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AU – chennai – Nov/Dec 2011

181202 – Mathematics II

(Regulation 2010)

Part – A

1) Find the particular integral of $(D^2-4D+4)y = x^2e^{2x}$.

Soln:

$$PI = \frac{1}{D^2-4D+4}x^2e^{2x} = e^{2x}\cdot\frac{1}{(D+2)^2-4(D+2)+4}x^2$$

$$= e^{2x}\cdot\frac{1}{D^2}x^2 = e^{2x}\cdot\frac{1}{D}\frac{x^3}{3} = e^{2x}\cdot\frac{x^4}{12}.$$

2) Transform the differential eqn $(x^2D^2+4xD+2)y = x+\frac{1}{x}$ to a differential eqn with constant coefficients.

Soln:

Put $x=e^z$, $z=\log x$.

$x\frac{dy}{dx} = D'y$, $x^2\frac{d^2y}{dx^2} = D'(D'-1)y$

$[D'(D'-1)+4D'+2]y = e^z+\frac{1}{e^z} \Rightarrow \boxed{(D'^2+3D'+2)y = e^z+e^{-z}}$

3) Find the directional derivative of $\phi(x,y,z) = xy^2+yz^2$ at the point $(2,-1,1)$ in the direction of the vector $\vec{i}+2\vec{j}+3\vec{k}$.

Soln: $D.D = \nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|}$.

$\nabla\phi = y^2\vec{i}+\vec{j}(2xy+z^2)+\vec{k}(2yz)$ | $\vec{a} = \vec{i}+2\vec{j}+3\vec{k}$, $|\vec{a}| = \sqrt{14}$.

$(\nabla\phi)_{(2,-1,1)} = \vec{i}-3\vec{j}-\vec{k}$

$$D.D = (\vec{i}-3\vec{j}-\vec{k})\cdot\frac{(\vec{i}+2\vec{j}+3\vec{k})}{\sqrt{14}} = \frac{-8}{\sqrt{14}}$$

4) Evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$ where $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ and V is the volume enclosed by the cube $0\le x,y,z\le 1$.

Soln: $\nabla\cdot\vec{F} = 2x+2y+2z$.

$$\iiint \nabla\cdot\vec{F}\,dv = 2\int_0^1\int_0^1\int_0^1(x+y+z)\,dx\,dy\,dz = 3.$$

5) Are $|z|$, $Re(z)$, $Im(z)$ analytic? Give reason.

Soln:

(i) $|z| = |x+iy|^2 = \sqrt{x^2+y^2}$

$u = \sqrt{x^2+y^2}$, $v = 0$.

$u_x = \frac{1}{2\sqrt{x^2+y^2}} 2x = \frac{x}{\sqrt{x^2+y^2}}$ | $v_y = 0$

$u_y = \frac{1}{2\sqrt{x^2+y^2}} 2y = \frac{y}{\sqrt{x^2+y^2}}$ | $v_x = 0$.

$\therefore$ $|z|$ is not analytic fn since its not satisfies CR eqns

(ii) $Re(z)$

$z = x+iy$

$Re(z) = x = u$

$u_x = 1$ | $v_x = 0$

$u_y = 0$. | $v_y = 0$.

$u_x \neq v_y$.

(iii) $Im(z)$

$z = x+iy$

$Im(z) = y = v$.

$u_x = 0$ | $v_x = 0$.

$u_y = 0$ | $v_y = 1$.

$u_x \neq v_y$

It does not satisfies C-R eqns.

$\therefore$ $|z|$, $Re(z)$, $Im(z)$ are not analytic fns.

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6) Define conformal.

A transformation that preserves angle between every pair of curves through a point, both in magnitude and sense is said to be conformal at that point.

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7) Using Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z+2)} dz$ where c is $|z| = \pi/2$.

Soln: $z=-1$ is a simple pole lies inside $|z| = \pi/2 = \frac{3.14}{2} = 1.5$

$z=-2$ " " " " outside $|z| = \pi/2$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z+2)} dz = \int_c \frac{\frac{\sin\pi z^2+\cos\pi z^2}{z+2}}{z+1} dz = 2\pi i f(-1) = 2\pi i(-1)$$

$$= -2\pi i.$$

8) classify the singularity of $f(z) = e^{e^{1/z}}$.

Soln:

(i) $\frac{1}{z} = 0$ is a singular point

(ii) $\underset{z\to\infty}{lt} f(z) = \underset{z\to\infty}{lt} e^{e^{\frac{1}{z^2}}} = e^{\infty} = \infty$ it

$z = 0$ is a essential singularity.

9) Find the laplace transform of $f(t) = \begin{cases} 0 & , t < 2\pi/3 \\ \cos(t - 2\pi/3), & t > 2\pi/3. \end{cases}$

Soln: By second shifting property

$L[f(t)] = F(s)$ and $G(t) = \begin{cases} 0, & t < a \\ F(t-a), & t > a \end{cases}$

$L[G(t)] = e^{-as} F(s)$ —— ①

$F(t-a) = \cos(t - 2\pi/3)$

$F(t) = \cos t$ and $a = \frac{2\pi}{3}$ —— ②

$\therefore L[F(t)] = L[\cos t] = \frac{s}{s^2+1}$ —— ③

Sub ② and ③ in ①,

$L[G(t)] = e^{-\frac{2\pi s}{3}} \cdot \frac{s}{s^2+1}$.

10) Verify the final value theorem for $f(t) = 3e^{-t}$.

Proof: Final Value theorem is $\underset{t\to\infty}{lt} f(t) = \underset{s\to 0}{lt} sF(s)$

L.H.S

$\underset{t\to\infty}{lt} f(t) = \underset{t\to\infty}{lt} 3e^{-t} = 0 \quad [\because e^{-\infty} = 0$.

R.H.S

$\underset{s\to 0}{lt} sF(s) = \underset{s\to 0}{lt} s \cdot L(e^{-t}) = \underset{s\to 0}{lt} s \cdot \frac{1}{s+1} = 0$.

L.H.S = RHS

Hence Final Value theorem is verified,

Part - B

11) a) (i) Solve $(D^2+4D+3)y = 6e^{-2x}\sin x \sin 2x$.

Soln: $(D^2+4D+3)y = 0$.

Put $D=m$, The A.E is $m^2+4m+3=0 \Rightarrow (m+1)(m+3)=0$.

$m_1=-1,\ m_2=-3$.

$$\boxed{C.F = C_1e^{-x}+C_2e^{-3x}}$$

$$P.I = \frac{1}{D^2+4D+3}\,6e^{-2x}\sin x\sin 2x = \frac{-6}{2}\cdot\frac{1}{D^2+4D+3}e^{-2x}[\cos 3x-\cos x]$$

$$=-3\left[\frac{1}{D^2+4D+3}e^{-2x}\cos 3x - \frac{1}{D^2+4D+3}e^{-2x}\cos x\right].$$

$$=-3\left[e^{-2x}\cdot\frac{1}{(D-2)^2+4(D-2)+3}\cos 3x - e^{-2x}\cdot\frac{1}{(D-2)^2+4(D-2)+3}\cos x\right).$$

$$=-3\left[e^{-2x}\cdot\frac{1}{D^2-1}\cos 3x - e^{-2x}\cdot\frac{1}{D^2-1}\cos x\right].$$

$$=-3\left[e^{-2x}\cdot\frac{1}{-10}\cos 3x - e^{-2x}\cdot\frac{1}{-2}\cos x\right].$$

$$= +\frac{3e^{-2x}}{10}\cos 3x - \frac{3e^{-2x}}{2}\cos x.$$

$y = C.F+P.I$

$$= C_1e^{-x}+C_2e^{-3x}+\frac{3e^{-2x}}{+10}\cos 3x - \frac{3}{2}e^{-2x}\cos x.$$

a) (ii) Using variation of parameters, solve $(2D^2-D-3)y = 25e^{-x}$.

Soln: Put $D=m$,

$2m^2-m-3=0 \Rightarrow (2m-3)(m+1)=0$.

$m=3/2,\ m=-1$

$C.F = C_1e^{3/2x}+C_2e^{-x}$.

(3)

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_2 f_1'}\,dx = -\int \frac{e^{-x}\cdot 25e^{-x}}{(-5/2)(e^{x/2})}\,dx = 10\int e^{-5/2 x}\,dx$$

$$\boxed{P = -4e^{-5/2 x}}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'}\,dx = \int \frac{e^{3x/2}\cdot 25e^{-x}}{(-5/2)(e^{x/2})}\,dx = -10\int dx.$$

$$\boxed{Q = -10x.}$$

$$P.I = Pf_1 + Qf_2 = (-4e^{-5/2 x})(e^{3/2 x}) - 10x(e^{-x})$$

$$= -4e^{-x} - 10xe^{-x}.$$

$$\boxed{y = c_1 e^{3/2 x} + c_2 e^{-x} - 4e^{-x} - 10xe^{-x}}.$$

11) b)
(ii) Solve $\frac{dx}{dt} + 2x - 3y = t$ and $\frac{dy}{dt} - 3x + 2y = e^{2t}$.

Soln:
$(D+2)x - 3y = t$ —— (1)
$-3x + (D+2)y = e^{2t}$ —— (2)

(1) $\times 3 \Rightarrow 3x(D+2) - 9y = 3t$
(2) $\times D+2 \Rightarrow -3x(D+2) + (D+2)^2 y = e^{2t}(D+2)$

$$(D^2+4D-5)y = 4e^{2t} + 3t$$

A.E is $m^2 + 4m - 5 = 0 \Rightarrow m = -5, 1$

$$C.F = c_1 e^{-5t} + c_2 e^{t}$$

$$P.I = \frac{4}{D^2+4D-5}e^{2t} + \frac{3}{D^2+4D-5}t$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[1 + \left(\frac{D^2+4D}{5}\right)\right]t$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$$

$$y = c_1 e^{-5t} + c_2 e^{t} + \frac{4}{7} e^{2t} - \frac{3}{5} t - \frac{12}{25} \quad \text{--- (3)}$$

Put (3) in (2), $3x = -e^{2t} + (D+2)y$

$$x = \frac{1}{3}\Big[e^{2t} + \frac{d}{dt}\left(c_1 e^{-5t} + c_2 e^{t} + \frac{4}{7}e^{2t} - \frac{3}{5}t + \frac{12}{25}\right)$$
$$+ 2\left(c_1 e^{-5t} + c_2 e^{t} + \frac{4}{7} - \frac{3}{5}t - \frac{12}{25}\right)\Big]$$

$$x = -c_1 e^{-5t} + c_2 e^{t} - \frac{13}{25} - \frac{2t}{5} + \frac{3}{7}e^{2t}.$$

12) a)

(i) Determine $f(r)$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, if $f(r)\vec{r}$ is solenoidal and irrotational.

Soln:
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$f(r)\vec{r} = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}.$$

$$\nabla[f(r)\vec{r}] = \frac{\partial}{\partial x}[f(r)x] + \frac{\partial}{\partial y}[f(r)y] + \frac{\partial}{\partial z}[f(r)z]$$

Now $\frac{\partial}{\partial x}[f(r)x] = f(r) + x f'(r)\frac{\partial r}{\partial x} = f(r) + x f'(r) \cdot \frac{x}{r}$

$|||^{ly}$ $\frac{\partial}{\partial y}[f(r)y] = f(r) + y f'(r) \cdot \frac{y}{r}$

$\frac{\partial}{\partial z}[f(r)z] = f(r) + z f'(r)\, z/r$.

$$\nabla[f(r)\vec{r}] = 3f(r) + \frac{f'(r)}{r}(x^2+y^2+z^2) = 3f(r) + r f'(r).$$

If $f(r)\vec{r}$ is solenoidal then $\operatorname{div}(f(r)\vec{r}) = 0$.

$$\therefore 3f(r) + r f'(r) = 0 \Rightarrow \frac{f'(r)}{f(r)} = -\frac{3}{r}.$$

Integrating w.r.to r, we get

$$\log f(r) = -3\log r + \log c = \log\left(\frac{c}{r^3}\right)$$

$\therefore f(r) = \frac{c}{r^3}$, where c is a constant.

(4)

Now $\text{curl}\vec{F} = \text{curl}\left(f(r)\vec{r}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$

$$= \Sigma\,\vec{i}\left\{z\frac{\partial}{\partial y}f(r) - y\frac{\partial}{\partial z}f(r)\right\}$$

$$= \Sigma\,\vec{i}\left\{z\,f'(r)\,\frac{y}{r} - y\,f'(r)\,\frac{z}{r}\right\}$$

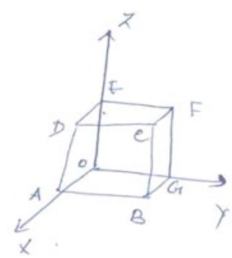
$$= \Sigma\,\vec{i}\,f'(r)\left\{\frac{yz}{r} - \frac{yz}{r}\right\} = \vec{0}, \text{ for all } f(r)$$

$\therefore f(r)\vec{r}$ is both solenoidal and irrotational.

(ii) Verify stoke's theorem for the vector $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$ where S is the surface bounded by the planes $x=0, y=0, z=0$, $x=1, y=1, z=1$ and C is the square boundary on the xoy plane.

Soln:

Stoke's theorem $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$



$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BG}\vec{F}\cdot d\vec{r} + \int_{GO}\vec{F}\cdot d\vec{r}$$

Along OA, $y=0, dy=0$, $z=0, dz=0$ | Along AB, $x=1, dx=0$, $z=0, dz=0$

Along BG, $y=1, dy=0$, $z=0, dz=0$ | Along GO, $x=0, dx=0$, $z=0, dz=0$

$$L.H.S = \int_0^1 (y-z)\,dx + \int_0^1 yz\,dy + \int_1^0 (y-z)\,dx + \int_1^0 yz\,dy = 1 \quad \text{--- (1)}$$

R.H.S

$$\text{curl}\,\vec{F} = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{x=1\\ \hat{n}=\vec{i}}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=1\\ \hat{n}=\vec{j}}} \nabla\times\vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{\substack{z=1\\ \hat{n}=\vec{k}}} \nabla\times\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 y\,dy\,dz + \int_0^1\int_0^1 (-y)\,dy\,dz + \int_0^1\int_0^1 (z-1)\,dx\,dz + \int_0^1\int_0^1 -(z-1)\,dx\,dz$$

$$+ \int_0^1\int_0^1 dx\,dy$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + 1$$

$$= 1. \quad \text{--- } (2)$$

LHS = RHS

Stoke's theorem is verified.

b)(ii) If $\vec{F}$ is a vector point function prove that
curl (curl $\vec{F}$) $= \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$.

Proof:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\nabla\times(\nabla\times\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial^2 F_2}{\partial x\partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x\partial z}\right]$$

$$= \sum \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z} - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial x}(\nabla\cdot\vec{F}) - \nabla^2 F_1\right]$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\nabla\cdot\vec{F}) - \nabla^2\left[F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right]$$

$$= \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

(5)

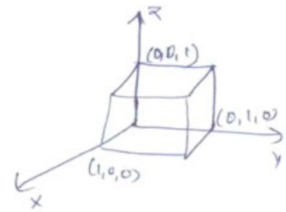
b)

(ii) Verify Gauss's theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ over the rectangular parallelopiped formed by $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$.

Soln:

Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$



$$\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$$

$$\nabla\cdot\vec{F} = 2(x+y+z)$$

R.H.S

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 2(x+y+z)\,dx\,dy\,dz = 2\int_0^1\int_0^1\left(\tfrac{1}{2}+y+z\right)dy\,dz$$

$$= 2\int_0^1\left(\tfrac{1}{2}+\tfrac{1}{2}+z\right)dz = 2\left(z+\tfrac{z^2}{2}\right) = 3 \quad \text{--- (1)}$$

L.H.S

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{x=1\\ \hat{n}=\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=1\\ \hat{n}=\vec{j}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{z=0\\ \hat{n}=-\vec{k}}} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{\substack{z=1\\ \hat{n}=\vec{k}}} \vec{F}\cdot\hat{n}\,ds.$$

$$= \int_0^1\int_0^1 -(x^2-yz)\,dy\,dz + \int_0^1\int_0^1 (x^2-yz)\,dy\,dz + \int_0^1\int_0^1 -(y^2-zx)\,dx\,dz + \int_0^1\int_0^1 (y^2-zx)\,dx\,dz$$

$$+ \int_0^1\int_0^1 -(z^2-xy)\,dx\,dy + \int_0^1\int_0^1 (z^2-xy)\,dx\,dy$$

$$= \frac{3}{4}+\frac{1}{4}+\frac{3}{4}+\frac{1}{4}+\frac{3}{4}+\frac{1}{4} = 3 \quad \text{--- (2)}$$

(1) = (2)

Hence Gauss divergence theorem is verified.

13) a)

(i) If $f(z)$ is analytic fn of z in any domain prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = p^2\,|f'(z)|^2\,|f(z)|^{p-2}.$$

Proof: $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^p$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left[\left(f(z)\, f(\bar{z})\right)^{1/2}\right]^P$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left(f(z)\, f(\bar{z})\right)^{P/2}$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left(f(z)\right)^{P/2} \left(f(\bar{z})\right)^{P/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[\left(f(z)\right)^{P/2} \cdot P/2 \left(f(\bar{z})\right)^{P/2-1} f'(\bar{z})\right]$$

$$= 2P\, f(\bar{z})^{\frac{P-2}{2}} f'(\bar{z}) \frac{\partial}{\partial z}\left(f(z)\right)^{P/2}$$

$$= 2P\, f(\bar{z})^{\frac{P-2}{2}} f'(\bar{z}) \frac{P}{2}\left(f(z)\right)^{\frac{P-2}{2}} f'(z)$$

$$= P^2\left[\left(f(z)\, f(\bar{z})\right)^{1/2}\right]^{P-2}\left(f'(z) f'(\bar{z})\right)^{1/2}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^P = P^2 |f(z)|^{P-2} |f'(z)|^2$$

Hence proved.

(ii) Show that the transformation $w = \frac{1}{z}$ transforms, in general circles and straight line into circles and straight lines that are transformed into straight lines and circles.

Soln:

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Q.No 13 (ii)

(or)

b(i) Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $w = e^{z^2}$.

Soln:

$$u+iv = e^{z^2} = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2} \cdot e^{i2xy}$$

$$u+iv = e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$$

$$u = e^{x^2-y^2}\cos 2xy = c_1$$

Diff w.r.t x

$$u_x = e^{x^2-y^2}[-\sin 2xy]\left[2x\frac{dy}{dx} + 2y\right] + \cos 2xy\, e^{x^2-y^2}\left[2x - 2y\frac{dy}{dx}\right] = 0$$

$$(-2x\sin 2xy - 2y\cos 2xy)\frac{dy}{dx} = 2y\sin 2xy - 2x\cos 2xy$$

(6)

$$m_1 = \frac{dy}{dx} = \frac{x\cos 2xy - y\sin 2xy}{x\sin 2xy + y\cos 2xy} \quad \text{——— (1)}$$

$v = e^{x^2-y^2} \sin 2xy = c_2$

$$v_x = e^{x^2-y^2}\left[\cos 2xy\left(2x\frac{dy}{dx} + 2y\right)\right] + \sin 2xy\, e^{x^2-y^2}\left[2x - 2y\frac{dy}{dx}\right] = 0$$

$$\frac{dy}{dx}\left[2x\cos 2xy + (-2y\sin 2xy)\right] = -2y\cos 2xy - 2x\sin 2xy$$

$$m_2 = \frac{dy}{dx} = \frac{-(x\sin 2xy + y\cos 2xy)}{x\cos 2xy - y\sin 2xy} \quad \text{——— (2)}$$

$$\therefore m_1 m_2 = \frac{x\cos 2xy - y\sin 2xy}{x\sin 2xy + y\cos 2xy} \times \frac{-(x\sin 2xy + y\cos 2xy)}{x\cos 2xy - y\sin 2xy}$$

$$= -1$$

Hence its cut orthogonally.

(ii) Find the Bilinear transformation that maps the pts $1+i, -i, 2-i$ of the z-plane into the points $0, 1, i$ of the w-plane.

Soln:

Bilinear transformation is $\frac{w-w_1}{w_1-w_2} \cdot \frac{w_2-w_3}{w_3-w} = \frac{z-z_1}{z_1-z_2} \cdot \frac{z_2-z_3}{z_3-z}$

$$\frac{w-0}{0-1} \cdot \frac{1-i}{i-w} = \frac{z-(1+i)}{1+i+i} \cdot \frac{-i-2+i}{2-i-z}$$

$$\frac{w-i}{w} = \frac{(z-2+i)(-1-2i)(1-i)}{(z-1-i)(-2)}$$

$$w = \frac{2z-2-2i}{-i\{-(1+i)z+5-3i\}}$$

$$w = \frac{2z-2-2i}{(i-1)z-3-5i}$$

14) a)

(i) Find the Laurent's series expansion of $f(z) = \frac{1}{z(1-z)}$ valid in the regions $|z+1|<1$, $1<|z+1|<2$ and $|z+1|>2$.

Soln:

$$f(z) = \frac{1}{z(1-z)}$$

Put $z+1 = u$ (or) $z = u-1$

$$f(z) = \frac{1}{(u-1)(2-u)}$$

By using partial fraction

$$f(z) = \frac{1}{u-1} + \frac{1}{2-u}$$

(i) $|z+1|<1$

$$f(z) = \frac{-1}{1-u} + \frac{1}{2(1-u/2)} = -(1-u)^{-1} + \frac{1}{2}(1-u/2)$$

$$= -\sum_{n=0}^{\infty} u^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{u^n}{2^n} = \sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)(z+1)^n.$$

The Laurent's expansion is valid if $|u|<1$ & $|u|<2$ (ie) $|z+1|<1$

(ii) $1<|z+1|<2$.

$$f(z) = \frac{1}{u(1-\frac{1}{u})} + \frac{1}{2(1-u/2)} = \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} + \frac{1}{2}\left(1-\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{u^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z+1)^n$$

The Laurent's expansion is valid if $|\frac{1}{u}|<1$ and $|\frac{u}{2}|<1$ (ie) $|u|>1$ & $|u|<1$

$1<|z+1|<2$.

(iii) $|z+1|>2$.

$$f(z) = \frac{1}{u(1-\frac{1}{u})} - \frac{1}{u(1-2/u)} = \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\frac{1}{u^n} - \frac{1}{u}\sum_{n=0}^{\infty}\frac{2^n}{u^n} = \sum_{n=0}^{\infty}(1-2^n)\frac{1}{(z+1)^{n+1}}$$

$|\frac{1}{u}|<1$ and $|\frac{2}{u}|<1$ (ie) $|u|>1$ & $|u|>2$

(ie) $|z+1|>2$.

a)
(ii) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ $(a>b>0)$ using contour integration.

Soln:

$z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{dz/iz}{a+\frac{b}{2}\left(\frac{z^2+1}{z}\right)} = \frac{1}{i}\int_C \frac{dz}{z\left[\frac{2az+bz^2+b}{2z}\right]}$$

$$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b} = \frac{2}{i}\int_C f(z)\,dz$$

The poles of $f(z)$ are $bz^2+2az+b=0$.

$$z = \frac{-a+\sqrt{a^2-b^2}}{b},\ \frac{-a-\sqrt{a^2-b^2}}{b}$$

$z=\alpha$, $z=\beta$.

where $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

$$\operatorname{Res} f(z)\Big|_{z=\alpha} = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} (z-\alpha)\cdot\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b\left[\frac{-a+\sqrt{a^2-b^2}}{b}+\frac{a+\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

$$\int_C f(z)\,dz = 2\pi i(\text{sum of the residue}) = 2\pi i\left(\frac{1}{2\sqrt{a^2-b^2}}\right) = \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\int_C f(z)\,dz$$

$$= \frac{2}{i}\left[\frac{\pi i}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

b)

(i) If $f(a) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$, where c is the circle $|z|=2$, find the values of $f(3)$, $f'(1+i)$ and $f''(1+i)$.

Soln:

$$f(a) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$$

$$f(3) = \int_C \frac{3z^2+7z+1}{z-a}\,dz$$

$z=3$ lies outside the circle $x^2+y^2=4$, $|z|=2$

$\therefore f(3) = 0$.

$$f'(1+i) = +\int_C \frac{3z^2+7z+1}{(z-(1+i))^2}\,dz = +\int_C \frac{\phi(z)\,dz}{(z-(1+i))^2}$$

The pt $z=1+i$ lies inside the circle $x^2+y^2=4$

$$f'(1+i) = -2\pi i\,\phi'(1+i) = -2\pi i(6z+7) = -(13+6i)2\pi i$$

$$f''(1+i) = 2\int_C \frac{3z^2+7z+1}{(z-(1+i))^3}\,dz = 2\int_C \frac{\phi(z)\,dz}{(z-(1+i))^3}$$

The pt $z=1+i$ lies inside the circle $x^2+y^2=4$

$$\therefore f''(1+i) = 2\pi i[\phi''(z)] = 2\pi i(6) = 12\pi i.$$

(ii) Evaluate $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$ using contour integration where $a>b>0$

Soln: Let $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx.$$

As $R\to\infty$, $\int_\Gamma f(z)\,dz = 0$

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

By residue theorem $\int_C f(z)dz = 2\pi i \sum \text{Res} f(z)$

Poles of $f(z)$ are $z^2+a^2=0$, $z=\pm ai$ | $z^2+b^2=0$, $z=\pm bi$

The pole $z=ia$, $z=ib$ lies in the upper hemisphere

$$\text{Res} f(z)\Big|_{z=ia} = \lim_{z\to ai} (z-ai) f(z)$$

$$= \lim_{z\to ai} (z-ai) \frac{e^{iz}}{(z-ai)(z+ai)(z^2+b^2)} = \frac{e^{-a}}{2ai(b^2-a^2)}$$

$$\text{Res} f(z)\Big/_{z=ib} = \lim_{z\to bi} (z-bi) \cdot \frac{e^{iz}}{(z^2+a^2)(z-bi)(z+bi)} = \frac{e^{-b}}{2bi(a^2-b^2)}$$

$$\sum R = \frac{1}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{2\pi i}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right] = \frac{\pi}{a^2-b^2}\left[\frac{ae^{-b}-be^{-a}}{ab}\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left[\frac{ae^{-b}-be^{-a}}{ab}\right]$$

Equating real part

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left[\frac{ae^{-b}-be^{-a}}{ab}\right]$$

15 a)

(i) Find the Laplace transform of $f(t) = \begin{cases} t, & 0<t<1 \\ 0, & 1<t<2 \end{cases}$ and $f(t+2) = f(t)$ for $t>0$.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

As $T=2$, $L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st} t\, dt + \int_1^2 0\right]$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-e^{-s}}{s} - \frac{1}{s^2}(e^{-s}-1)\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}(1+s)}{s^2}\right].$$

(ii) Solve $y'' - 3y' + 2y = 4e^{2t}$, $y(0) = -3$, $y'(0) = 5$ using Laplace Transform.

Soln:

$$y'' - 3y' + 2y = 4e^{2t}$$

Taking Laplace on both sides

$$s^2 L(y) - s\,y(0) - y'(0) - 3[s\,L(y) - y(0)] + 2L(y) = 4L[e^{2t}]$$

$$L(y)\,(s^2 - 3s + 2) = \frac{4}{s-2} - 3s + 14.$$

$$L(y) = \frac{-3s^2 + 20s - 24}{(s-2)^2(s-1)} \quad \text{— (1)}$$

Consider $\dfrac{-3s^2+20s-24}{(s-2)^2(s-1)} = \dfrac{A}{s-2} + \dfrac{B}{(s-2)^2} + \dfrac{C}{s-1}$

$$-3s^2 + 20s - 24 = A(s-2)(s-1) + B(s-1) + C(s-2)^2$$

Solving this we get

$$\boxed{A = 4,\ B = 4,\ C = -7}$$

(1) $\Rightarrow$ $y(t) = L^{-1}\left[\frac{4}{s-2}\right] + L^{-1}\left[\frac{4}{(s-2)^2}\right] + L^{-1}\left[\frac{-7}{s-1}\right]$

$$= 4e^{2t} + 4te^{2t} - 7e^{t}.$$

15) b)

(i) Find $L^{-1}\left[\frac{1}{s}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$.

Soln:

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \int_0^t L^{-1}(F(s))\,dt$$

$$= \int_0^t L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]dt \quad \text{— (1)}$$

Consider

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = f(t) \Rightarrow L[f(t)] = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \log(s^2+a^2) - \log(s^2+b^2)$$

$$L[t f(t)] = -\frac{d}{ds}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$$

$$= \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

$$t f(t) = L^{-1}\left[\frac{2s}{s^2+b^2}\right] - L^{-1}\left[\frac{2s}{s^2+a^2}\right] = 2\cos bt - 2\cos at$$

$$L^{-1}\left[\log\left[\frac{s^2+a^2}{s^2+b^2}\right]\right] = f(t) = \frac{2\cos bt - 2\cos at}{t}$$

(ii) Using convolution theorem find $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$.

Soln:

$$F(s) = \frac{1}{s^2+1} \text{ and } G(s) = \frac{1}{s+1} \Rightarrow f(t) = \sin t \ \& \ g(t) = e^{-t}$$

$$L^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right] = L^{-1}[F(s)\,G(s)] = \int_0^t f(y)\,g(t-y)\,dy$$

$$= \int_0^t \sin y \, e^{-(t-y)}\,dy$$

$$= e^{-t}\left[\frac{e^y}{2}(\sin y - \cos y)\right]_0^t$$

$$= \frac{e^{-t}}{2} + \frac{\sin t}{2} - \frac{\cos t}{2}$$

$$L^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right] = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right]$$