UNIIT -I Matrices

Part –A (2 marks)

1. Two eigen values of the matrix A= $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ **are equal to 1 each. Find the eigen values of** A^{-1}**.**

Ans: Let the third Eigen value be λ. The other two eigen values are 1, 1.

W.K.T sum of the eigen values = Trace of A.

$1 + 1 + \lambda = 2+3+2=7$

$\Rightarrow \lambda=5$

The eigen values of A are 1, 1, 5

The eigen values of A^{-1} are 1, 1, 1/5.

2. The product of two eigen values of the matrix

A= $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

is 16. Find the third eigen value.

Ans: Let $\lambda_1, \lambda_2 and \lambda_3$ be the eigen values of a

Given $\lambda_1, \lambda_2 = 16$

W.K.T $\lambda_1, \lambda_2 \ \lambda_3 =$ A

A = $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **= 32**

$$\lambda_3 = \frac{32}{\lambda_1 \lambda_2} = \frac{32}{16} = 2$$

3. If $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ **are the eigen values of an n x n matrix A, then show that** $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots \lambda_n^3$ **are the eigen values of** A^3.

Ans: Let X. be the eigen vector for the eigen value λ_r, then $Ax_r = \lambda_r x_r$ (1)

Premultiplying (1) by A and using (1)

$A^2 X_r = \lambda_r (AX_r) = \lambda_r^2 X_r$

Premultiplying (2) by A and using (1)

$A^2 X_r = \lambda_r^2 (AX_r) = \lambda_r^3 X_r$

From (3), $\lambda_1^3, \lambda_2^3, \ldots\ldots\ldots \lambda_n^3$ are the eigen values of A^3.

4.If A is an orthogonal matrix, show that A^{-1} is also orthogonal.

Ans: A is an orthogonal matrix $\Rightarrow$ $A^{-1} = A^T$

B = If A^{-1}, $B^T = (A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1} = B^{-1}$
$\Rightarrow B = A^{-1}$ is an orthogonal matrix.

5.Find the constants a and b such that the matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and –2 as its eigen values.

Ans: a + b = 3+(-2) = 1

$|A| = 3(-2) = -6$

$ab - 4 = -6 \Rightarrow ab = -2$

$(a-b)^2 = (a+b)^2 - 4ab = 1 + 8 + 9$

$\therefore$ a - b = ±3. Taking a + b = 1 and a - b = 3,

solution is a = 2, b = -1

Also if a + b = 1 and a - b = -3 gives a = -1, b = 2

Hence a = 2 and b = -1 or a = -1 and b = 2

6. If the system of equations x + 2y + z =0,5x +y + z = 0 and x +5y + λz = 0 has a non-trivial solution find the value of λ.

Ans : for non-trail solution $|A|$ = 0 where A is the coefficient matrix the system.

$\begin{vmatrix} 1 & 2 & 1 \\ 5 & 1 & -1 \\ 1 & 5 & \lambda \end{vmatrix}$ =0 $\Rightarrow$ λ. + 5 -10λ -2 +24 =0

9λ = 27

$\therefore \lambda = 3$

7. Find the sum and product of the eigen values of the matrix

$\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$

Ans : Sum of the eigen values = trace of A=2+2+2=6

Product of eigen values = $|A|$ = 6.

8. One of the eigen values of $\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is –9, find the other two eigen values.

Ans: If a, b are the other two eigen values, then

a+b –9 =-9 $\Rightarrow$ a + b = 0

-9 ab = $|A|$ =(63) 7-4 (0-28) –4 (28)

$\Rightarrow$ -9ab = 441

$\therefore ab = -49$

$(a-b)^2 = (a+b)^2 - 4ab \Rightarrow (a-b)^2 = 196$

$\therefore a - b = \pm 14.$

$Solving, a + b = 0$ and $a - b = 14, a = 7, b = -7$

$a + b = 0, a - b = -14, a = -7, b = 7$

$\therefore$ The other two eigen values are 7 and -7

9.Prove that eigen values of $-3A^{-1}$ are the same as those of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Ans : The characteristic equation of A is $|A - \lambda I = 0|$

$\lambda = 1,3.$

$\therefore$ The eigen values of A are -1,3

The eigen values of A^{-1} are -1,1/3

The eigen values of $3A^{-1}$ are -3(-1),-3(1/3)

$\therefore$ The eigen values of $-3A^{-1}$ are 3, -1.

10. If the sum of two eigen values and trace of 3 x 3 matrix A are equal, find the value of $|A|$.

Ans: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of the given matrix A

Given sum of two eigen values = trace of the matrix A

ie $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3$

$\lambda_3 = 0$

Product of the eigen values $= |A|$

$\lambda_1 \lambda_2 \lambda_3 = |A|$

$\Rightarrow |A| = 0$

11.Two of the eigen values of A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **are –1 and –1. Find the eigen values of A^{-1}**

Ans: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 1 + 1 = 3$

$(-1) + (-1) + \lambda_3 = 3$

$\lambda_3 = 5$

$\therefore$ The eigen values of $A^{-1} = -1, -1, 1/5$

12. Find the sum of the squares of the eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Ans : If the given matrix is the upper triangular matrix then the eigen values are the leading diagonal elements.

$\therefore$ The eigen values of the given matrix are 1,4,6

$\therefore$ The sum of squares of the eigen values are

$=1^2+4^2+6^2=1+16+36=53$

13. Find the eigen values of the matrix $\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$ **Hence form the matrix whose eigen values are –1/3 and ½**

Ans: The characteristic equation of A is $|A-\lambda r| = 0$

$$ie\begin{vmatrix} -2 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow \lambda = -3 \text{ or } 2$$

$\therefore$ The eigen values are - 3 and 2. Now the matrix whde eigen values are the reciprocals of A is given by A^{-1}

$W.K.T$ $$A^{-1} = \frac{adj\ A}{|A|}$$

$$|A| = -6.$$

$$\text{adj A} = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$$

$$\therefore A^{-1} = -\frac{1}{6}\begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$$

14. Find the eigen values of the matrix $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ **and A-3I.**

Ans: The characteristic equation is $\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3 - \lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)\ (\lambda - 5) = 0$$

$$\therefore \lambda = 1,5.$$

W.k.t.if $\lambda_1\lambda_2$ are the eigen values of A, then A - kI has the eigen values $\lambda_1 - k, \lambda_2 - k$.

$\therefore$ The eigen values of A - 3I are I - 3,5 - 3

$\therefore$ The eigen values of A - 3I are - 2,2

$[\because K = 3, \lambda = 1, \lambda = 5]$

15. Two eigen values of $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ **are –4 and 3.Find the third eigen value.**

Ans: Sum of the eigen values = sum of the diagonal elements =2+1-3=0
Since the sum of the 2 given eigen values is –1, the third eigen value is 1.

16. If 3 and 15 are the two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **the values of the determinant is ______________**

Ans: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values.

$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3 = 18$

$3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$

$\therefore$ The value of the determinant = product of the eigen values = 0

$\therefore$ Value of the determinant is zero.

17. 6,3,1 are the eigen values of $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **if** $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **are the two eigen vectors then find the third eigen vector?**

Ans: Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the third eigen vector

$X_1^T X_3 = 0 \Rightarrow a + 2a + 0c = 0$

$X_1^T X_3 = 0 \Rightarrow 0a + 0b + c = 0$

Solving we get $X_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

18. One of the eigen values $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is –9.

Find the other two eigen values.

Ans: $\lambda_1 + \lambda_2 + \lambda_3 = 7 - 8 - 8 = -9$

$\lambda_1 \lambda_2 \lambda_3 = |A| = 441$

since $\lambda_3 = -9 \Rightarrow \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -\lambda_1$

$\lambda_1 \lambda_2 = \frac{441}{\lambda_3} = \frac{441}{-9} = -49$

$\lambda_1^2 = 49 \Rightarrow \lambda_1 = \pm 7$

$\therefore$ The other two eigen values are 7, - 7.

19. The eigen values of the matrix $\begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & 6 & 3 \end{bmatrix}$ are distince. If the eigen vectors of the given matrix are $\begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} and \begin{pmatrix} 2 \\ -2 \\ b \end{pmatrix}$ then find the value of 'a' and 'b'?

Ans: $X^T X_2 = 0 \Rightarrow 2a + 2 - 4 = 0 \quad a = 1 \quad X_2^T X_3 = 0 \Rightarrow 4 - 2 - 2b = 0$

b=1

20. Write the matrix of the quadratic form $3x^2 - 2y^2 - z^2 + 12yz + 8zx\ 4xy$

Ans : A=$\begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$

21. If the characteristic equation of $\begin{pmatrix} 5 & 2 & -3 \\ 0 & 0 & 8 \\ 0 & 0 & 7 \end{pmatrix}$ is λ^3-12λ^2+35λ-k=0, then find k

Ans:

We know that β_3=|A|

Therefore |A| = k= 0

22. Define orthogonal matrix

A suare matrix A is said to be an orthogonal matrix if $A^{-1} = A^T$

That is A. A^T = A^T.A=I

23. Find the eigen values of A^2 and A^{-1}. Given the matrix A= $\begin{pmatrix} 3 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Ans:
Since A is a triangular matrix, the eigen values are 3, 2 and 5
Therefore eigenvalues of A^{-1} are 1/3, 1/2 , 1/5
And eigen values of A^2 are 9,4,25

24. Find the eigen values of 3 A^2 if A= $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Ans:
The characteristic equation of A is $\lambda^2 - c_1\lambda - c_2 = 0$

$C_1 = 6$ and $c_2 = 5$
Therefore $\lambda^2 - 6\lambda - 5 = 0$ and $\lambda = 1,5$
Therefore eigen values of 3 A^2 are 3 and 75.

25. Write down the matrix of quadratic frm of $ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$
Ans:

A= $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

26. Write down the matrix of quadratic form $3x^2_1 + 4x^2_2 + 4x_1 x_2 - 4x_2 x_3$
Ans:

A= $\begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 0 \end{pmatrix}$

27. Show that $\begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}$ **is orthogonal**

Ans:

Let A= 

Then $A^T.A = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Similarly A. A^T= I= A^T.A. Therefore A is orthogonal.

28. Determine the nature of the given quadtratic form f(x)= $x_1^2 + 2x_2^2$

Ans: A= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ The eigen values are 1,2,0.

The nature of the quadratic form is positive semi-definite.

29. State Cayley Hamilton theorem

Ans: Every square matrix satisfies its own characteristic equation.

30.If A is an orthogonal matrix show that A^{-1} is also orthogonal.

Ans: Since A is orthogonal $A^{-1} = A^T$

If $B=A^{-1}$ then $B = A^{-1}$ then $B^T = \left(A^{-1}\right)^T = \left(A^T\right)^{-1} = \left(A^{-1}\right)^{-1} = B^{-1}$

Therefore A^{-1} is orthogonal.

UNIT – II SEQUENCES AND SERIES

PART - A

1. Test the convergence of the sequence $\dfrac{2n^3 + 7n}{5n^3 + 3n^2}$.

Solution:

$$S_n = \frac{2n^3 + 7n}{5n^3 + 3n^2} = \frac{n^3\left(2 + \frac{7}{n^2}\right)}{n^3\left(5 + \frac{3}{n}\right)} = \frac{\left(2 + \frac{7}{n^2}\right)}{\left(5 + \frac{3}{n}\right)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \frac{\left(2 + \frac{7}{n^2}\right)}{\left(5 + \frac{3}{n}\right)} = \frac{2}{5}$$

Therefore the given sequence is convergent and converges to $\frac{2}{5}$.

2. Test the convergence of the sequence $\dfrac{3n}{7n^2 + n}$.

Solution:

$$S_n = \frac{3n}{7n^2 + n} = \frac{3n}{n(7n + 1)} = \frac{3}{(7n + 1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \frac{3}{(7n + 1)} = \frac{3}{\infty} = 0$$

Therefore the given sequence is convergent and converges to 0.

3. Test the convergence of the sequence $S_n = \dfrac{n^2 - n}{2n^2 + n}$.

Solution:

$$S_n = \frac{n^2 - n}{2n^2 + n} = \frac{n^2\left(1 - \frac{1}{n^2}\right)}{n^2\left(2 + \frac{1}{n^2}\right)} = \frac{\left(1 - \frac{1}{n^2}\right)}{\left(2 + \frac{1}{n^2}\right)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \frac{\left(1 - \frac{1}{n^2}\right)}{\left(2 + \frac{1}{n^2}\right)} = \frac{1}{2}$$

Therefore the given sequence is convergent and converges to $\frac{1}{2}$.

4. Test the convergence of the sequence $S_n = 3 + (-1)^n$.

Solution:

$$S_n = 3 + (-1)^n = \begin{cases} 3 + 1, & n \text{ is even} \\ 3 - 1, & n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 4 = 4, \quad if \quad n \text{ is even}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 = 2, \quad if \quad n \text{ is odd}$$

Since the limit is not unique, the sequence is oscillatory.

5. Test the convergence of the sequence $S_n = \frac{n - 2}{n + 2}$.

Solution:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{n - 2}{n + 2}\right) = \lim_{n\to\infty} \left(\frac{n\left(1 - \frac{2}{n}\right)}{n\left(1 + \frac{2}{n}\right)}\right) = \lim_{n\to\infty} \left(\frac{1 - \frac{2}{n}}{1 + \frac{2}{n}}\right) = 1$$

Therefore the given sequence is convergent and converges to 1.

6. Test the convergence of the series $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!}$.

Solution:

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Take $\sum V_n = \sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ which is convergent $[\because k - \text{series}, k = \frac{1}{2}]$

Since $u_n < v_n$ by comparison test, $\sum u_n$ is a convergent series.

7. Test the convergence of the series $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\cdots$

Solution:

$$\sum u_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\cdots$$

Take $\sum v_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ which is convergent $[\because k - \text{series}, k = 1]$

Since $u_n > v_n$ by comparison test, the given series is a divergent series.

8. Test the convergence of the series $\sum \frac{1}{n2^n}$

Solution:

$$\text{Since} \quad n2^n > 2^n, \quad \frac{1}{n2^n} < \frac{1}{2^n}$$

$$\text{Take} \quad u_n = \frac{1}{n2^n} \; and \; v_n = \frac{1}{2^n}$$

$u_n > v_n$

But $\sum v_n = \sum \frac{1}{2^n}$ is convergent $[\because$ Geometric series with $a = 1, r = 1/2]$

By comparison test, $\sum \frac{1}{n2^n}$ is a convergent series.

9. Test the convergence of the series $\frac{1}{4.7.10}+\frac{4}{7.10.13}+\frac{9}{10.13.16}+\cdots$

Solution:

$$n^{th} term = u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

$$v_n = \frac{1}{n} \; (Divergent)$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{(3n+1)(3n+4)(3n+7)} \times \frac{n}{1}$$

$$= \lim_{n\to\infty}\frac{n^3}{n^3\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)} = \frac{1}{\left(3+\frac{1}{\infty}\right)\left(3+\frac{4}{\infty}\right)\left(3+\frac{7}{\infty}\right)} = \frac{1}{27}$$

$$\neq 0$$

By comparison test, $\sum u_n$ is a divergent series.

10. Test the convergence of the $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ where a, b, p, q are all positive.

Solution:

$$\sum v_n = \sum \frac{1}{n^{p+q}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(a+n)^p(b+n)^q} \times n^{p+q} = \lim_{n\to\infty} \frac{1}{\left(\frac{a}{n}+1\right)^p \left(\frac{b}{n}+1\right)^q} = 1$$

Hence $\sum u_n$ and $\sum v_n$ are both convergent and divergent together.

But $\sum v_n$ is convergent for $p + q > 1$ and divergent if $p + q \leq 1$.

$\therefore \sum u_n$ is convergent for $p + q > 1$ and divergent if $p + q \leq 1$.

11. Test the convergence of the series $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots$

Solution:

$$\text{Take } f(x) = \frac{1}{2x-1}$$

$$\text{By integral test, } \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{2x-1}dx = \frac{1}{2}[log(2x-1)]_1^{\infty} = \infty$$

$\therefore$ the given series is divergent.

12. Test the convergence of the series $\sin\pi + \frac{1}{4}\sin\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{3} + \dots\dots\dots$

Solution:

$$Un = \frac{1}{n^2} sin\frac{\pi}{n}$$

$$\text{Let } f(x) = \frac{1}{x^2} sin\frac{\pi}{x}$$

$$\text{By integral test } \int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^2} sin\frac{\pi}{x} dx \quad put \quad \frac{\pi}{x} = t \ , \quad x = 1, \ t = \pi$$

$$-\frac{\pi}{x^2}\, dx = dt, \ x = \infty, \ t = 0$$

$$= \int_{\pi}^{0} - sin\, t\, dx = \int_{0}^{\pi} sin\, t\, dt \quad = (-cos\, t)0\, \pi \ = 1 + 1 = 2.$$

Hence the series is convergence.

13. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+9}$.

Solution:

$u_n = \frac{e^n}{e^{2n}+9}$. *Let* $f(x) = \frac{e^x}{e^{2x}+9}$.

By integral test $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{e^x}{e^{2x}+9} dx$

put $e^x = t,\ e^x dx = dt\ \ x = 1,\ \ t = e\ \ and\ \ x = \infty,\ \ t = \infty$

$$\int_1^{\infty} \frac{e^x}{e^{2x}+9} dx = \int_e^{\infty} \frac{dt}{t^2+9} dx = \left[\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)\right]_e^{\infty} = \frac{1}{3}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{e}{3}\right)\right]$$

$=$ convergent

14. Test the convergence of the series $\sum_{n=0}^{\infty} e^{-n^2}$.

Solution:

Let $f(x) = e^{-x^2}$

$\int_0^{\infty} f(x)\,dx = \int_0^{\infty} e^{-x^2}\,dx$ which cannot be evaluated

By comparing with $\int_0^{\infty} f(x)\,dx = \int_0^{\infty} e^{-x^2}\,dx$ is finite.

Hence the given series is convergent.

15. Test the convergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$.

Solution:

Let $f(x) = x\,e^{-x^2}$

By integral test $\int_1^{\infty} f(x)dx = \int_1^{\infty} x\,e^{-x^2}\,dx$

put $x^2 = t,\ \ 2x\,dx = dt,\ \ x = 1,\ \ t = 1$ and $x = \infty,\ \ t = \infty$

$$\int_1^\infty f(x)\,dx = \frac{1}{2}\int_1^\infty e^{-t}\,dt = \frac{1}{2}[-e^{-t}]_1^\infty = \frac{1}{2}[-e^{-\infty} + e^{-1}] = \frac{1}{2}[-0 + e^{-1}] = \frac{1}{2e}$$

Hence the given series is convergent.

16. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{2n^3}{n^4+1}$.

Solution:

$$u_n = \frac{2n^3}{n^4+1}$$

Let $f(x) = \frac{2x^3}{x^4+1}$

By integral test $\int_1^\infty f(x)dx = \int_1^\infty \frac{2x^3}{x^4+1}dx$

put $x^4 + 1 = t$, $4x^3dx = dt$, $x = 1$, $t = 2$ and $x = \infty$, $t = \infty$

$$\int_1^\infty f(x)\,dx = \int_2^\infty \frac{2x^3}{x^4+1}dx = \frac{1}{2}\int_2^\infty \frac{dt}{t} = \frac{1}{2}[\log t]_2^\infty = \frac{1}{2}[\log \infty - \log 2]$$

$$= \infty$$

Hence the given series is divergent.

17. Test the convergence of the series whose nth term is $\frac{(n+3)!}{(3!)(n!)(3^n)}$.

Solution:

$$u_n = \frac{(n+3)!}{(3!)(n!)(3^n)}$$

$$u_{n+1} = \frac{(n+4)!}{(3!)(n+1)!\,(3^{n+1})}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+4)!}{(3!)(n+1)!\,(3^{n+1})} * \frac{(3!)(n!)(3^n)}{(n+3)!}$$

$$= \lim_{n\to\infty} \frac{n+4}{(n+1).3} = \frac{1}{3}\lim_{n\to\infty} \frac{1+\frac{4}{n}}{1+\frac{1}{n}} = \frac{1}{3} < 1$$

This series is convergent.

18. Test the convergence of the series $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$

Solution:

$$u_n = \frac{n!}{1+2^n} \quad and \quad u_{n+1} = \frac{(n+1)!}{1+2^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{(n+1)!}{1+2^{n+1}} * \frac{1+2^n}{n!} \right)$$

$$= \lim_{n\to\infty} \frac{n+1}{1+2^{n+1}} * \frac{1+2^n}{n} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{2^n\left(2+\frac{1}{2^n}\right)n} = \frac{1}{2} < 1$$

This series is convergent.

19. Test the convergence of the series $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots$

Solution:

$$u_n = \frac{n^2(n+1)^2}{n!} \quad and \quad u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2(n+2)^2}{(n+1)!} * \frac{n!}{n^2(n+1)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2(n+2)^2}{(n+1)!} * \frac{n!}{n^2(n+1)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2\left(1+\frac{1}{n}\right)^2 n^2\left(1+\frac{2}{n}\right)^2}{(n+1)n!} * \frac{n(n+1)!}{n^2n^2\left(1+\frac{1}{n}\right)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(1+0)^2(1+0)^2}{\infty} * \frac{1}{(1+0)^2} \right) = \frac{1}{\infty} = 0 <$$

This series is convergent.

20. Test the convergence of the series $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \cdots$

Solution:

$$u_n = \frac{n}{5n+1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n}{5n+1} \right) = \frac{1}{5} \neq 0$$

By Leibnitz's test, series is convergent.

21. Test the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$.

Solution:

Given the series is alternating series with

$$u_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{\infty} = 0$$

By Leibnitz's test, series is convergent.

22. Test whether the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$ is convergent or not.

Solution:

Given the series is alternating series with

$$u_n = \frac{1}{2n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{1}{2n-1}\right) = \frac{1}{\infty} = 0$$

By Leibnitz's test, series is convergent.

UNIIT -III Differential calculus

Part –A (2 marks)

1. Define-Curvature and radius of curvature.

Ans:The rate of bending of the curve with respect to actual distance at'p' is called the curvature of the curve, which is denoted by 'k'.Therefore, $k=\delta_{S\to 0}\ \partial\psi/\partial s=d\psi/ds$.

Radius of curvature is reciprocal of curvature and it is denoted by ρ. Therefore, $\rho=1/k$.

2. What is the curvature of $x^2+y^2-4x-6y+10=0$ at any point on it?

Ans: The given equation is a circle, with radius $\sqrt{u^2+v^2-d}=\sqrt{2^2+3^2-10}=\sqrt{3}$.

W.K.T for a circle k=1/radius,$k=1/\sqrt{3}$.

3. Find the radius of curvature at (3,-4) to the curve $x^2+y^2=25$.

Ans; The given equation is a circle with radius r=5.therefore k=1/radius=1/5.

ρ=1/k=1/5=5.

4. Find the curvature of the curve $2x^2+2y^2+5x-2y+1=0$

Ans: $2x^2+2y^2+5x-2y+1=0$, $x^2+y^2+(5/2)x-y+(1/2)=0$,which is a equation of a circle.

Therefore, $k=1/radius=1/(\sqrt{21}/4)=4/\sqrt{21}$

5. Find the radius of curvature at x=π/2 on the curve y=4sinx-sin2x.
Ans: y=4sinx-sin2x. x=π/2. Therefore, 4sin(π/2)-sin2(π/2)=4 ,
Therefore, the point is (π/2,4). $\rho=[1+y']^{3/2}/y''$
$y'=dy/dx=4\cos x-2\cos 2x$, $y'_{(\pi/2,4)}=2$,similarly, $y''=d^2y/dx^2=-4\sin x+4\sin 2x$, $y''_{(\pi/2,4)}=-4$
Therefore, $\rho=5\sqrt{5}/4$.

6. What is the curvature of
a)straight line b)circle of radius 2 units
Ans: a)for straight lines k=0 b)for circle of radius 2 units,k=2

7.Find the radius of curvature of any point(x,y) on y=a log seec(x/a)
Ans: $y'=\tan(x/a)$,$y''=(1/a)\sec^2(x/a)$, $\rho=a.\sec(x/a)$.

8.Find the radius of curvature of the curve y=a cos h(x/a) at any point on it.
Ans: $y'=\sinh(x/a)$, $y''=(1/a)\cosh(x/a)$,
$\rho=\{1+\sinh^2(x/a)\}^{3/2}/(1/a)\cosh(x/a)$
$=[\cosh^2(x/a)]^{3/2} / (1/a)\cosh(x/a)$
$=a.\cosh(x/a)=a.y^2/a^2=y^2/a$.

9.Find the radius of curvature at y=2a on the curve y^2=4ax
Ans: y=2a,x=a
Differentiating the given eqn,2y.y′=4a
$y'=4a/2y=2ay^{-1}$, $y'_{(a,2a)}=2a/2a=1$.
$y''=-2a/y^2$; $y''_{(a,2a)}=-2a/4a^2=-1/2a$
$\rho=[1+y']^{3/2} / y''$
$\rho_{(a,2a)}=\{1+1\}^{3/2}/(-1/2a)=2a.2^{3/2}=2^{5/2}.a$

10.For the curve x^2 = 2c(y-c),find the radius of curvature at (0,c).
Ans: $x^2=2c(y-c)$.--------(1).
Differentiate with respect to x.
2x = 2cy’.
y’ = (x /c).
y” = 1/c.
Find the envelope of the family of straight lines xcosα + ysinα = P,where α is the parameter.
$\rho=[1+y']^{3/2} / y''$
$=[1+(x/c)]^{3/2} / (1/c)$. $P_{(0,c)}=c$.

11.Write the formula for radius of curvature in Cartesian form, parametric form, and polar form.
Ans:
(i) Cartesian form :
$\rho=[1+y']^{3/2} / y''$
(ii) Parametric form :
$\rho=[x'^2+y'^2]^{3/2} / [x'y''-x''y']$

(iii) polar form :
$\rho = [r'^2+r^2]^{3/2} / [r^2 + 2r'-rr'']$.

12.Find the envelope of the family of straight lines $y = mx \pm (m^2 -1)^{1/2}$,where m is the parameter

Ans: $y = mx \pm (m^2 -1)^{1/2}$.
$m^2 -1 = y^2 + m^2x^2-2mxy$.
$(x^2-1) m^2 – 2xym + y^2 +1 = 0$.
The envelope is given by equation
$4 x^2 y^2 - 4(x^{2-1}) (y^2+1) = 0$.
$(x^2 /1)- (y^2 / 1) = 1$.

13. Find the envelope of the family of straight lines y = mx + (a/m).
,where m is the parameter
Ans: y = mx + (a/m).
y = mx + (a/m).
$m^2x-my+a = 0$.
The envelope is given by equation.,$y^2 -4ax = 0$, which is a parabola.

14. Find the envelope of the family of straight lines $y = mx +am^2$,where m is the parameter
Ans: $y = mx + am^2$.
$am^2 + mx –y =0$.
The envelope is given by $x^2 + 4xy =0$.

15. Find the envelope of the family of straight lines xcosα + ysinα = P,where α is the parameter.
Ans: xcosα + ysinα = P -----------(1)
Diffrentiate (1) patialy with respect to α ,
-xsinα + ycosα =0 .-----------(2)
$(1)^2 + (2)^2$ gives,$x^2+y^2=p^2$.

16. Find the envelope of the family of circles $(x-a)^2+y^2 = 4a$.
Ans: $(x-a)^2+y^2 = 4a$.---------(1)
Diffrentiate (1) patialy with respect to α ,
2(x-a)(-1)+0 = 4.
a = x+2.
(1) Implies $y^2-4x = 4$.

17. Find the envelope of the family of straight lines xcosα + ysinα = a secα ,where α is the parameter.
Ans: xcosα + ysinα = a secα .
Divide by cosα
$x + y \tan\alpha = a \sec^2\alpha$.
$a \tan^2\alpha – y\tan\alpha+(a-x) = 0$.
The envelope is given by $y^2 = 4a(a-x)$.

18. Find the envelope of $(x/t) + yt = 2c$, t is the parameter.

Ans: $(x/t) + yt = 2c$

$yt^2+x = 2ct$.

$yt^2+x - 2ct =.0$.

The envelope is given by $xy = c^2$.

19. Show that the family of circles $(x-a)^2+y^2=a^2$, a is the parameter has no envelope.

Ans: $(x-a)^2+y^2=a^2$.-------(1)

Diffrentiate (1) patialy with respect to α ,

$-2(x-a) = 2a$.

$x = 2a$.

Therefore $y = 0$.

20.If the centre of curvature is $((c/a)\cos^3 t, (c/a)\sin^3 t)$,find the evolute of curve.

Ans: $x = (c/a)\cos^3 t$, $y = (c/a)\sin^3 t$, $(ax)^{2/3}+(ay)^{2/3} = c^{2/3}$.

21.Define envelope

A curve which touches each member of a family of curves is called the envelope of that family of curves.

22. Define circle of curvature

The circle whose centre is the centre of curvature and whose radius is equal to the radius of curvature ρ is called the circle of curvature. Its equation is $(x-\bar{x})^2 +(y-\bar{y})^2 = \rho^2$, where $\bar{x} = x-(1+y'^2)y'/y''$, $\bar{y} = y+(1+y'^2)/y''$ and $\rho = (1+y'^2)^{(3/2)}/y''$.

23. Find ρ for the catenary whose intrinsic equation is s = a tab φ.

Solution:

$\rho = ds/ d\varphi = a \sec^2\varphi$.

24. Find ρ for the cycloid s= 4a sin φ

Solution:

$\rho = ds/ d\varphi = 4a \cos \varphi$.

25. What is the radius of curvature at (3,4) on $x^2+y^2 = 25$?

Solution:

Radius of curvature = Radius of the circle

= 5

26. Find the curvature at x = 0 on e^x?

Solution:

when $x = 0$, $y = e^0 = 1$

$y' = e^x$ and $y'' = e^x$

$y'_{(0,1)} = 1$ and $y''_{(0,1)} = 1$

Therefore $\rho = (1+1)^{(3/2)}/1 = 2\sqrt{2}$

Therefore curvature $= 1/ \rho = 1/2\sqrt{2}$.

27. Find the curvature for e^x at the where the curve cuts the y-axis?

Solution:

When the curve cuts the y-axis, x = 0.

Therefore $y = e^0 = 1$. Hence the point of contact is (0,1)

$y' = e^x$ and $y'' = e^x$

$y'_{(0,1)} = 1$ and $y''_{(0,1)} = 1$

Therefore $\rho = (1+1)^{(3/2)}/1 = 2\sqrt{2}$.

28. Find the envelope of the family of curves $(x-\alpha)^2+y^2 = 4\alpha$

Solution:

$(x-\alpha)^2+y^2 = 4\alpha$

$x^2-2x\,\alpha+\alpha^2+y^2-4\,\alpha = 0$

$\alpha^2-(2x+4)\,\alpha+y^2+x^2 = 0$

This is a quadratic equation in α, therefore the envelope of the given family of curves is

$B^2-4AC = 0$ with $A=1$, $B=-(2x+4)$ and $C= y^2+x^2$

$(-(2x+4))^2-4\,(y^2+x^2)=0$

$y^2-4x-4=0$

29. Find the envelope of the family of curves $y = mx+\sqrt{(a^2m^2+b^2)}$, where m is the parameter.

Solution:

Given $y = mx+\sqrt{(a^2m^2+b^2)}$

$y-mx = \sqrt{(a^2m^2+b^2)}$

$(y-mx)^2 = a^2m^2+b^2$

$m(x^2-a^2)-2mxy+y^2-b^2=0$, a quadratic equation in m with $A= x^2-a^2$, $B=-2xy$ and $C= y^2-b^2$

Therefore $B^2-4AC=4xy-4(x^2-a^2)(y^2-b^2)=0$

$x^2 b^2+y^2 a^2=a^2 b^2$

$x^2/a^2 +y^2/b^2 =1$ which is the required envelope.

30. Find the maxima and minima of the functions $f(x,y)= x^2 +y^2 -3x$?

Solution:

$f_x=3x^2-3$ $\quad$ $f_y=2y$ $\quad$ $f_{xy}=0$

$f_x{}^2=6x$ $\quad$ $f_y{}^2=2$

For extreme values, $f_x=0$ and $f_y=0$

Therefore $3x^2-3=0$ and $2y=0$

$x^2=1 \Rightarrow x=\pm1$ and $y=0$

The extreme points are (1,0) and (-1,0)

At (1,0), $f_x{}^2\, f_y{}^2- (f_{xy})=6*2=12>0$

At (1,0) the function is minimum.

At (-1,0), $f_x{}^2\, f_y{}^2- (f_{xy})=-12<0$ and also $f_x{}^2=-6<0$.

Therefore the point (-1,0) is a saddle point.

UNIIT –IV FUNCTIONS OF SEVERAL VARIABLES

Part –A (2 marks)

1. **If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$,prove that $x(\partial u/\partial x) + y(\partial u/\partial y) = 0$.**

   Ans:If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$,prove that $x(\partial u/\partial x) + y(\partial u/\partial y) = 0$.

   Here u is homogenous function of degree n = 0.By Euler's theorem,

   $x(\partial u/\partial x) + y(\partial u/\partial y) = n*0 = 0$.

2. **If $u = (x/y)+(y/z)+(z/x)$,find $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)$.**

   Ans:If $u = (x/y)+(y/z)+(z/x)$,find $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)$.

   Here u is homogenous function of degree n = 0.By Euler's theorem,

   $x(\partial u/\partial x) + y(\partial u/\partial y) = n*0 = 0$.

3. **$e^y-e^x+xy =0$,find dy/dx**

   Ans:.$e^y-e^x+xy =0$,find dy/dx.

   Let $f(x,y) = e^y-e^x+xy$

   $\partial f/\partial x= -e^x+y$

   $\partial f/\partial y= e^y+x$

   $dy/dx= - (\partial f/\partial x)/(\partial f/\partial y) = (e^x-y)/ (e^y+x)$.

4. **If $f(x,y)= \log (x^2+y^2) + \tan^{-1}(y/x)$,find dy/dx.**

   Ans:.If $f(x,y)= \log (x^2+y^2) + \tan^{-1}(y/x)$,find dy/dx.

   $\partial f/\partial x = (2x-1)/ (x^2+y^2)$

   $\partial f/\partial y = (2y+x)/ (x^2+y^2)$

   $dy/dx= - (\partial f/\partial x)/(\partial f/\partial y) = (1-2x)/(2y+x)$.

5. **If $u = e^xyz^2$,find du.**

   Ans:.If $u = e^xyz^2$,find du.

   We know that $du = (\partial u/\partial x)dx + (\partial u/\partial y)dy +(\partial u/\partial z)dz$.

   $du = (e^xyz^2)dx + (e^xz^2) dy+ (2e^xyz) dz$.

6. **If u = f(x-y,y-z,z-x), find $(\partial u/\partial x) + (\partial u/\partial y) + (\partial u/\partial z)$.**

Ans:If u = f(x-y,y-z,z-x), find $(\partial u/\partial x) + (\partial u/\partial y) + (\partial u/\partial z)$.

Let r = x-y, s = y-z, t= z-x.

Therefore u = f(r,s,t).

$\partial u/\partial x = (\partial u/\partial r)(\partial r/\partial x) + (\partial u/\partial s)(\partial s/\partial x) + (\partial u/\partial t)(\partial t/\partial x)$.

$= (\partial u/\partial r)(1) + (\partial u/\partial s)(0) + (\partial u/\partial t)(-1)$.

$= (\partial u/\partial r) - (\partial u/\partial t)$

Similarly $\partial u/\partial y = -(\partial u/\partial r) + (\partial u/\partial s)$

$\partial u/\partial z = (\partial u/\partial t) - (\partial u/\partial s)$

$\partial u/\partial x) + (\partial u/\partial y) + (\partial u/\partial z) = 0$.

7. **If $u = x^3y^2 - x^2y^3$, where $x=at^2$, $y=2at$, find du/dt.**

Ans: If $u = x^3y^2 - x^2y^3$, where $x=at^2$, $y=2at$, find du/dt.

$\partial u/\partial x = 3x^2-2xy^3$, $\partial u/\partial y = 2x^3y - 3x^2y^2$, $dx/dt = 2at$, $dy/dt=2a$.

$du/dt = (\partial u/\partial x)dx/dt + (\partial u/\partial y)dy/dt + (\partial u/\partial z)dz/dt$

$= 32a^5t^7 - 56a^5t^6$.

8. **Expand e^x log (1+y) in powers of x and y up to second degree.**

Ans: Expand e^x log(1+y) in powers of x and y up to second degree.

Let $f(x,y) = e^x \log(1+y)$ and (a,b) = (0,0).

$f(x,y) = e^x \log(1+y)$; $f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$; $f_x(0,0) = 0$

$f_{xx}(x,y) = e^x \log(1+y)$; $f_{xx}(0,0) = 0$.

$f_{xy}(x,y) = e^x /(1+y)$; $f_x(0,0) = 1$

$f_y(x,y) = e^x /(1+y)$; $f_y(0,0) = 1$.

$f_{yy}(x,y) = -e^x/(1+y)^2$; $f_{yy}(x,y) = -1$.

Therefore $f(x,y) = y + xy - (y^2/2)$.

9. **Expand e^xsiny in powers of x and y as for as terms of second degree.**

Ans: Expand e^xsiny in powers of x and y as for as terms of second degree.

Let $f(x,y) = e^x \sin y$ and (a,b) = (0,0).

$f(x,y) = e^x \sin y$; $f(0,0) = 0$

$f_x(x,y) = e^x \sin y$; $f_x(0,0) = 0$

$f_{xx}(x,y) = e^x \sin y$; $f_{xx}(0,0) = 0$.

$f_{xy}(x,y) = e^x \cos y$; $f_x(0,0) = 1$

$f_y(x,y) = e^x \sin y$; $f_y(0,0) = 1$.

$f_{yy}(x,y) = -e^x \sin y$; $f_{yy}(x,y) = 0$.

Therefore f(x,y) = y +xy.

10. Expand xy+2x-3y+2 in powers of (x-1) & (y+2) using tailors theorem upto first degree terms.

Ans: Expand xy+2x-3y+2 in powers of (x-1) & (y+2) using tailors theorem upto first degree terms.

Let f(x,y) = xy+2x-3y+2 and (a,b) = (1,-2).

f(x,y) = xy+2x-3y+2; f(1,-2) = 8

$f_x(x,y) = y+2$; $f_x(1,-2) = 0$

$f_y(x,y) = x-3$; $f_x(1,-2) = -2$.

Therefore f(x,y) = 8 -2y -4 = 4-2y.

11. If x = r cosθ,y = rsinθ ,find ∂(r, θ) / ∂ (r,y).

Ans:If x = r cosθ,y = rsinθ ,find ∂(r,θ) / ∂ (r,y).

Let J' = ∂(r,θ) / ∂ (r,y). ∂x/∂r

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

J' = 1/ J = 1 / r.

12. If u= x+y,y=uv, find ∂(x, y) / ∂ (u,v).

Ans:.If u= x+y,y=uv, find ∂(x, y) / ∂ (u,v).

$$\partial(x, y) / \partial (u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

13. If u,v,w are functions of independent variables x,y,z and ∂(u,v,w) / ∂ (x,y,z) = 4,

find the value of ∂(2u,2v,2w)/ ∂ (x,y,z).

Ans:If u,v,w are functions of independent variables x,y,z and ∂(u,v,w) / ∂ (x,y,z) = 4,find the value of ∂(2u,2v,2w)/ ∂ (x,y,z).

$\partial(r,s,t) / \partial (x,y,z) = [\partial(r,s,t) / \partial(u,v,w)] \times [\partial(u,v,w) / \partial(x,y,z)]$

$= 4 \times 8 = 32.$

14. Find the possible extreme points of f(x,y)= $x^2+y^2+(2/x)+(2/y)$.

Ans:Find the possible extreme points of f(x,y)= $x^2+y^2+(2/x)+(2/y)$.

$\partial f/\partial x = 2x-(2/x^2)$

$\partial f/\partial y = 2y-(2/y^2)$.

Let $\partial f/\partial x = 2x-(2/x^2) = 0$

$\partial f/\partial y = 2y-(2/y^2) = 0$.

Therefore x = 1,y =1 are the extreme points.

15. Find the stationary points of the function f(x,y)= x^3+y^3-12xy.

Ans:Find the stationary points of the function f(x,y)= x^3+y^3-12xy.

$\partial f/\partial x = 3x^2-(12y)$

$\partial f/\partial y = 3y^2-(12x)$.

Let $\partial f/\partial x = 3x^2-(12y) = 0$

$\partial f/\partial y = 3y^2-(12x).= 0$.

Therefore (0,0),(2,2)are the extreme points.

16. Find the stationary points of f(x,y) = $x^3+3xy^2-15x^2-15y^2+72x$.

Ans:Find the stationary points of f(x,y) = $x^3+3xy^2-15x^2-15y^2+72x$.

$\partial f/\partial x = 3x^2+3y^2-30x+72$

$\partial f/\partial y = 6yx-30y$.

Let $\partial f/\partial x = 3x^2+3y^2-30x+72 = 0$

$\partial f/\partial y = 6yx-30y.= 0$.

Therefore (4,0), (6,0) (5,1) and (5,-1)are the extreme points.

17. If u = (x+y)/(1-xy),v = $\tan^{-1}x + \tan^{-1}y$ then prove that u and v are functionally related.

Ans:

If $u = (x+y)/(1-xy), v = \tan^{-1}x + \tan^{-1}y$ then prove that u and v are functionally related.

$$J = \partial(u, v) / \partial (x,y) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} (1-y^2)/((1-xy)^2) & (1-x^2)/((1-xy)^2) \\ 1/(1+x^2) & 1/(1+x^2) \end{vmatrix} \neq 0.$$

Therefore u and v are functionally related.

18. State Euler's theorem.

Ans: If u is a Homogeneous function of degree 'n', then $x(\partial u/\partial x) + y(\partial u/\partial y) = n.u.$

19. Write Taylor's series.

Ans: The Taylor series expansion for two variable at (a,b) is

$$f(x,y) = f(a,b) + (1/1!)[f_x(a,b)(x-a)+ f_y(a,b)(y-b)] + (1/2!) [f_{xx}(a,b)(x-a)^2+2 f_{xy}(a,b))(x-a)(y-b) + f_{yy}(a,b)(y-b)^2]+..$$

20. Write the conditions about extreme values.

Ans: Find $r = \partial^2 f/\partial x^2, s = \partial^2 f/\partial x\partial y$ and $t = \partial^2 f/\partial y^2$.

If $rt - s^2 > 0$ and $r < 0$,then'f' is maximum.

If $rt - s^2 > 0$ and $r >0$, then'f' is minimum.

If $rt - s^2 < 0$,then 'f is neither maximum nor minimum.

If $rt - s^2 = 0$,then nothing can be said whether maximum or minimum.

21. If $u = (y/z)+(z/x)$, find the value of $x\partial u/\partial x+y\partial u/\partial y+z\partial u/\partial z$

Ans: Given $u = (y/z)+(z/x)$

$\partial u/\partial x = -(z/x^2)$, $\partial u/\partial y = 1/z$ and $\partial u/\partial z = -(y/z^2)+(1/x)$

Therefore $x\partial u/\partial x+y\partial u/\partial y+z\partial u/\partial z = -(z/x)+(y/z)-(y/z)+(z/x)$.

22. If $u=(y^2/x)$, $v= (x^2/y)$, find $\partial(u,v)/ \partial(x,y)$.

Ans: $\partial(u,v)/ \partial(x,y)= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

$u=(y^2/x) \Rightarrow \partial u/\partial x = -y^2 /x^2$ and $\partial u/\partial y= 2y/x$

$v= (x^2/y) \Rightarrow \partial v/\partial x= 2x/y$ and $\partial v/\partial y= -x^2 /y^2$

Therefore $\partial(u,v)/\partial(x,y) = \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix}$

$= -3$

23. If u = f(r,s) r=x+y s=x-y prove that $\partial u/\partial x + \partial u/\partial y = 2\partial u/\partial r$

Ans: $\partial u/\partial x = (\partial u/\partial r)(\partial r/\partial x) + (\partial u/\partial s)(\partial s/\partial x)$

$= (\partial u/\partial r)(1) + (\partial u/\partial s)(1)$

$= (\partial u/\partial r) + (\partial u/\partial s)$

$\partial u/\partial y = (\partial u/\partial r)(\partial r/\partial y) + (\partial u/\partial s)(\partial s/\partial y)$

$= (\partial u/\partial r)(1) + (\partial u/\partial s)(-1)$

$= (\partial u/\partial r) - (\partial u/\partial s)$

Therefore $\partial u/\partial x + \partial u/\partial y = 2\partial u/\partial r$

24. What is the total derivative of u ?

Ans: If **u** is a homogeneous function of x and y , then the total differential is given by

$du = (\partial u/x).dx + (\partial u/y).y$

25. $\partial(u,v)/\partial(x,y)\partial(x,y)/\partial u,v) = ?$

Ans: 1

26. If $V=(x^3 y^3)/(x^3 + y^3)$, then find $x\partial V/\partial x + y\partial V/\partial y$.

Ans: $x\partial V/\partial x + y\partial V/\partial y = 3V$

27. If J_1 is the Jacobian of u(x,y) and v(x,y) and J_2 is the Jacobian of x(u,v) and y(u,v) then find $J_1 J_2$.

Ans: By property $J_1 J_2 = 1$.

28. If $z = e^{(ax+by)} f(ax-by)$, prove that $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$

Ans: $\partial z/\partial x = a\, e^{(ax+by)} f(ax-by) + a\, e^{(ax+by)} f'(ax-by)$

$\partial z/\partial y = b\, e^{(ax+by)} f(ax-by) - b\, e^{(ax+by)} f'(ax-by)$

Therefore $b(\partial z/\partial x) + a(\partial z/\partial y) = ab\, e^{(ax+by)} f(ax-by) + ab\, e^{(ax+by)} f'(ax-by) + ab\, e^{(ax+by)} f(ax-by)$

$- \quad ab\, e^{(ax+by)} f'(ax-by)$

$= 2ab\, e^{(ax+by)} f(ax-by)$

$= 2abz.$

29. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1,\ find\ \frac{dy}{dx}.$

Ans: Let f(x,y)= $x^3+3x^2y+6xy^2+y^3-1$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2+6xy+6y^2}{3x^2+12xy+3y^2}.$$

30. Find the total differential of the function $u=\tan(3x-y)+6^{y+z}$ **.**

Ans: Given $u=\tan(3x-y)+6^{y+z}$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$=3\sec^2(3x-y)dx+[-\sec^2(3x-y)+6^{y+z}\log 6]dy+6^{y+z}\log 6dz$$

UNIIT – V MULTIPLE INTEGRALS

Part –A (2 marks)

1. Evaluate $\int_1^2\int_0^3 (x+y)^2 dxdy$

$$\int_1^2\int_0^3 (x+y)^2 dxdy = \int_1^2 (\frac{x^2}{2}+y^2x)_0^3 dy$$

$$= \left(\frac{9}{2}y+y^3\right)_1^2$$

= 23/2

2. Evaluate $\int_1^3\int_{x^2}^{x+2} dxdy$

$$\int_1^3\int_{x^2}^{x+2} dxdy = \int_1^3 (y)_{x^2}^{x+2}dx$$

$$= \int_1^3 (x+2-x^2)dx$$

$$= \left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)_1^3 = -2/3$$

3. Evaluate $\int_0^{\pi}\int_0^{1-\cos\theta} rdrd\theta$

$$\int_0^{\pi}\int_0^{1-\cos\theta} rdrd\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{1=\cos\theta} d\theta$$

$$= ½ \int_0^{\pi}(1+\cos^2\theta-2\cos\theta)d\theta$$

$$= \frac{\pi}{2} + \frac{1}{4}(\theta)_0^{\pi} + \frac{1}{4}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

4. Evaluate $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\,dxdy$

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\,dxdy = \int_0^{\frac{\pi}{2}} [\sin(x+y)]_{\frac{\pi}{2}}^{\pi}\,dy$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin(\pi+y) - \sin(\frac{\pi}{2}+y) \right) dy$$

$$= \int_0^{\frac{\pi}{2}} (-\sin y - \cos y)\,dy = -2$$

5. Evaluate $\int_1^2\int_x^{2x} \frac{1}{x^2+y^2}\,dydx$

$$\int_1^2\int_x^{2x} \frac{1}{x^2+y^2}\,dydx = \int_1^2 \left(\frac{1}{x}\tan^{-1}\frac{y}{x} \right)_x^{2x} dx$$

$$= \left(\tan^{-1}2 - \tan^{-1}1\right)(\log x)_1^2$$

$= (\tan^{-1}2 - \tan^{-1}1)\log 2$

6. Evaluate $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$ **.**

$$\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)}} \int_0^1 \frac{dy}{\sqrt{(1-y^2)}}$$

$$= (\sin^{-1}x)_0^1(\sin^{-1}y)_0^1$$

$$= \pi^2/4.$$

7. Evaluate $\int_1^2\int_2^3\int_1^3 (x^2y+z)\,dzdydx$ **.**

$$\int_1^2\int_2^3\int_1^3 (x^2y+z)\,dzdydx = \int_1^2\int_2^3 (x^2yz + z^2/2)_1^3\,dydx$$

$$= \int_1^2 (x^2y^2 + 4y)_2^3\,dx$$

$= (5x^3/3 + 4x)_1^2 = 47/3.$

8. Write the substitution to change the Cartesian coordinates to polar co ordinates.

$x = r\cos\theta$

$y = r\sin\theta$

9. Write the substitution to change the Cartesian coordinates to spherical polar co ordinates.

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

10. Write the substitution to change the Cartesian coordinates to cylindrical co ordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$z = r$

11. **Find the limits of integration with double integral , $\iint_R f(x,y)dxdy$ where R is the first quadrant bounded by x=0 , y = 0 ,and x+y = a**.

$$\iint_R f(x,y)dxdy = \int_0^a \int_0^{a-y} f(x,y)dxdy$$

12. Evaluate $\iint_A dxdy$, where A is the upper half of the circle $x^2 + y^2 = 1$.

The area of the semi circle = $\iint_A dxdy = \pi\frac{(1)^2}{2} = \frac{\pi}{2}$.

13. **Evaluate** $\int_1^2 \int_0^x \frac{dxdy}{x^2+y^2}$.

$$\int_1^2 \int_0^x \frac{dxdy}{x^2+y^2} = \int_1^2 \left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)_0^x dx$$

$$= \int_1^2 \left(\frac{\pi}{4}\frac{1}{x}\right)dx = \frac{\pi}{4}\log 2.$$

14. **Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx = \int_0^a \sqrt{(a^2-x^2)}dx$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi a^2}{4}.$$

15. Evaluate $\int_0^2\int_0^{x^2} e^{\frac{y}{x}}dydx$ **.**

$$\int_0^2\int_0^{x^2} e^{\frac{y}{x}}dydx = \int_0^2 x(e^x-1)dx$$

$$= 2e^2-e^2+1-2=e^2-1 .$$

16. **Evaluate** $\int_0^{\pi}\int_0^1 x\cos(xy)dydx$.

$$\int_0^{\pi}\int_0^1 x\cos(xy)dydx = \int_0^{\pi}\left[\frac{x\sin(xy)}{x}\right]_0^1 dx$$

$$= \int_0^{\pi}\sin x dx$$

$$= (-\cos x)_0^{\pi} = 2 .$$

17. **Evaluate** $\int_0^a\int_0^{\sqrt{ay}} xydxdy$.

$$\int_0^a\int_0^{\sqrt{ay}} xydxdy = \int_0^a\left[\frac{x^2y}{2}\right]_0^{\sqrt{ay}} dy$$

$$=1/2\int_0^a ay^2dy$$

$$=a^4/6.$$

18. **Change the integral** $\int_{-1}^{1}\int_0^{\sqrt{(1-x^2)}} dydx$ **into polar co-ordinate and evaluate**.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=1$ becomes $r^2 = 1$ and hence r = 1

r varies from 0 to 1

θ varies from 0 to π

$$\int_{-1}^{1}\int_0^{\sqrt{(1-x^2)}} dydx = \int_0^{\pi}\int_0^1 rdrd\theta$$

$$= \int_0^{\pi}\frac{1}{2}d\theta = \frac{1}{2}(\theta)_0^{\pi} = \frac{\pi}{2}$$

19. Change the order of integration and evaluate $\int_0^1\int_0^x dydx$

$$\int_0^1\int_0^x dydx = \int_0^1\int_0^y dxdy$$

$$= \int_0^1 (1-y)dy = 1-\frac{1}{2} = \frac{1}{2}$$

20. **Change the order of integration** $\int_0^\infty\int_0^\infty f(x,y)dxdy$

$$\int_0^\infty\int_0^\infty f(x,y)dxdy = \int_0^\infty\int_x^\infty f(x,y)dydx$$

$$= \int_0^\infty\int_0^y f(x,y)dxdy$$

21.**Express the volume bounded by** $x \geq 0, y \geq 0, z \geq 0$ **and** $x^2+y^2+z^2=1$ **in triple integration**.

$$V = \int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} dzdydx$$

22. **Express the volume bounded by** $0 \leq x \leq 1, 1 \leq y \leq 2, 2 \leq z \leq 3$ **in triple integration**.

$$V = \int_0^1\int_1^2\int_2^3 dzdydx$$

23. **Express the volume bounded by x = 0, y = 0, z = 0, x + y + z = 1 in triple integration.**

$$V = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dzdydx$$

24.Transform into polar co-ordinates the integral $\int_0^a\int_y^a f(x,y)dxdy$ **.**

$$\int_0^{\frac{\pi}{4}}\int_0^{a\sec\theta} f(r,\theta)rdrd\theta$$

25. Express the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ in double integration.

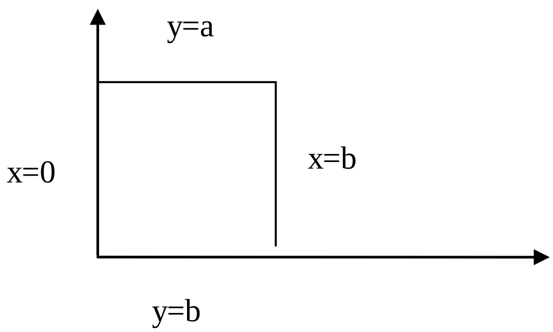
$$A = \int_0^\pi\int_{2\sin\theta}^{4\sin\theta} drd\theta$$

26. **Change the order of integration in** $\int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} x\,dx\,dy$ **.**

$$\int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} x\,dx\,dy = \int_{0}^{a}\int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} x\,dx\,dy\,.$$

27. **Sketch roughly the region of integration for the following double integral** $\int_{0}^{a}\int_{0}^{b} f(x,y)\,dx\,dy$ **.**

The region R is



28. **Change the order of integration in** $\int_{1}^{3}\int_{0}^{\frac{6}{x}} x^2\,dx\,dy$

$$\int_{1}^{3}\int_{0}^{\frac{6}{x}} x^2\,dx\,dy = \int_{1}^{3}\int_{0}^{2} x^2\,dx\,dy + \int_{1}^{3}\int_{2}^{\frac{6}{x}} x^2\,dx\,dy$$

29. **Change the order of integration in** $\int_{0}^{a}\int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} dx\,dy$

$$\int_{0}^{a}\int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} dx\,dy = \int_{0}^{2a}\int_{0}^{\sqrt{(a^2-(x-a)^2)}} dy\,dx$$

30. **Change the order of integration in** $\int_{0}^{\infty}\int_{0}^{y} y e^{\frac{-y^2}{x}}\,dx\,dy$

$$\int_{0}^{\infty}\int_{0}^{y} y e^{\frac{-y^2}{x}}\,dx\,dy = \int_{0}^{\infty}\int_{x}^{\infty} y e^{\frac{-y^2}{x}}\,dy\,dx$$

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