Eigen values are $1, 1 \,\&\, 5$.

To find eigen vectors corresponding to each eigen values.

$(A - \lambda I)X = 0$.

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i) when $\lambda = 1$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 + x_3 = 0$; $x_1 + 2x_2 + x_3 = 0$; $x_1 + 2x_2 + x_3 = 0$

put $x_3 = 0$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2 \Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$.

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ when } \lambda = 1.$$

put $x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3$.

$\frac{x_1}{-1} = \frac{x_3}{1}$.

$$\therefore X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \text{ when } \lambda = 1.$$

case (ii) when $\lambda = 5$,

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$-3x_1 + 2x_2 + x_3 = 0$ ✓

$x_1 - 2x_2 + x_3 = 0$ ✓

$x_1 + 2x_2 - 3x_3 = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2 & 1 & -3 & 2 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2} \Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$X_3 = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1) a) (ii) Using Cayley - Hamilton theorem, find A^{-1} when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Sln.

Characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

where $S_1 = 2+2+2 = 6$

$$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$S_2 = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 6 - 1 - 2$$

$$S_3 = 3$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0 \longrightarrow (1)$$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+9 & -7-12-9 & 14+6+18 \\ -10-6-6 & 5+12+6 & -10-6-12 \\ 10+5+7 & -5-10-7 & 10+5+14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

~~put~~

Now,

$$A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$- \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence C-H theorem verified.

To find A^{-1}.

Now, consider

$$A^3 - 6A^2 + 8A - 3I = 0$$

premultiplying A^{-1} on bothsides,

$$A^2 - 6A + 8I - 3A^{-1} = 0.$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$A^{-1} = \frac{1}{3}\left\{\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & +2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

(OR)

b) Reduce the Q.F $2x^2 + 5y^2 + 3z^2 + 4xy$ to the canonical form by orthogonal reduction and state its nature.

Sln.

Matrix of the quadratic form $= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

To find the characteristic eqn.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2+5+3 = 10$

$$S_2 = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$$

$= 15 + 6 + 10 - 4.$

$S_2 = 27$

$S_3 = 2(15-0) - 2(6-0) + 0 = 30 - 12 = 18$

$\therefore \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0.$

$\lambda = 1 \Rightarrow 1 - 10 + 27 - 18 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -10 & 27 & -18 \\ & 0 & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$\lambda = 6, 3$

Eigen values are $1, 3$ & 6.

To find eigen vectors:

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

case (i): when $\lambda = 1$;

$x_1 + 2x_2 + 0x_3 = 0$ ✓

$2x_1 + 4x_2 + 0x_3 = 0 \Rightarrow \div 2 \Rightarrow x_1 + 2x_2 = 0.$

$2x_3 = 0.$ ✓

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ when } \lambda = 1.$$

case (ii) $\lambda = 3$.

$-x_1 + 2x_2 = 0 \Rightarrow x_1 - x_2 = 0$ ✓

$2x_1 + 2x_2 = 0 \Rightarrow x_1 + x_2 = 0$. ✓

$0x_3 = 0$.

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{array}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{2} \quad \therefore X_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ when } \lambda = 3.$$

case (iii) $\lambda = 6$.

$-4x_1 + 2x_2 = 0$; $2x_1 - x_2 = 0$; $-3x_3 = 0$

$\Rightarrow 2x_1 - x_2 = 0$ & $x_3 = 0$.

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ when } \lambda = 6.$$

Normalized modal matrix

$$N = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

To find $N^T A N$.

$$N^T A N = \cancel{\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D.$$

To find $Y^T (N^T A N) Y$

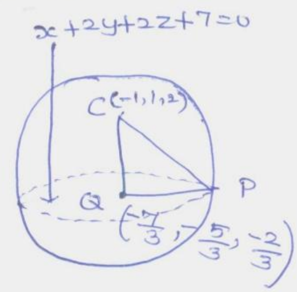
$$Y^T (N^T A N) Y = (y_1 \; y_2 \; y_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 3y_2^2 + 6y_3^2$$

The Q.F is positive definite.

12) a) (i) Find the centre and radius of the circle given by $x^2+y^2+z^2+2x-2y+4z-19=0$ and $x+2y+2z+7=0$

Sln :

The centre of the given sphere is $(-1, 1, 2)$



Radius is $CP = \sqrt{1+1+4+19} = 5$

Let $Q(x, y, z)$ be the centre of the circle.

Then Q is the foot of the $\perp$r from $C(-1,1,2)$ to the plane $x+2y+2z+7=0$.

The d.r's of CQ are $(x+1, y-1, z-2)$.

The d.r's of the normal to the plane are $1, 2, 2$

we have, $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r$

Any point on CQ is $(r-1, 2r+1, 2r+2)$.

If this point lies on $x+2y+2z+7=0$

we get,
$$r-1+2(2r+1)+2(2r+2)+7=0$$
$$r-1+4r+2+4r+4+7=0$$
$$9r+12=0$$
$$\Rightarrow r = -4/3$$

$\therefore$ The coordinates of Q are $\left(-\frac{4}{3}-1, 2\left(-\frac{4}{3}\right)+1, 2\left(\frac{-4}{3}\right)+2\right)$

ie) $\left(\frac{-7}{3}, \frac{-5}{3}, -\frac{2}{3}\right)$.

$C(-1,1,2)$ $\qquad\qquad$ $Q\left(\frac{-7}{3}, \frac{-5}{3}, \frac{-2}{3}\right)$

The centre of the circle is $\left(-\frac{7}{3}, \frac{-5}{3}, -\frac{2}{3}\right)$.

Length of $CQ = \sqrt{\left(-\frac{7}{3}+1\right)^2 + \left(-\frac{5}{3}-1\right)^2 + \left(-\frac{2}{3}-2\right)^2}$

$$= \sqrt{\frac{16}{9} + \frac{64}{9} + \frac{64}{9}} = \sqrt{\frac{144}{9}} = \frac{12}{3} = 4.$$

Radius of the circle is $QP^2 = \sqrt{CP^2 - CQ^2}$

$$= \sqrt{5^2 - 4^2} \quad \left[\because \begin{array}{l} CP = 5 \\ CQ = 4 \end{array}\right]$$

$$= 1.$$

$$\text{Area} = \pi r^2 = \pi (1)^2 = \pi.$$

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12) a) (ii) Find the equation of the cone whose vertex is the point $(1,1,0)$ and whose base is the curve $y=0$, $x^2+z^2=4$.

Sln:

Given base of the cone is

$$x^2 + z^2 = 4, \quad y = 0 \quad \longrightarrow (1)$$

Any line passes through the vertex $(1,1,0)$ is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z}{n} \quad \longrightarrow (2).$$

It meets the plane $y=0$

$$\frac{x-1}{l} = \frac{0-1}{m} = \frac{z}{n}.$$

$$\frac{x-1}{l} = \frac{-1}{m} = \frac{z}{n}$$

$$\frac{x-1}{l} = \frac{-1}{m} \qquad \frac{-1}{m} = \frac{z}{n}$$

$$x-1 = -\frac{l}{m} \qquad \boxed{z = \frac{-n}{m}}$$

$$\boxed{x = 1-\frac{l}{m}}$$

This point lies on the ellipse $x^2+z^2=4$

$$\left(1-\frac{l}{m}\right)^2 + \left(-\frac{n}{m}\right)^2 = 4 \qquad \longrightarrow (3)$$

From (2), we get, $\frac{l}{m} = \frac{x-1}{y-1}$ and $\frac{n}{m} = \frac{z}{y-1}$.

Sub these values in (3) we get,

$$\left[1-\frac{x-1}{y-1}\right]^2 + \left[-\frac{z}{y-1}\right]^2 = 4$$

$$\left[\frac{y-\cancel{1}-x+\cancel{1}}{y-1}\right]^2 + \left[\frac{z^2}{(y-1)^2}\right] = 4$$

$$(y-x)^2 + z^2 = 4(y-1)^2$$

$$x^2+y^2+z^2+2xy = 4[y^2-2y+1]$$

$$x^2-3y^2+z^2+2xy+8y-4=0$$

Which is the required equation of the cone.

(OR)

12) b) (i) Find the equation to the sphere passing through the circle $x^2+y^2+z^2=9$, $x+y+z=1$ and cuts orthogonally the sphere

$x^2+y^2+z^2+2x-4y-16z+17=0$

Sln.

The equation of a sphere through the given circle is

$$x^2+y^2+z^2+k(x+y+z-1)=0 \longrightarrow (1)$$

Centre of the sphere $x^2+y^2+z^2+2x-4y-16z+17=0$

is $\left(-\frac{1}{2}(2), -\frac{1}{2}(-4), -\frac{1}{2}(-16)\right) = (-1,2,8)$

Since sphere (1) passes through $(-1,2,8)$ we have,

$$1+4+64+k(-1+2+8-1)=0.$$

$$69+k(8)=0.$$

$$k=\frac{-69}{8}.$$

Sub $k=\frac{-69}{8}$ in (1) we get,

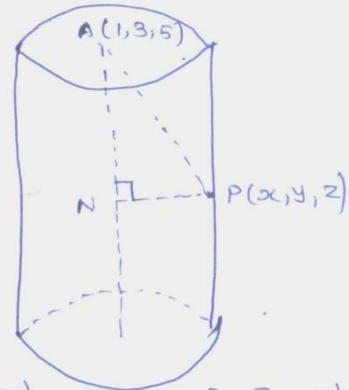
$$x^2+y^2+z^2-\frac{69}{8}(x+y+z-1)=0$$

$$8x^2+8y^2+8z^2-69x-69y-69z+69=0.$$

Which is the required eqn. of the sphere.

17) b) (ii) Find the equation of the right circular cylinder of radius 3 and axis

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}.$$



Sln: Given radius $= 3$

Axis AN is $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$

d.r's of AN are $2, 2, -1$.

d.c's of AN are $\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$ ie) $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

From figure, we have

$$PN^2 = AP^2 - AN^2$$

$$= \left[(x-1)^2 + (y-3)^2 + (z-5)^2\right] - \left[\text{projection of } AP \text{ on } AN\right]^2$$

$$(3)^2 = x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 10z + 25\Big] - \left[\frac{2}{3}(x-1) + \frac{2}{3}(y-3) - \frac{1}{3}(z-5)\right]^2$$

$$9 = x^2 - 2x + 1 + y^2 - 6y + 9 + z^2 - 10z + 25 - \frac{1}{9}\left[2x + 2y - z - 3\right]^2$$

$$9 = x^2 + y^2 + z^2 - 2x - 6y - 10z + 35 - \frac{1}{9}\left[(2x-z) + (2y-3)\right]^2$$

$$9 = x^2 + y^2 + z^2 - 2x - 6y - 10z + 35 - \frac{1}{9}\left[(2x-z)^2 + (2y-3)^2 + 2(2x-z)(2y-3)\right]$$

$$81 = 9x^2 + 9y^2 + 9z^2 - 18x - 54y - 90z + 315 - \left[4x^2 + z^2 - 4xz + 4y^2 + 9 - 12y + 8xy - 12x - 4yz + 6z\right]$$

$81 = 9x^2+9y^2+9z^2-18x-54y-90z+315$

$-4x^2-z^2+4xz-4y^2-9+12y-8xy+12x$

$+4yz-6z$

$5x^2+5y^2+8z^2-8xy+4xz+4yz-6x-42y-96z+225=0$

13) a) (i) Find the radius of curvature at any point of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

Sln:

$x=a(\theta+\sin\theta) \qquad y=a(1-\cos\theta)$

$\frac{dx}{d\theta}=a(1+\cos\theta) \qquad \frac{dy}{d\theta}=a\sin\theta$

$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}$

$=\frac{2a\sin\theta/2\cos\theta/2}{a\,2\cos^2\theta/2}$

$\frac{dy}{dx}=\tan\theta/2$

$\frac{d^2y}{dx^2}=\sec^2\theta/2\left(\frac{1}{2}\cdot\frac{d\theta}{dx}\right)$

$=\frac{1}{2}\sec^2\theta/2\,\frac{1}{a(1+\cos\theta)}$

$=\frac{1}{2a}\sec^2\theta/2\,\frac{1}{2\cos^2\theta/2}$

$\frac{d^2y}{dx^2}=\frac{1}{4a}\sec^4\theta/2=\frac{1}{4a\cos^4\theta/2}$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + \tan^2\frac{\theta}{2}\right]^{3/2}}{\frac{1}{4a\cos^4\theta/2}} = \left[\sec^{\cancel{2}}\theta/2\right]^{3/\cancel{2}} \cdot 4a\cos^4\theta/2$$

$$= \frac{4a}{\cos^3\theta/2}\cos^4\theta/2$$

$$\rho = 4a\cos\theta/2.$$

(ii) Find the circle of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$ on $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Sln: Given $\sqrt{x} + \sqrt{y} = \sqrt{a}$. $\longrightarrow$ (1)

Diff (1) w.r. to 'x'

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}} 2\sqrt{y} = \frac{-\sqrt{y}}{\sqrt{x}}.$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{4}, \frac{a}{4}\right)} = -1.$$

$$\frac{d^2y}{dx^2} = -\frac{\left[\sqrt{x}\,\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\,\frac{1}{2\sqrt{x}}\right]}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{a}{4},\frac{a}{4}\right)} = -\frac{\left[\frac{1}{2}(-1) - \frac{1}{2}\right]}{\frac{a}{4}} = \frac{4}{a}$$

$$\bar{x} = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= \frac{a}{4} - \frac{(-1)(2)}{\frac{4}{a}}$$

$$= \frac{a}{4} + \frac{2a}{4}$$

$$\bar{x} = \frac{3a}{4}$$

$$\bar{y} = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = \frac{a}{4} + \frac{[2]}{\frac{4}{a}} = \frac{a}{4} + \frac{2a}{4}$$

$$\bar{y} = \frac{3a}{4}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{2^{3/2}}{\frac{4}{a}} = \frac{2a\sqrt{2}}{4} = \frac{a}{\sqrt{2}}.$$

Hence the equation of the circle of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$ is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2} \;//$$

(OR)

13) b) (i) Show that the evolute of the parabola $y^2 = 4ax$ is the curve $27ay^2 = 4(x-2a)^3$.

Sln:

The parametric equations of the given parabola

$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$ $\qquad \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$.

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$ $\qquad \frac{d^2y}{dx^2} = \frac{-1}{t^2}\frac{dt}{dx} = \frac{-1}{t^2}\frac{1}{2at}$

$$\frac{d^2y}{dx^2} = \frac{-1}{2at^3}$$

$$\bar{X} = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

$$= at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= at^2 + \frac{2at^{\cancel{3}\,2}}{\cancel{t}}\left(1+\frac{1}{t^2}\right).$$

$$\bar{X} = 3at^2 + 2a$$

$$\frac{\bar{X} - 2a}{3a} = t^2 \Rightarrow t^6 = \frac{(\bar{X} - 2a)^3}{27a^3} \longrightarrow (1)$$

$$\bar{Y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 2at + \frac{\left[1+\frac{1}{t^2}\right]}{\frac{-1}{2at^3}}.$$

$$= 2at - 2at^3 - 2at.$$

$$\bar{Y} = -2at^3 \Rightarrow \frac{-\bar{Y}}{2a} = t^3 \Rightarrow t^6 = \frac{\bar{Y}^2}{4a^2} \longrightarrow (2)$$

From (1) & (2)

$$\frac{(\bar{x}-2a)^3}{27a^3} = \frac{\bar{y}^2}{4a^2}$$

$$27a\bar{y}^2 = 4(\bar{x}-2a)^3$$

The locus of $(\bar{x}, \bar{y})$ is $27ay^2 = 4(x-2a)^3$

(ii) Find the envelope of the straight line $\frac{x}{a}+\frac{y}{b}=1$, where a and b are connected by the relation $ab=c^2$, c is a constant.

Sln:

Given $\frac{x}{a}+\frac{y}{b}=1 \longrightarrow (1)$

$$ab = c^2 \longrightarrow (2)$$

Diff (1) w.r. to 'a'

$$\frac{-x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \Rightarrow -\frac{y}{b^2}\frac{db}{da} = \frac{x}{a^2} \longrightarrow (3)$$

Diff (2) w.r. to 'a'

$$a\frac{db}{da} + b(1) = 0 \Rightarrow \frac{db}{da} = \frac{-b}{a} \longrightarrow (4)$$

Sub (4) in (3), $-\frac{y}{b^2}\left(\frac{-b}{a}\right) = \frac{x}{a^2}$

$$\Rightarrow \frac{\frac{x}{a}}{1} = \frac{\frac{y}{b}}{1} = \frac{\frac{x}{a}+\frac{y}{b}}{1+1} = \frac{\frac{x}{a}+\frac{y}{b}}{2}$$

$$\Rightarrow \frac{x}{a} = 2 \Rightarrow a = \frac{x}{2} \text{ and } \frac{y}{b} = 2.$$

$$b = \frac{y}{2}$$

$\therefore$ (2) becomes,

$$ab = c^2 \Rightarrow 4xy = c^2$$

14) a) (i) If $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$.

prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

Sln:

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$; $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\left(\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \longrightarrow (1)$$

$$\left(\frac{\partial u}{\partial r}\right)^2 = \left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]^2$$

$$\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta \longrightarrow (2)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = r\left[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right]$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}$$

$$\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \sin^2\theta\left(\frac{\partial u}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta \longrightarrow (3)$$

(3) + (4) $\Rightarrow$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + \left(\frac{\partial z}{\partial y}\right)^2 \left(\cos^2\theta + \sin^2\theta\right)$$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

---

14) b) (i) Find the maximum and minimum values of $x^2 - xy + y^2 - 2x + y$.

Sln:

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$\frac{\partial f}{\partial x} = 2x - y - 2$	$\frac{\partial f}{\partial y} = -x + 2y + 1$
$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x - y - 2 = 0$.	$\frac{\partial f}{\partial y} = 0 \Rightarrow x - 2y = 1 \rightarrow$ (ii)
$2x - y = 2 \longrightarrow$ (i)	

Solving (i) & (ii) we get, $x = 1$, $y = 0$.

Turning point is $(1, 0)$.

$r = \frac{\partial^2 f}{\partial x^2} = 2$. $\quad (r)_{(1,0)} = 2$.

$s = \frac{\partial^2 f}{\partial x \partial y} = -1 \quad (s)_{(1,0)} = -1$.

$t = \frac{\partial^2 f}{\partial y^2} = 2 \quad (t)_{(1,0)} = 2$

$rt - s^2$ at $(1, 0)$ $\quad 3 > 0$.

Since $r > 0$, $rt - s^2 > 0$.

$\therefore (1, 0)$ - minimum point.

The minimum value of $f(1, 0) = +1 - 0 + 0 - 2 + 0 = -1$.

(ii) A rectangular box open at the top, is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction.

Sln:

Let x, y, z be the length, breadth & height of the box respectively.

The surface area $S = xy + 2yz + 2xz$

Hence we have to minimize 'S' s. to the condition that the volume of the box $xyz = 32$.

$$F = xy + 2yz + 2xz + \lambda(xyz - 32) \longrightarrow (1)$$

$\frac{\partial F}{\partial x} = 0$

$y + 2z + \lambda yz = 0$

$-\lambda = \frac{1}{z} + \frac{2}{y} \rightarrow (2)$

$\frac{\partial F}{\partial y} = 0$

$x + 2z + \lambda zx = 0$

$-\lambda = \frac{1}{z} + \frac{2}{x} \rightarrow (3)$

$\frac{\partial F}{\partial z} = 0$

$2x + 2z + \lambda xy = 0$

$-\lambda = \frac{2}{y} + \frac{2}{x} \rightarrow (4)$

$$\frac{\partial F}{\partial \lambda} = xyz - 32 = 0 \longrightarrow (5)$$

From (2) & (3), $\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$

$$x = y \longrightarrow (6)$$

From (3) & (4), $\frac{1}{z} + \cancel{\frac{2}{x}} = \frac{2}{y} + \cancel{\frac{2}{x}}$

$$y = 2z \longrightarrow (7)$$

$$x = y = 2z \longrightarrow (8)$$

Sub (8) in (5),

$$(2z)(2z)(z) = 32$$

$$4z^3 = 32$$

$$z^3 = 8$$

$$z = 2.$$

$y = 4$, $x = 4$ [From (8)]

$\therefore$ The dimensions of the box are 4, 4, 2.

15) a) (i) Change the Order of integration in the integral $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$ and hence evaluate it

Sln:

$$\text{Let } I = \int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$$

The limits for y varies from $y = \frac{x^2}{a}$ to $y = 2a - x$

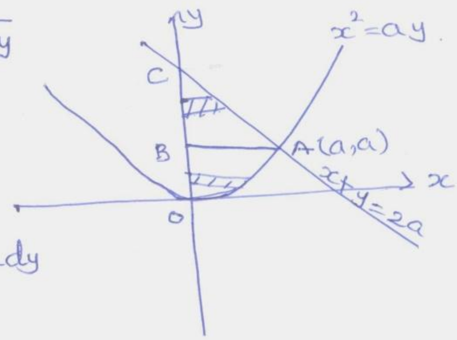
& the limits for x varies from $x = 0$ to $x = a$

Here, $y = \frac{x^2}{a}$ (or) $x^2 = ay \Rightarrow x = \sqrt{ay}$

$$y = 2a - x \text{ (or) } x + y = 2a$$

The region of integration can be split up into two parts viz (i) OAB (ii) ABC

(i) In the region OAB the limits for



x : $x = 0$ to $x = \sqrt{ay}$

y : $y = 0$ to $y = a$

$$\iint_{OAB} xy\,dx\,dy = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y\,dy = \frac{1}{2}\int_0^a ay^2\,dy = \frac{a}{2}\left[\frac{y^3}{3}\right] = \frac{a^4}{6} \longrightarrow (1)$$

In the region ABC, the limits for

x : $x = 0$ to $x = 2a - y$

y : $y = a$ to $y = 2a$.

$$\therefore \iint_{ABC} xy\,dx\,dy = \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy.$$

$$= \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_a^{2a} y\,(2a-y)^2\,dy$$

$$= \frac{1}{2}\int_a^{2a} y\,[4a^2 + y^2 - 4ay]\,dy$$

$$= \frac{1}{2}\int_a^{2a} [4a^2y + y^3 - 4ay^2]\,dy$$

$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left[\frac{4a^2 \cdot 4a^2}{2} + \frac{16a^4}{4} - \frac{32a^4}{3} - 2a^4 - \frac{a^4}{4} + \frac{4a^4}{3}\right]$$

$$= \frac{1}{2}\left[10a^4 - \frac{28a^4}{3} - \frac{a^4}{4}\right]$$

$$\iint_{ABC} xy\,dx\,dy = \frac{5a^4}{24} \longrightarrow (2)$$

Adding (1) & (2) we get,

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy = \iint_{OAB} xy\,dx\,dy + \iint_{ABC} xy\,dx\,dy$$

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy = \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}$$

(ii) By transforming into polar coordinates, evaluate

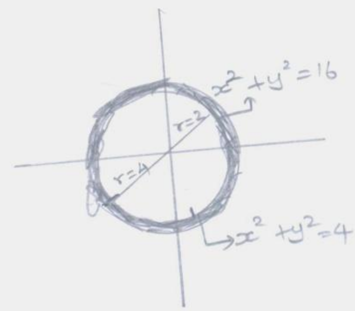
$\iint\left(\frac{x^2y^2}{x^2+y^2}\right)dx\,dy$ over annular region between

the circles $x^2+y^2=16$ and $x^2+y^2=4$.

Sln :

Let $x = r\cos\theta$; $y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta$



To cover the annular region,

$r=2$ to $r=4$

$\theta=0$ to $\theta=2\pi$

$$\therefore \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_2^4 \frac{r^2\cos^2\theta\, \cancel{r^2}\sin^2\theta}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_2^4 r^3\,dr\right)\sin^2\theta\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_2^4 \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{240}{16}\int_0^{2\pi}\sin^2 2\theta\,d\theta \qquad \left[\because 2\sin\theta\cos\theta = \sin 2\theta\right]$$

$$= 15\left[\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta\right]$$

$$= \frac{15}{2} \int_0^{2\pi} (1 - \cos 4\theta) d\theta$$

$$= \frac{15}{2} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{2\pi}$$

$$= \frac{15}{2} \left[2\pi - \frac{\sin 8\pi}{4} - 0 \right]$$

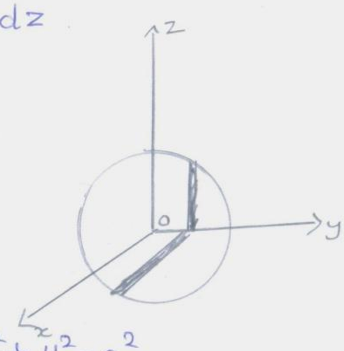
$$= \underline{\underline{15\pi}}$$

(OR)

b) (i) Find the value of $\iiint xyz \, dx dy dz$ through the positive spherical Octant for which $x^2+y^2+z^2 \le a^2$.

Sln :

Let $I = \iiint xyz \, dx \, dy \, dz$



The projection of the sphere $x^2+y^2+z^2 \le a^2$ which lies in the first octant on XOY - plane is a circle $x^2+y^2=a^2$

In the positive Octant, the limits are

z : $z=0$ to $z=\sqrt{a^2-x^2-y^2}$

y : $y=0$ to $y=\sqrt{a^2-x^2}$

x : $x=0$ to $x=a$

$$\therefore I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} z\,dz \right) xy\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(a^2xy - x^3y - xy^3\right) dy\,dx$$

$$= \frac{1}{2} \int_0^a \left[a^2x\,\frac{y^2}{2} - \frac{x^3y^2}{2} - x\frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a \left[\frac{a^2x}{2}(a^2-x^2) - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_0^a \left[\frac{x}{2}(a^2-x^2)\left[a^2-x^2\right] - \frac{x}{4}(a^2-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_0^a \frac{x}{2}(a^2-x^2)^2 \left[1-\frac{1}{2}\right] dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2-x^2)^2\left(\frac{1}{2}\right)dx$$

$$I = \frac{1}{8}\int_0^a x(a^2-x^2)^2\,dx$$

put $a^2 - x^2 = t$ | $x=0 \Rightarrow t=a^2$.

$-2x\,dt = dt$. | $x=a \Rightarrow t=0$.

$$I = \frac{1}{8}\int_{a^2}^{0} t^2\left(-\frac{dt}{2}\right) = \frac{1}{16}\int_0^{a^2} t^2\,dt.$$

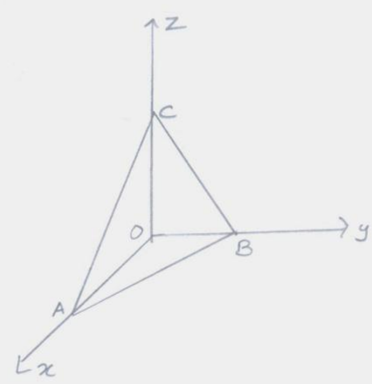
$$= \frac{1}{16}\left[\frac{t^3}{3}\right]_0^{a^2}$$

$$I = \frac{a^6}{48}$$

(ii) Find the Volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate plane $x=0,\ y=0,\ z=0$.

Sln.

Volume $= \iiint dz\,dy\,dx$



Here

z Varies from

0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y varies from 0 to $b\left(1-\frac{x}{a}\right)$.

x varies from 0 to a

$$V = \iiint_V dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz \right) dy\,dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right) - \frac{xb}{a}\left(1-\frac{x}{a}\right) - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2 - \frac{b}{2\cancel{b}}\left(1-\frac{x}{a}\right)^2 \right] dx$$

$$= bc \int_0^a \left(1-\frac{x}{a}\right)^2 \left[1-\frac{1}{2}\right]$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= \frac{-bca}{2\cdot 3}\left[1-\frac{x}{a}\right]^3\Big|_0^a$$

$$= \frac{bc}{6}a\,(-1)$$

$$V = -\frac{abc}{6}$$ cub units