Anna University - Chennai

B.E / B.Tech. Degree Examination, January 2009

First Semester

MA1101 - Mathematics - I

Part - A

1 For a given matrix A of order 3, $|A| = 32$ and two of its eigen values are 8 and 2. Find the sum of the eigen values.

Soln:

$$\lambda_1 \lambda_2 \lambda_3 = |A|.$$

$$8 \cdot 2 \cdot \lambda_3 = 32$$

$$16\lambda_3 = 32$$

$$\boxed{\lambda_3 = 2}$$

sum of the eigen values = 12.

2 Check whether the matrix B is orthogonal? Justify.

$$B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

$$BB^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ B is orthogonal.

3 Write the equation of the tangent plane at $(1,5,7)$ to the sphere $(x-2)^2+(y-3)^2+(z-4)^2 = 14$.

Soln: Given $(x-2)^2+(y-3)^2+(z-4)^2 = 14$

$$x^2+y^2+z^2-4x-6y-8z+15 = 0.$$

$2u = -4$, $u = -2$ | $2v = -6$, $v = -3$ | $2w = -8$, $w = -4$ | $d = 15$

Point of contact is $(x_1, y_1, z_1) = (1, 5, 7)$

Tangent plane is $xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$

$x+5y+7z-2(x+1)-3(y+5)-4(z+7)+15=0$

$\Rightarrow -x+2y+3z-30=0$

$x-2y-3z+30=0$.

4 Find the equation of the right circular cone whose vertex is at the origin and axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ having semivertical angle of 45°.

Soln:

$$\cos\theta = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{1+4+9}}$$

$$\frac{1}{\sqrt{2}} = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{14}}.$$

$$\sqrt{7}\sqrt{x^2+y^2+z^2} = x+2y+3z$$

$$7(x^2+y^2+z^2)=(x+2y+3z)^2$$

$$6x^2+3y^2-2z^2-4xy-12yz-6xz=0$$

5 Find the envelope of the lines $y=mx \pm \sqrt{a^2m^2+b^2}$ where m is the parameter.

Soln: $y=mx \pm \sqrt{a^2m^2+b^2}$

$(y-mx)=\sqrt{a^2m^2+b^2}$

Squaring on both sides

$(y-mx)^2=(a^2m^2+b^2)$

$y^2+m^2x^2-2xmy=a^2m^2+b^2$

$(x^2-a^2)m^2-2xym+(y^2-b^2)=0$.

This is a quadratic eqn in m, with $A=x^2-a^2$, $B=-2xy$, $C=y^2-b^2$

$\therefore$ The envelope is $B^2-4AC=0$.

$4x^2y^2-4(x^2-a^2)(y^2-b^2)=0$.

$b^2x^2+a^2y^2=a^2b^2$

$\div a^2b^2$ $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is an ellipse.

6 Define the circle of curvature of a point $p(x_1, y_1)$ on the curve $y = f(x)$.

Ans: The circle which touches the curve at P and whose radius is equal to the radius of curvature is known as circle of curvature.

7 Using Euler's theorem, given $u(x,y)$ is a homogeneous function of degree n, prove that $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = n(n-1)u$.

Soln:

By Eulers theorem $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$. —— (1)

Diff p.w.r.t x & y we get

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x} \quad \text{—— (2)}$$

$$x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \quad \text{— (3)}$$

$\times$ (2) by x and (3) by y and adding

$$x^2 u_{xx} + 2xy u_{xy} + u_{yy} = n(n-1)u.$$

8 Using the definition of total derivative, find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$.

Soln: $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (-4a)(2at) + (2y)(2a)$

$= -8a^2t + 8a^2t$

$= 0.$

9 Write down the double integral, to find the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\text{Area} = 2\int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta. = 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

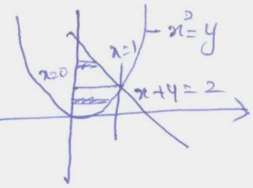
$$= \int_0^{\pi/2}(16\sin^2\theta - 4\sin^2\theta)\,d\theta = \int_0^{\pi/2} 12\sin^2\theta\,d\theta$$

$$= 12\int_0^{\pi/2} \sin^2\theta \, d\theta = 12 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 3\pi .$$

10 Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\,dx\,dy$.

Soln:

$$\int_0^1 \int_{x^2}^{2-x} f(x,y)\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-y} f(x,y)\,dy\,dx.$$



Part B

11) a) Find the characteristic equation of the matrix A given

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence find A^{-1} and A^4.

soln:

The char. eqn is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

$s_1 = 6, \quad s_2 = -9, \quad s_3 = 4$.

C.E is $\lambda^3 - 6\lambda^2 - 9\lambda + 4 = 0$.

By using Cayley-Hamilton theorem $A^3 - 6A^2 - 9A + 4I = 0$ — ①

Pre multiplying ① by A^{-1}, $A^2 - 6A + 9I - 4A^{-1} = 0$.

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$= \frac{1}{4}\left\{ \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + \begin{pmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} \right\}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Pre-multiplying ① by A, $A^4 = 6A^3 - 9A^2 + 4A$

$$A^4 = 6\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 9\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}.$$

(ii) Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Soln:

The characteristic eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 5$

$S_2 = 7$

$S_3 = 3$

$\therefore$ (ie) $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

The eigen values are $\lambda = 1, 1, 3$.

The eigen vector is $(A - \lambda I)X = 0$.

The eigen vectors are $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(or)

b) Reduce the given quadratic form Q to its canonical form using orthogonal transformation. $Q = x^2 + 3y^2 + 3z^2 - 2yz$.

Soln: The given quadratic form is $Q = x^2 + 3y^2 + 3z^2 - 2yz$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic eqn of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$.

The eigen values are $\lambda = 1, 2, 4$.

The eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Modal matrix $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Normalised modal matrix $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$D = N^{T}AN$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Canonical form $Q = y^{T}Dy$

$$= (y_1 \;\; y_2 \;\; y_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= y_1^2 + 2y_2^2 + 4y_3^2 .$$

12) a) (i) Obtain the equation of the sphere having the circle $x^2+y^2+z^2=9$, $x+y+z=3$ as a great circle.

Soln:

$S : x^2+y^2+z^2-9=0$, $U = x+y+z-3=0$.

The required sphere is $S_1 = S+kU=0$.

$S_1 : x^2+y^2+z^2-9+k(x+y+z-3)=0$.

$2u = k$, $u = k/2$

$2v = k$, $v = k/2$

$2w = k$, $w = k/2$,

Centre $C(-u,-v,-w) = \left(\frac{-k}{2}, -\frac{k}{2}, \frac{-k}{2}\right)$

$\therefore x+y+z=3 \Rightarrow -\frac{k}{2}-\frac{k}{2}-\frac{k}{2}=3 \Rightarrow \boxed{k=-2}$

(ie) $x^2+y^2+z^2-2x-2y-2z-3=0$.

(ii) Find the equation of the right circular cylinder whose axis is the line $x=2y=-z$ and radius 4.

Soln:

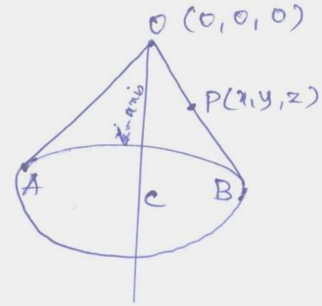
Let $P(x,y,z)$ be any point on the surface of the cylinder.

In right triangle AMP, $AP^2 = AM^2+MP^2$

$$(x-0)^2+(y-0)^2+(z-0)^2 = \left[\frac{(x-0)(1)+(y-0)(1/2)+(z-0)(-1)}{\sqrt{1^2+(\frac{1}{2})^2+1^2}}\right]+16$$

$$5x^2+8y^2+5z^2-xy+yz+2xz-36=0.$$

b) (i) Find the equation of the right circular cone whose vertex is at the origin and axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has semivertical angle of 30°.



Soln:

Direction ratio for $OP = x, y, z$

Direction ratio for $OC = 1, 2, 3$

$$\cos\theta = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{1+4+9}}$$

$$\cos 30^\circ = \frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}\sqrt{14}}$$

$$19x^2+13y^2+3z^2-8xy-24yz-12zx=0.$$

which is the required cone.

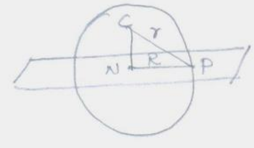
(ii) Find the equation of the sphere described on the line joining the points $(2,-4,4)$ and $(-2,2,-2)$ as diameter. Find the area of the circle in which this sphere is cut by the plane $2x+y-z=3$.

Soln:

$$S = (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$$

$$S = x^2+y^2+z^2-y-2z-14$$

centre $(0, 1/2, 1)$



$$r = \sqrt{0+\frac{1}{4}+1+14} = \frac{\sqrt{61}}{2}$$

CN = ⊥r distance from centre to the plane $= \frac{-7}{2\sqrt{6}}$

R = radius of the circle $= \sqrt{r^2 - CN^2} = \sqrt{\frac{317}{24}}$

Area $= \pi R^2 = \pi\left(\frac{317}{24}\right)$ sq. units.

13) a)(i) Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ considering it as the envelope of its normal.

Soln:

Any point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(a\sec\theta, b\tan\theta)$

$x = a\sec\theta$, $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$, $\frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b}{a}\operatorname{cosec}\theta = m.$$

The eqn of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$$(y - b\tan\theta) = -\frac{a}{b\operatorname{cosec}\theta}(x - a\sec\theta)$$

$$\Rightarrow \sec\theta = \left(\frac{ax}{a^2+b^2}\right)^{1/3}, \quad \tan\theta = \left(\frac{-by}{a^2+b^2}\right)^{1/3}$$

$$\Rightarrow (ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

(ii) Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Soln: $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\sqrt{y} = \sqrt{a} - \sqrt{x} \quad \text{——} \quad (1)$$

Diff (1) w.r.to x.

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$y_1\left(\frac{a}{4}, \frac{a}{4}\right) = \frac{-\sqrt{a/4}}{\sqrt{a/4}} = -1$$

Diff ① w.r.to x,

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} - \sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right]$$

$$y_2(a/4, a/4) = -\left[\frac{\sqrt{a/4}\ \frac{1}{2\sqrt{a/4}}(-1) - \sqrt{a/4}\cdot\frac{1}{2\sqrt{a/4}}}{a/4}\right] = -\frac{\left[-\frac{1}{2}-\frac{1}{2}\right]}{\frac{a}{4}}$$

$$y_2(a/4, a/4) = \frac{4}{a}.$$

$\therefore$ The radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+(-1)^2]^{3/2}}{4/a} = \frac{\sqrt{2}a}{2}$.

(or)

b(i) Find the equation of circle of curvature of the parabola $y^2=12x$ at the point $(3,6)$.

Soln: $y^2=12x$

$$y_1 = \frac{dy}{dx} = \frac{6}{y} \Rightarrow y_1(3,6) = 1$$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{6}{y^2}\frac{dy}{dx} \Rightarrow y_2(3,6) = \frac{-1}{6}.$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = 15, \quad \bar{y} = y + \frac{1+y_1^2}{y_2} = -6$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = -12\sqrt{2}.$$

Circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

(ie) $x^2+y^2+12y-57=0$.

(ii) Find the envelope of the family of the lines $\frac{x}{a}+\frac{y}{b}=1$ subject to the condition that $a+b=1$.

Soln:

$$\frac{x}{a}+\frac{y}{b} = 1 \Rightarrow \frac{da}{db} = -\frac{y}{x}\frac{a^2}{b^2}$$

$a+b=1 \Rightarrow \frac{da}{db}=-1$

$\therefore \frac{-y}{x}\frac{a^2}{b^2}=-1 \Rightarrow \frac{x}{a^2}=\frac{y}{b^2}$

$$\frac{\frac{x}{a}}{a}=\frac{\frac{y}{b}}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{1} \Rightarrow \frac{x}{a^2}=1,\ \frac{y}{b^2}=1$$

$\Rightarrow a=\sqrt{x},\ b=\sqrt{y}$

$\therefore a+b=1$

$\boxed{\sqrt{x}+\sqrt{y}=1}$

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14)a) (i) If $u=x^y$ show that $u_{xxy}=u_{xyx}$.

Soln: $u=x^y \Rightarrow \frac{\partial u}{\partial y}=x^y \log x$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=yx^{y-1}\log x+x^y\left(\frac{1}{x}\right)$$
$$=x^{y-1}(y\log x+1)$$

$$\frac{\partial^3 u}{\partial x^2\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x\partial y}\right)=\frac{\partial}{\partial x}\left(x^{y-1}(y\log x+1)\right)$$

Now $\frac{\partial u}{\partial x}=yx^{y-1}$

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=x^{y-1}+yx^{y-1}\log x=x^{y-1}(y\log x+1).$$

$$\frac{\partial^3 u}{\partial x\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\partial x}\right)=\frac{\partial}{\partial x}\left(x^{y-1}(y\log x+1)\right)$$

$\therefore u_{xxy}=u_{xyx}$.

(ii) If $u=\log(x^2+y^2)+\tan^{-1}\left(\frac{y}{x}\right)$ prove that $u_{xx}+u_{yy}=0$.

Soln: $u=\log(x^2+y^2)+\tan^{-1}(y/x)$

$$u_x=\frac{2x}{x^2+y^2}+\frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right)=\frac{2x-y}{x^2+y^2}$$

$$u_{xx}=\frac{(x^2+y^2)(2)-(2x-y)2x}{(x^2+y^2)^2}=\frac{-2x^2+2y^2+2xy}{(x^2+y^2)^2} \quad \text{—(1)}$$

$$u_y = \frac{2y}{x^2+y^2} + \frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) = \frac{2y+x}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(2)-(2y+x)2y}{(x^2+y^2)^2} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

$\therefore u_{xx}+u_{yy}=0.$

(iii) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, and $z=r\cos\theta$.

Soln:

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\left[\sin^3\theta+\cos^2\theta\sin\theta\right] = r^2\sin\theta.$$

b)
(i) Find the maximum value of $x^m y^n z^p$ subject to the condition $x+y+z=a$.

Soln: Let $f(x,y,z) = x^m y^n z^p$

$\phi(x,y,z) = x+y+z-a$

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial\phi}{\partial x} = 0 \Rightarrow m x^{m-1}y^n z^p+\lambda=0 \quad \text{——} (1)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial\phi}{\partial y} = 0 \Rightarrow n y^{n-1}x^m z^p+\lambda=0 \quad \text{——} (2)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial\phi}{\partial z} = 0 \Rightarrow p z^{p-1}x^m y^n+\lambda=0 \quad \text{——} (3)$$

From (1), (2) & (3) we get

$$-\lambda = m x^{m-1}y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$$

(ie) $\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}.$

Hence the maximum value of f occurs when

$$x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p}$$

$\therefore$ The maximum value of $f = a^{m+n+p} \cdot \dfrac{m^n n^n p^p}{(m+n+p)^{m+n+p}}$.

(ii) Find the Taylors series expansion of $e^x \sin y$ at the point $\left(-1, \frac{\pi}{4}\right)$ upto 3rd degree term.

Soln: $f(x,y) = e^x \sin y$

$f_x = e^x \sin y$

$f_y = e^x \cos y$

$f_{xx} = e^x \sin y$

$f_{xy} = e^x \cos y$

$f_{yy} = -e^x \sin y$

$f_{xxx} = e^x \sin y$

$f_{xxy} = e^x \cos y$

$f_{xyy} = -e^x \sin y$

$f_{yyy} = -e^x \cos y$

$f(-1, \pi/4) = \frac{1}{e\sqrt{2}}, \quad f_x\left(-1, \frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}}, \quad f_y(-1, \pi/4) = \frac{1}{e\sqrt{2}}, \ldots$

The Taylor's series is

$$f(x,y) = \frac{1}{e\sqrt{2}} + \frac{1}{1!}\left[(x+1)\frac{1}{e\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\frac{1}{e\sqrt{2}}\right]$$
$$+ \frac{1}{2!}\left[(x+1)^2\frac{1}{e\sqrt{2}} - \left(y-\frac{\pi}{4}\right)^2\frac{1}{e\sqrt{2}} + 2(x+1)\left(y-\frac{\pi}{4}\right)\frac{1}{e\sqrt{2}}\right]$$
$$+ \frac{1}{3!}\left[(x+1)^3\frac{1}{e\sqrt{2}} + 3(x+1)^2\left(y-\frac{\pi}{4}\right)\frac{1}{e\sqrt{2}} - 3(x+1)\left(y-\frac{\pi}{4}\right)^2\frac{1}{e\sqrt{2}}\right.$$
$$\left. - \left(y-\frac{\pi}{4}\right)^3\frac{1}{e\sqrt{2}}\right] + \cdots$$

15] a

(i) Find the area inside the circle $r = a\sin\theta$ but lying outside the cardioid $r = a(1-\cos\theta)$.

Soln:

$r = a\sin\theta$ — ①

$r = a(1-\cos\theta)$ — ②

Solving ① & ② we get $a\sin\theta = a(1-\cos\theta)$

$$\sin\theta + \cos\theta = 1$$

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{\pi}{4}+\theta\right) = \sin\frac{\pi}{4} \Rightarrow \frac{\pi}{4}+\theta = \frac{\pi}{4} \text{ (or) } \pi - \frac{\pi}{4}$$

$$\theta = 0 \text{ or } \theta = \pi/2$$

$$\text{Area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2}\left(a^2\sin^2\theta - a^2(1-\cos\theta)^2\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(\sin^2\theta - \cos^2\theta - 1 + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2}\right) - \frac{\pi}{2} + 2\right]$$

$$= \frac{a^2}{4}(4-\pi).$$

(ii) Evaluate $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$.

Soln:

$$\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^b\left(\frac{x^3}{3}+y^2x+z^2x\right)_0^c dy\,dz$$

$$= \int_0^a\int_0^b\left(\frac{c^3}{3}+cy^2+cz^2\right)dy\,dz$$

$$= \int_0^a\left(\frac{c^3y}{3}+\frac{cy^3}{3}+cz^2y\right)_0^b dz$$

$$= \int_0^a\left(b\frac{c^3}{3}+\frac{b^3c}{3}+bcz^2\right)dz$$

$$= \left[\frac{bc^3}{3}z+\frac{b^3cz}{3}+bc\frac{z^3}{3}\right]_0^a$$

$$= \frac{abc}{3}(a^2+b^2+c^2)$$

b)

(i) Change to spherical polar coordinates and hence evaluate $\iiint_V \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$, where V is the volume of the sphere $x^2+y^2+z^2=a^2$.

Soln :

$x = \rho \sin\phi \cos\theta$.

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$.

Given $x^2+y^2+z^2=a^2$.

$dxdydz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

limits of $\rho : \rho = 0$ to $\rho = a$.

$\theta : \theta = 0$ to $\theta = 2\pi$

$\phi : \phi = 0$ to $\phi = \pi$.

$$\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{\rho^2}$$

$$= \left(\int_0^a d\rho\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= (\rho)_0^a \, (-\cos\phi)_0^{\pi} \, (\theta)_0^{2\pi}$$

$$= 4\pi a.$$