**B.E./B.Tech. DEGREE EXAMINATION, JUNE 2011**

Common to all B.E./B.Tech

Second Semester

**181202 – MATHEMATICS – II**

**(Regulation 2010)**

**Part A**

**1.** $\boldsymbol{Solve : (D^3+1)y = 0}$

**Solution:**

Auxiliary equation is $m^3+1=0 \Rightarrow (m+1)(m^2-m+1)=0$

$$m=-1, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

The complementary function is

$$y = Ae^{-x} + e^{\frac{x}{2}}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)$$

**2.** $\boldsymbol{Reduce\ the\ equation\ x^4\, y''' - x^3\, y'' + x^2\, y' = 1\ into\ linear\ equation\ with}$ $\boldsymbol{constant}$ **coefficients.**

**Solution:**

Let $z = \log x \Rightarrow x = e^z$

$$x^4\, y''' - x^3\, y'' + x^2\, y' = 1 \Rightarrow x^3\, y''' - x^2\, y'' + xy' = \frac{1}{x} \ldots (1)$$

$$x\frac{dy}{dx} = D'y \qquad where\ D' = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y, x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$$

Equation (1) reduces to

$$D'(D'-1)(D'-2)y - D'(D'-1)y + D'y = e^{-z}$$

$$((D')^3 - 4(D')^2 + 4D')y = e^{-z}$$

**3.** ***Find the unit vector normal to the surface of the sphere*** $\boldsymbol{x^2 + y^2 + z^2 = 1}$.

**Solution:**

$$Let\ \phi = x^2 + y^2 + z^2 - 1$$

Unit normal vector is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(2x\vec{\imath} + 2y\,\vec{\jmath} + 2z\,\vec{k}\right)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{\left(x\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)}{\sqrt{x^2 + y^2 + z^2}}$$

**4.** ***Show that*** $\boldsymbol{\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{\imath} + (3xz + 2xy)\,\vec{\jmath} + (3xy - 2xz + 2z)\vec{k}}$ ***is both solenoidal and irrotational.***

**Solution:**

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x - 2x + 2 = 0$$

$$\nabla.\vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right)$$

$$-\vec{\jmath}\left(\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right)$$

$$= \vec{\imath}(3x - 3x) + \vec{\jmath}(3y - 2z + 2z - 3y) + \vec{k}(3z + 2y - 2y - 3z)$$

$$= 0\,\vec{\imath} + 0\,\vec{\jmath} + 0\,\vec{k} = \vec{0}$$

$\therefore \vec{F}$ is both solenoidal and irrotational.

**5. *Find the analytic function $w = u + iv$ whose imaginary part is given by***

$$\boldsymbol{v = e^x\,(x \sin y + y \cos y)\,.}$$

**Solution:**

$$v = e^x\,(x \sin y + y \cos y)\,.$$

$$v_x = e^x \sin y + e^x\,(x \sin y + y \cos y)$$

$$v_x(z, 0) = e^z \sin 0 + e^z\,(z \sin 0 + 0 \cos 0) = 0$$

$$v_y = e^x\,(x \cos y + \cos y - y \sin y)$$

$$v_y(z, 0) = e^z\,(z \cos 0 + \cos 0 - 0 \sin 0) = e^z\,(z + 1)$$

$$f(z) = \int \Big(v_y(z, 0) + i v_x(z, 0)\Big)\,dz + C = \int e^z\,(z + 1)dz + C$$

$$= (z + 1)e^z - e^z + C$$

$$f(z) = ze^z + C$$

**6. *Find the image of $|z + 1| = 1$ under the mapping $w = \frac{1}{z}$.***

**Solution:**

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$|z + 1| = 1 \Rightarrow \left|\frac{1}{w} + 1\right| = 1 \Rightarrow \left|\frac{1 + w}{w}\right| = 1 \Rightarrow \frac{|1 + w|}{|w|} = 1 \Rightarrow |1 + w| = |w|$$

$$|(u + 1) + iv| = |u + iv| \Rightarrow (u + 1)^2 + v^2 = u^2 + v^2 \Rightarrow (u + 1)^2 = u^2$$

$$u^2 + 2u + 1 = u^2 \Rightarrow 2u + 1 = 0 \Rightarrow u = -\frac{1}{2}$$

*The image of* $|z + 1| = 1$ *under the mapping* $w = \frac{1}{z}$ *is* $u = -\frac{1}{2}$

**7. *Find the residue of $\frac{(1 - e^{2z})}{z^4}$ at its pole.***

**Solution:**

Here $z = 0$ is a simple pole of order 4.

$$Residue = \frac{1}{(4 - 1)!} \lim_{z \to 0} \frac{d^{4-1}}{dz^{4-1}} z^4 \frac{(1 - e^{2z})}{z^4} = \frac{1}{3!} \lim_{z \to 0} \frac{d^3}{dz^3}(1 - e^{2z}) = \frac{1}{6} \lim_{z \to 0}(-8e^{2z}) = -\frac{8}{6}$$

$$Residue = -\frac{4}{3}$$

**8.** $\boldsymbol{Expand\ \frac{1}{z+2}\ at\ z = 1\ as\ a\ Taylor's\ series.}$

**Solution:**

| | |
|---|---|
| $f(z) = \frac{1}{z+2}$ | $f(1) = \frac{1}{1+2} = \frac{1}{3}$ |
| $f'(z) = -\frac{1}{(z+2)^2}$ | $f'(1) = -\frac{1}{(1+2)^2} = -\frac{1}{9}$ |
| $f''(z) = \frac{2}{(z+2)^3}$ | $f''(1) = \frac{2}{(1+2)^3} = \frac{2}{27}$ |
| $f'''(z) = \frac{-6}{(z+2)^4}$ | $f'''(1) = \frac{-6}{(1+2)^4} = -\frac{2}{27}$ |
| $f'^{v}(z) = \frac{24}{(z+2)^5}$ | $f'^{v}(z) = \frac{24}{(1+2)^5} = \frac{8}{81}$ |

**The Taylor's series is**

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(z) + \frac{(z-a)^3}{3!}f'''(z) + \cdots$$

$$f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \frac{(z-1)^3}{3!}f'''(1) + \frac{(z-1)^4}{4!}f'^{v}(1) + \cdots$$

$$f(z) = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{1}{27}(z-1)^2 - \frac{1}{81}(z-1)^3 + \frac{1}{243}(z-1)^4 + \cdots$$

**9.** $\boldsymbol{Find\ the\ inverse\ Laplace\ transform\ of\ \frac{e^{-2s}}{s-3}.}$

**Solution:**

$$L^{-1}\left[\frac{e^{-2s}}{s-3}\right] = \left\{L^{-1}\left[\frac{1}{s-3}\right]\right\}_{t\to t-2}$$

$$= \{e^{3t}\}_{t\to t-2} = e^{3(t-2)}$$

$$L^{-1}\left[\frac{e^{-2s}}{s-3}\right] = e^{3(t-2)}$$

**10.** $\boldsymbol{If\ L[f(t)] = \frac{1}{s(s^2+a^2)}, find\ \lim_{t\to 0} f(t)\ and\ \lim_{t\to\infty} f(t).}$

**Solution:**

$\phi(s) = L[f(t)] = \frac{1}{s(s^2+a^2)}$

By initial value theorem,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\phi(s) = \lim_{s\to\infty} s\frac{1}{s(s^2+a^2)} = \lim_{s\to\infty}\frac{1}{(s^2+a^2)} = 0$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\phi(s) = \lim_{s\to 0} s\frac{1}{s(s^2+a^2)} = \lim_{s\to 0}\frac{1}{(s^2+a^2)} = \frac{1}{a^2}$$

**PART B – ( 5 x 16 = 80 marks )**

**11. (a) (i) Solve :** $\boldsymbol{(D^2 + 3D + 2)\,y = sin\,x + x^2}$.

Solution:

The Auxiliary Equation is $m^2 + 3m + 2 = 0$

$$m = -2, -1$$

The complementary function is $y = Ae^{-2x} + Be^{-x}$

$$P.I_1 = \frac{sin\,x}{D^2 + 3D + 2} = \frac{sin\,x}{-1^2 + 3D + 2} = \frac{sin\,x}{3D + 1} = \frac{(3D-1)sin\,x}{(3D-1)(3D+1)}$$

$$P.I_1 = \frac{3\cos x - sin\,x}{(9D^2-1)} = \frac{3\cos x - sin\,x}{(9(-1^2)-1)}$$

$$= -\frac{1}{10}(3\cos x - sin\,x)$$

$$P.I_2 = \frac{x^2}{D^2 + 3D + 2} = \frac{x^2}{2\left(1+\left(\frac{D^2+3D}{2}\right)\right)} = \frac{\left(1+\left(\frac{D^2+3D}{2}\right)\right)^{-1} x^2}{2}$$

$$= \frac{1}{2}\left(1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right)x^2 = \frac{1}{2}\left(x^2-\left(\frac{D^2}{2}x^2+\frac{3D}{2}x^2\right)+\frac{9D^2}{4}x^2\right)$$

$$P.I_2 = \frac{1}{2}\left(x^2-(1+3x)+\frac{9}{2}\right) = \frac{1}{2}\left(x^2-3x+\frac{7}{2}\right)$$

The general solution is $Complementary\ Function\ +\ Particular\ Integral$

$$y = Ae^{-2x} + Be^{-x} - \frac{1}{10}(3\cos x - sin\ x) + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

**(ii) Solve for $x$ from the equations $D^2\,x + y = 3e^{2t}, Dx - Dy = 3e^{2t}$ .**

**Solution:**

$$D^2\,x + y = 3e^{2t} \dots (1)$$

$$Dx - Dy = 3e^{2t} \dots (2)\ where\ D = \frac{d}{dt}$$

$$(2) \times D \Rightarrow D^2x - D^2y = 3De^{2t} \Rightarrow D^2x - D^2y = 6e^{2t} \dots (3)$$

$$(1) - (3) \Rightarrow (D^2 + 1)y = -3e^{2t}$$

The Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

The complementary function is $y = A\cos t + B\sin t$

$$P.I = \frac{-3e^{2t}}{D^2 + 1} = \frac{-3e^{2t}}{2^2 + 1} = -\frac{3}{5}e^{2t}$$

The general solution is $Complementary\ Function\ +\ Particular\ Integral$

$$y = A\cos t + B\sin t - \frac{3}{5}e^{2t}$$

$$Dy = -A\sin t + B\cos t - \frac{6}{5}e^{2t}$$

$$Dx - Dy = 3e^{2t} \Rightarrow Dx + A\sin t - B\cos t + \frac{6}{5}e^{2t} = 3e^{2t}$$

$$Dx = -A\sin t + B\cos t - \frac{6}{5}e^{2t} + 3e^{2t} = -A\sin t + B\cos t + \frac{9}{5}e^{2t}$$

$$x = \int\left(-A\sin t + B\cos t + \frac{9}{5}e^{2t}\right)dt$$

$$x = A\cos t + B\sin t + \frac{9}{10}e^{2t}$$

**(b) Solve, by the method of variation of parameters, the equation**

$$\frac{d^2y}{dx^2} + a^2y = \tan ax$$

**Solution:**

The Auxiliary Equation is $m^2 + a^2 = 0$

$$m = \pm ai$$

The complementary function is $y = A\ cos\ ax + B\ sin\ ax$

Here $f_1 = \cos ax\,, f_2 = \sin ax$

$$f_1' = -a\ \sin ax\,, f_2{}' = a\cos ax$$

$$P.I = Pf_1 + Qf_2$$

$$where \qquad P = -\int \frac{f_2 f(x)\,dx}{f_1 f_2' - f_1' f_2}\ ,\quad Q = \int \frac{f_1 f(x)\,dx}{f_1 f_2' - f_1' f_2}$$

$$P = -\int \frac{\sin ax\,(\tan ax)\,dx}{\cos ax\,(a\cos ax) - (-a\sin ax)\sin ax} = -\int \frac{\sin ax\,(\tan ax)\,dx}{a(\cos^2 ax + \sin^2 ax)}$$

$$= -\int \frac{\sin^2 ax\ dx}{a\cos ax}$$

$$P = -\int \frac{1-\cos^2 ax\ dx}{a\cos ax} = -\frac{1}{a}\int (\sec ax - \cos ax\,)dx$$

$$= -\frac{1}{a}\left[\frac{1}{a}\log(\sec ax\ + \tan ax) - \frac{\sin ax}{a}\right]$$

$$P = -\frac{1}{a^2}[\log(\sec ax\ + \tan ax) - \sin ax]$$

$$Q = \int \frac{\cos ax\,(\tan ax)\,dx}{\cos ax\,(a\cos ax) - (-a\sin ax)\sin ax} = \int \frac{\sin ax\ dx}{a(\cos^2 ax + \sin^2 ax)} = \frac{1}{a}\int \sin ax\,dx$$

$$Q = -\frac{1}{a^2}\cos ax$$

$$P.I = Pf_1 + Qf_2 = -\frac{1}{a^2}[\log(\sec ax\ + \tan ax) - \sin ax]\cos ax - \frac{1}{a^2}\cos ax\sin ax$$

$$P.I = \left[-\frac{\log(\sec ax\ + \tan ax)}{a^2}\right]\cos ax$$

The general solution is $Complementary\ Function\ +\ Particular\ Integral$

$$y = A\ cos\ ax + B\ sin\ ax - \left[\frac{\log(\sec ax\ + \tan ax)}{a^2}\right]\cos ax$$

**12. (a) Verify Green's theorem in the plane for**

$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

**where C is the boundary of the region define by $x = 0, y = 0, x + y = 1$.**

**Solution:**

The Green's theorem is given by

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \ldots (1)$$

$$P = 3x^2 - 8y^2, Q = 4y - 6xy$$

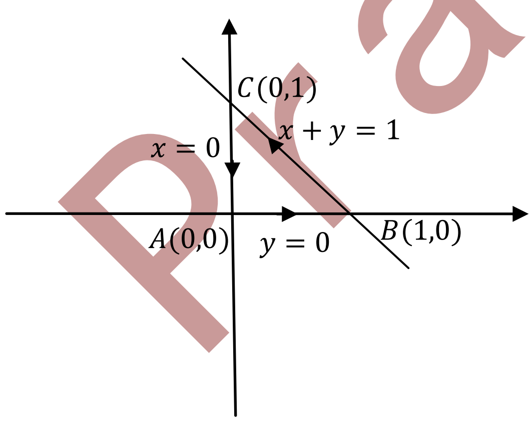
$$\frac{\partial P}{\partial y} = -16y, \frac{\partial Q}{\partial x} = -6y$$

R.H.S:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-y} (-6y + 16y)\, dx\, dy = 10 \int_0^1 \int_0^{1-y} y\, dx\, dy$$

$$= 10 \int_0^1 y\,(x)_0^{(1-y)}\, dy = 10 \int_0^1 y(1-y)\, dy = 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = 10\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{5}{3}$$

L.H.S:




$$\int_C Pdx + Qdy = \int_{AB} + \int_{BC} + \int_{CA}$$

On the line AB, $y = 0 \Rightarrow dy = 0$

$$\int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_0^1 (3x^2)dx = 3\left(\frac{x^3}{3}\right)_0^1 = 1$$

On the line BC, $x + y = 1, y = 1 - x \Rightarrow dy = -dx$

$$\int_{BC} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 (3x^2 - 8(1-x)^2)dx + (4(1-x) - 6x(1-x))(-dx)$$

$$= -\int_1^0 (-3x^2 + 8 + 8x^2 - 16x + 4 - 4x - 6x + 6x^2)dx = \int_0^1 (11x^2 + 6x - 4)dx$$

$$= \left[11\frac{x^3}{3} - 26\frac{x^2}{2} + 12x\right]_0^1 = \frac{11}{3} - 13 + 12 = \frac{8}{3}$$

On the line CA, $x = 0 \Rightarrow dx = 0$

$$\int_{CA} (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_1^0 4ydx = 4\left(\frac{y^2}{2}\right)_1^0 = -2$$

$$\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

L.H.S of (1) =R.H.S of (1)

Therefore Green's theorem is verified.

**12.(b) (i) Using Stoke's theorem prove that $\boldsymbol{curl\ (grad\ \phi) = 0}$.**

**Solution:**

Stoke's theorem is

$$\iint_S curl\,\vec{F}.\hat{n}\,ds = \oint_C \vec{F}.d\vec{r}$$

Taking $\vec{F} = grad\ \phi$, we get

$$\iint_S curl\,(grad\,\phi)\,.\hat{n}\,ds = \oint_C grad\,\phi.d\vec{r}$$

$$= \oint_C d\phi = 0$$

The above result is true for any open two sided surface $S$, provided it is bounded by the same simple closed curve $C$.

$$\therefore curl\,(grad\,\phi).\hat{n}\,ds = 0$$

$$\therefore curl\,(grad\,\phi) = 0$$

**12. (b)(ii) Evaluate $\iint_S \vec{F}.\hat{n}\,ds$ where $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and $S$ is the surface of the Parallelepiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1$ and $z = 3$.**

**Solution:**

By Gauss divergence theorem,

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V div\,\vec{F}\,dv$$

$$div\,\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz) = 2y + z^2 + x$$

$$\iiint_V div\,\vec{F}\,dv = \int_0^3\int_0^1\int_0^2 (2y + z^2 + x)dxdydz = \int_0^3\int_0^1 \left(2yx + z^2x + \frac{x^2}{2}\right)_0^2 dydz$$

$$= \int_0^3\int_0^1 (4y + 2z^2 + 2)dydz = \int_0^3 \left(4\frac{y^2}{2} + 2z^2y + 2y\right)_0^1 dz = \int_0^3 (2 + 2z^2 + 2)dz$$

$$= \int_0^3 (4 + 2z^2)dz = \left(4z + 2\frac{z^3}{3}\right)_0^3 = 12 + 18 = 30$$

$$\therefore \iint_S \vec{F}.\hat{n}\,ds = 30$$

**13. (a) (i) Prove that $u = e^{-y} \cos x$ and $v = e^{-x} \sin y$ satisfy Laplace equations, but that $u + iv$ is not an analytic function of $z$ .**

**Solution:**

$$u = e^{-y} \cos x, v = e^{-x} \sin y$$

$$u_x = -e^{-y} \sin x, u_y = -e^{-y} \cos x , u_{xx} = -e^{-y} \cos x , u_{yy} = e^{-y} \cos x$$

$$v_x = -e^{-x} \sin y, v_y = e^{-x} \cos y, v_{xx} = e^{-x} \sin y, u_{yy} = -e^{-x} \sin y$$

$$u_{xx} + u_{yy} = -e^{-y} \cos x + e^{-y} \cos x = 0$$

$$v_{xx} + v_{yy} = e^{-x} \sin y - e^{-x} \sin y = 0$$

∴ $u$ and $v$ satisfies Laplace equation.

Here $u_x \neq v_y$ and $u_y \neq -v_x$

∴CR equation is not satisfied.

∴ $u + iv$ is not an analytic function of $z$

**13. (a) (ii) Show that the families of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ cut orthogonally.**

**Solution:**

$$r^n = a \sec n\theta \Rightarrow r^n \cos n\theta = a \ldots (1)$$

$$r^n = b \operatorname{cosec} n\theta \Rightarrow r^n \sin n\theta = b \ldots (2)$$

Let $u(r, \theta) = a$ and $v(r, \theta) = b$

$$w = u(r, \theta) + iv(r, \theta) = a + ib = r^n \cos n\theta + ir^n \sin n\theta$$

$$w = r^n (\cos n\theta + i \sin n\theta) = r^n e^{in\theta} = \left(re^{i\theta}\right)^n = z^n$$

We know that $w = z^n$ is an analytic function.

Then by the property if $w = u(r, \theta) + iv(r, \theta)$ is an analytic function then the family of curves $u(r, \theta) = a$ and $v(r, \theta) = b$ cuts orthogonally where $a$ and $b$ are constants.

Applying this property to this problem we get,

The families of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ cut orthogonally.

**13. (b) Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$ . Hence find the image of $|z| < 1$.**

**Solution:**

The cross-ratio for bilinear transformation is given by

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Here $w_1 = w, w_2 = i, w_3 = 0, w_4 = -i, z_1 = z, z_2 = 1, z_3 = i, z_4 = -1$

$$\frac{(w - i)(0 + i)}{(w + i)(0 - i)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$-\frac{(w - i)}{(w + i)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$-\frac{(w - i) + (w + i)}{(w + i) - (w - i)} = \frac{(z - 1)(i + 1) + (z + 1)(i - 1)}{(z + 1)(i - 1) - (z - 1)(i + 1)} \qquad \left[= \frac{Nr + Dr}{Dr - Nr}\right]$$

$$-\frac{2w}{2i} = \frac{zi - i + z - 1 + zi + i - z - 1}{zi + i - z - 1 - (zi - i + z - 1)}$$

$$-\frac{w}{i} = \frac{2zi - 2}{2i - 2z} \Rightarrow w = \frac{-i(2zi - 2)}{2i - 2z} = \frac{2z + 2i}{2i - 2z}$$

$$w = \frac{z + i}{i - z}$$

$$wi - wz = z + i \Rightarrow wi - i = z + wz \Rightarrow z = \frac{i(w - 1)}{w + 1} \ldots (1)$$

$$|z| < 1 \Rightarrow \left|\frac{i(w - 1)}{w + 1}\right| < 1 \Rightarrow |w - 1| < |w + 1|$$

$$|u + iv - 1| < |u + iv + 1| \Rightarrow (u - 1)^2 + v^2 < (u + 1)^2 + v^2 \Rightarrow (u - 1)^2 < (u + 1)^2$$

$$u^2 - 2u + 1 < u^2 + 2u + 1 \Rightarrow -2u < 2u \Rightarrow 4u > 0 \Rightarrow u > 0$$

i.e., the right half of the $w$-plane.

∴ The image of $|z| < 1$ is right half of the w-plane.

**14. (a)(i) *Using Cauchy's integral formula evaluate* $\int_C \frac{z}{z^2 + 1} dz$ *, where C is the circle* $|z + i| = 1$ .**

**Solution:**

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{z}{(z+i)(z-i)}\,dz$$

$|z\ +\ i\,| =\ 1$ is a circle with centre at $(0,-1)$ and radius $1$.

The point $z = -i$ lies inside the circle $|z+i| = 1$ and the point $z = i$ lies ouside the circle $|z+i| = 1$.

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{\left(\frac{z}{z-i}\right)}{(z+i)}\,dz = \int_C \frac{f(z)}{(z+i)}\,dz \qquad \text{where } f(z) = \frac{z}{z-i}$$

By Cauchy's integral formula

$$\int_C \frac{z}{z^2+1}\,dz = 2\pi i\, f(-i) = 2\pi i \left(\frac{-i}{-i-i}\right) = 2\pi i \left(\frac{-i}{-2i}\right) = \pi i$$

**$14.(a)(ii)$ *Expand the function* $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ *in Laurent's series for the region* $|z|\ >\ 3$.**

**Solution:**

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{(5z+7)}{(z+2)(z+3)} \dots (1)$$

$$\frac{(5z+7)}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3} \dots (2)$$

$\Rightarrow 5z+7 = A(z+3) + B(z+2) \dots (3)$

Put $z = -3$ in (3)

$-8 = -B \Rightarrow B = 8$

Put $z = -2$ in (3)

$A = -3$

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \left(\frac{A}{z+2} + \frac{B}{z+3}\right) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Since the region of convergence is $|z| > 3$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1} = 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$f(z) = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}[3(2)^n - 8(3)^n] \ldots (4)$$

The Laurent's series (4) is valid if

$$\left|\frac{1}{z}\right| < 2 \text{ and } \left|\frac{1}{z}\right| < 3 \Rightarrow |z| > 2 \text{ and } |z| > 3$$

$(\boldsymbol{b})$ ***Evaluate, by contour integration, the integral*** $\int_0^{2\pi}\frac{d\theta}{1 - 2a\sin\theta + a^2}, 0 < a < 1.$

**Solution:**

On the circle $|z| = 1, z = e^{i\theta}, dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$

$$\int_0^{2\pi}\frac{d\theta}{1 - 2a\sin\theta + a^2} = \int_C \frac{\frac{dz}{iz}}{1 - 2a\left(\frac{z^2 - 1}{2iz}\right) + a^2}, \text{where } C \text{ is } |z| = 1$$

$$= \int_C \frac{dz}{iz - a(z^2 - 1) + iza^2} = \int_C \frac{dz}{iz - a z^2 + a + iza^2} = -\frac{1}{a}\int_C \frac{dz}{z^2 - i\left(a + \frac{1}{a}\right)z - 1}$$

$$= -\frac{1}{a}\int_C \frac{dz}{(z - ia)\left(z - \frac{i}{a}\right)}$$

To find poles:

$$(z - ia)\left(z - \frac{i}{a}\right) = 0 \Rightarrow z = ia, \frac{i}{a}$$

which are simple poles.

*The pole* $z = ia$ *lies inside* $C$, *but* $z = \frac{i}{a}$ *lies outside* $C$.

Residue at $z = ia$ is

$$= \lim_{z \to ia} \frac{(z - ia)}{(z - ia)\left(z - \frac{i}{a}\right)} = \frac{1}{ia - \frac{i}{a}} = \frac{ai}{1 - a^2}$$

By Cauchy's residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a \sin\theta + a^2} = -\frac{1}{a} 2\pi i \left(\frac{ai}{1 - a^2}\right) = \frac{2\pi}{1 - a^2}$$

**15. (a) Using convolution theorem, find the inverse Laplace transform of**

$$\boldsymbol{\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}}$$

**Solution:**

$$L^{-1}\left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right) = L^{-1}\left(\frac{s}{s^2 + a^2}\right) * L^{-1}\left(\frac{s}{s^2 + b^2}\right) = (\cos at) * (\cos bt)$$

$$= \int_0^t \cos au \cos a(t - u) du$$

$$= \frac{1}{2} \int_0^t (\cos at + \cos a(2u - t))\, du$$

$$= \frac{1}{2} \cos at \int_0^t du + \frac{1}{2} \int_0^t \cos a(2u - t)\, du$$

$$= \frac{1}{2} \cos at\, [u]_0^t + \frac{1}{2}\left[\frac{\sin a(2u - t)}{2a}\right]_0^t$$

$$= \frac{1}{2}[t \cos at\,] + \frac{1}{4a}[\sin at - \sin a(-t)] = \frac{1}{2}[t \cos at\,] + \frac{1}{4a}[\sin at + \sin at]$$

$$L^{-1}\left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right) = \frac{1}{2}[t \cos at\,] + \frac{1}{2a}[\sin at]$$

**15. (b) Solve, by Laplace transform method, the equation**

$$\boldsymbol{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t} \sin t, y(0) = 0, y'(0) = 1}$$

**Solution:**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}sin\, t \ldots (1)$$

Taking Laplace transforms on both sides of (1) we get,

$$L\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right] = L[e^{-t}sin\, t]$$

$$L\left[\frac{d^2y}{dt^2}\right] + 2L\left[\frac{dy}{dt}\right] + 5L[y] = \{L[sin\, t]\}_{s\to s+1}$$

$$s^2L[y] - sy(0) - y'(0) + 2(sL[y] - y(0)) + 5L[y] = \left\{\frac{1}{s^2+1}\right\}_{s\to s+1}$$

$$(s^2+2s+5)L[y] - 1 = \left\{\frac{1}{s^2+1}\right\}_{s\to s+1}$$

$$(s^2+2s+5)L[y] = 1 + \frac{1}{(s+1)^2+1} = 1 + \frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)L[y] = \frac{s^2+2s+3}{s^2+2s+2}$$

$$L[y] = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$y = L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$$

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

Equating the coefficients of $s^3, s^2, s$ and constant terms on both sides, we get

$$A + C = 0 \Rightarrow A = -C \ldots (1)$$

$$2A + B + 2C + D = 1 \ldots (2)$$

$$5A + 2B + 2C + 2D = 2 \ldots (3)$$

$$5B + 2D = 3 \ldots (4)$$

Substituting (1) in (2) we get

$$B + D = 1 \ldots (5)$$

$$(4) - 2 * (5) \Rightarrow 3B = 1 \Rightarrow B = \frac{1}{3}$$

$$B + D = 1 \Rightarrow D = 1 - B = 1 - \frac{1}{3} = \frac{2}{3}$$

$$5A + 2B + 2C + 2D = 2 \Rightarrow -5C + \frac{2}{3} + 2C + \frac{4}{3} = 2 \Rightarrow C = 0$$

$$A = 0$$

$$\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} = \frac{\frac{1}{3}}{(s^2 + 2s + 2)} + \frac{\frac{2}{3}}{(s^2 + 2s + 5)}$$

$$y = L^{-1}\left[\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right] = L^{-1}\left[\frac{\frac{1}{3}}{(s^2 + 2s + 2)} + \frac{\frac{2}{3}}{(s^2 + 2s + 5)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s^2 + 2s + 2)}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s^2 + 2s + 5)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s + 1)^2 + 1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s + 1)^2 + 2^2}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2 + 1}\right] + \frac{1}{3}e^{-t}L^{-1}\left[\frac{2}{s^2 + 2^2}\right]$$

$$y = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$