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Anna University of Technology, Chennai

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Regulation 2008

MA2161 – Mathematics II

Part - A

1) Reduce the eqn $(x^2D^2+xD+1)y = \log x$ into an ordinary diff. eqn with constant coefficients.

Soln:

$(x^2D^2+xD+1)\,y = \log x$.

Put $x = e^z$ (or) $z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$

$(D'(D'-1)+D'+1)\,y = z$.

$$\boxed{(D'^2+1)\,y = z}$$

2) Find the particular integral of $(D^2-2D+2)y = e^x\cos x$.

Soln:

$$P.I = \frac{1}{D^2-2D+2}\,e^x\cos x$$

$$= e^x \cdot \frac{1}{(D+1)^2-2(D+1)+2}\cos x$$

$$= e^x \cdot \frac{1}{D^2+1}\cos x = e^x \cdot \frac{1}{-1+1}\cos x$$

$$= e^x \cdot x \cdot \frac{1}{2D}\cos x = e^x \cdot \frac{x}{2}\sin x.$$

3) Prove that $\operatorname{div}\vec{r} = 3$ and $\operatorname{curl}\vec{r} = 0$.

Soln:

$$\operatorname{div}\vec{r} = \nabla\cdot\vec{r} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= 3.$$

$$\operatorname{curl}\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$

4) State Stoke's theorem.

The line integral of the tangential component of a vector function $\vec{F}$, around a simple closed curve C is equal to the surface integral of the normal component of curl $\vec{F}$ over any surface S having as its boundary

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

5) Verify whether the fn $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic

Soln:

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x \quad , \quad u_{xx} = 6x + 6$

$u_y = -6xy - 6y \quad , \quad u_{yy} = -6x - 6$

$u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

6) Verify whether $f(z) = \bar{z}$ is analytic function or not.

Soln:

$f(z) = \bar{z} = x - iy$

$= u + iv$

where $u = x \quad , \quad v = -y$

$u_x = 1 \quad\quad v_x = 0$

$u_y = 0 \quad\quad v_y = -1$

Here $u_x \neq v_y$ (ie) $f(z)$ is not satisfied by C-R eqns.

$\therefore f(z)$ is not analytic.

7) Evaluate $\int_C \frac{e^z}{z-1}\, dz$ if C is $|z| = 2$.

Soln: $z = 1$ is a pole of order 1 which lies inside $|z| = 2$

$\int_C \frac{e^z}{z-1}\, dz = 2\pi i f(1) = 2\pi i e^1$

8) If $f(z) = \frac{-1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \cdots\right]$ find the residue of $f(z)$ at $z = 1$.

Soln:

Since Residue of $f(z)$ at $z=1$ is the coeff of $\frac{1}{z-1}$

$$\text{Res } f(z)\Big|_{z=1} = -1.$$

9) Find Laplace transform of $t\sin 2t$.

Soln:

$$L[t.f(t)] = -\frac{d}{ds}F(s) = -\frac{d}{ds}\left[L(f(t))\right]$$

$$L[t\sin 2t] = -\frac{d}{ds}L(\sin 2t) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}.$$

10) Find $L^{-1}\left[\frac{1}{s^2+4s+4}\right]$.

Soln

$$L^{-1}\left[\frac{1}{s^2+4s+4}\right] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}.\,L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}.\,t$$

Part-B

11) a)

(i) Solve $(D^2+16)y = \cos^3 x$.

Soln:

A.E is $m^2+16 = 0$.

$m = \pm 4i$

$$\boxed{C.F = C_1\cos 4x + C_2\sin 4x}.$$

$$P.I = \frac{1}{D^2+16}\cos^3 x$$

$$= \frac{1}{D^2+16}\left[\frac{\cos 3x + 3\cos x}{4}\right]$$

$$= \frac{1}{4}\left[\frac{1}{D^2+16}\cos 3x + \frac{1}{D^2+16} 3\cos x\right]$$

$$= \frac{1}{4}\left[\frac{1}{7}\cos 3x + \frac{3}{15}\cos x\right]$$

$y = C.F + P.I$

$$y = c_1\cos 4x + c_2 \sin 4x + \frac{1}{4}\left(\frac{\cos 3x}{7} + \frac{3\cos x}{15}\right)$$

(ii) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 4y = \sec 2x.$$

Soln: A.E is $m^2+4=0 \Rightarrow m = \pm 2i$

$$\boxed{C.F = c_1\cos 2x + c_2 \sin 2x}$$

$$P.I = Pf_1 + Qf_2$$

$f_1 = \cos 2x$ | $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ | $f_2' = 2\cos 2x$.

$$f_1 f_2' - f_2 f_1' = 2\cos^2 2x + 2\sin^2 2x = 2.$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx = -\int \frac{\sin 2x \sec 2x}{2}\,dx = -\int \frac{\tan 2x}{2}\,dx$$

$$\boxed{P = \frac{1}{4}\log(\cos 2x)}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx = \int \frac{\cos 2x \sec 2x}{2} = \frac{1}{2}\int dx$$

$$\boxed{Q = \frac{x}{2}}$$

$$P.I = Pf_1 + Qf_2 = \frac{1}{4}\log(\cos 2x).\cos 2x + \frac{x}{2}\sin 2x$$

$y = C.F + P.I$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\log(\cos 2x).\cos 2x + \frac{x}{2}\sin 2x.$$

(or)

11) b)

(i) Solve $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$.

Soln:

Put $x = e^z$, $z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$

$[D'(D'-1) - 3D' + 4]\,y = e^{2z}\cos z$.

$(D'^2 - 4D' + 4)\,y = e^{2z}\cos z$.

A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2, 2$

$C.F = (C_1 z + C_2)e^{2z}$.

$$P.I = \frac{1}{D'^2 - 4D' + 4} e^{2z}\cos z$$

$$= e^{2z} \cdot \frac{1}{(D'+2)^2 - 4(D'+2) + 4}\cos z = e^{2z} \cdot \frac{1}{D^2}\cos z = e^{2z}(-\cos z)$$

$y = C.F + P.I$

$$\boxed{y = (C_1 z + C_2)e^{2z} - e^{2z}\cos z}$$

(ii) Solve $\frac{dx}{dt} + 2y = -\sin t$, $\frac{dy}{dt} - 2x = \cos t$ given $x = 1$ and $y = 0$ at $t = 0$

Soln:

$Dx + 2y = -\sin t$ — (1)

$-2x + Dy = \cos t$ — (2)

(1) × 2 ⇒ $2Dx + 4y = -2\sin t$

(2) × D ⇒ $-2Dx + D^2y = -\sin t$

$(D^2 + 4)y = -3\sin t$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$C.F = C_1\cos 2t + C_2\sin 2t$.

$$P.I = \frac{1}{D^2+4}(-3\sin t) = \frac{1}{-1+4}(-3\sin t) = -\sin t$$

$$\boxed{y = c_1 \cos 2t + c_2 \sin 2t - \sin t}$$

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t - \cos t$$

(2) $\Rightarrow$ $-2x = -Dy + \cos t$

$2x = Dy - \cos t$

$2x = -2c_1 \sin 2t + 2c_2 \cos 2t - \cos t - \cos t$

$$\boxed{x = -c_1 \sin 2t + c_2 \cos 2t - \cos t}$$

Given $x(0) = 2$ and $y(0) = 0$.

$c_1 = 0, \quad c_2 = 2$.

$\therefore x(t) = 2\cos 2t - \cos t$

$y(t) = 2\sin 2t - \sin t$.

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12) a)

(i) If $\vec{r}$ is the position vector of the point (x, y, z) prove that $\nabla^2 r^n = n(n+1) r^{n-2}$.

Soln:

W.K.T $\nabla(r^n) = n r^{n-2} \vec{r}$

$$\nabla^2(r^n) = \nabla \cdot \nabla(r^n) = \nabla \cdot (n r^{n-2} \vec{r})$$

$$= n\left[\nabla(r^{n-2}) \cdot \vec{r} + r^{n-2}(\nabla \cdot \vec{r})\right]$$

$$= n\left[(n-2) r^{n-4} \vec{r} \cdot \vec{r} + 3 r^{n-2}\right]$$

$$= n\left[(n-2) r^{n-4} r^2 + 3 r^{n-2}\right]$$

$$= n r^{n-2}\left[n - 2 + 3\right]$$

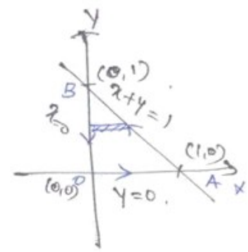
$$\boxed{\nabla^2(r^n) = n(n+1) r^{n-2}}$$

12) a)

(ii) Verify Green's theorem in the xy plane for $\int_C (3x^2-8y^3)dx + (4y-6xy)dy$, where c is the boundary of the region given by $x=0,\ y=0,\ x+y=1$

Soln:

Green's theorem is



$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$u = 3x^2 - 8y^2$	$v = 4y - 6xy$
$\frac{\partial u}{\partial y} = -16y$	$\frac{\partial v}{\partial x} = -6y$

L.H.S

$$\int_C u\,dx + v\,dy = \int_{OA} + \int_{AB} + \int_{BC} \quad \text{--- (1)}$$

(i) Along OA,

$y=0,\ dy=0$.

$$\int_C (3x^2-8y^2)dx + (4y-6xy)dy = \int_0^1 3x^2 dx = 1$$

(ii) Along AB, $x+y=1 \Rightarrow x = 1-y$

$dx = -dy$

$$\int_C (3x^2-8y^2)dx + (4y-6xy)dy = \int_C \left(3(1-y)^2 - 8y^2\right)(-dy) + (4y - 6(1-y)y)dy$$

$$= \int_0^1 \left[-(3+3y^2-6y-8y^2) + 4y - 6y + 6y^2\right] dy$$

$$= \int_0^1 (-3+5y^2+6y-2y+6y^2)dy$$

$$= \int_0^1 (11y^2+4y-3)dy = \left(\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{11}{3} - 1 = \frac{8}{3}.$$

(iii) Along BO, $x=0, dx=0$.

$$\int_C (3x^2-8y^2)dx + (4y-6xy)dy = \int_1^0 4y\,dy = -2.$$

$$\therefore \int_C u\,dx + v\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \text{---} \quad (1)$$

R.H.S

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$u = 3x^2 - 8y^2$ | $v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y$ | $\frac{\partial v}{\partial x} = -6y$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-y} (-6y+16y)\,dx\,dy = 10\int_0^1\int_0^{1-y} y\,dx\,dy$$

$$= 10\int_0^1 y\,(x)_0^{1-y}\,dy = 10\int_0^1 (y-y^2)\,dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = \frac{5}{3} \quad \text{---} \quad (2)$$

(1) = (2).

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

(or)

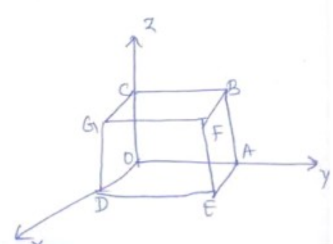
b)

(i) Verify the Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln:

Gauss divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$



L.H.S

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{x=1\\ \hat{n}=\vec{i}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=1\\ \hat{n}=\vec{j}}} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{\substack{z=0\\ \hat{n}=\vec{k}}} \vec{F}\cdot\hat{n}\,ds + \iint_{\substack{z=1\\ \hat{n}=\vec{k}}} \vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 0\,dy\,dz + \int_0^1\int_0^1 4z\,dy\,dz + \int_0^1\int_0^1 0\,dx\,dz + \int_0^1\int_0^1 -y^2\,dx\,dz$$

$$+ \int_0^1\int_0^1 0\,dx\,dy + \int_0^1\int_0^1 y\,dx\,dy$$

$$= 0+2+0-1+0+\frac{1}{2} = \frac{3}{2} \quad \text{——} (1)$$

R.H.S

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz = \int_0^1\int_0^1 (4zx-xy)_0^1\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)\,dy\,dz = \int_0^1 \left(4yz-\frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 \left(4z-\frac{1}{2}\right)dz = \frac{3}{2} \quad \text{——} (2)$$

(1) = (2)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Hence Gauss divergence theorem is verified.

13] a]
(i) Prove that $u = e^x(x\cos y - y\sin y)$ is harmonic and hence find the analytic fn $f(z) = u+iv$.

Soln

$$u = e^x(x\cos y - y\sin y)$$

$$= xe^x\cos y - ye^x\sin y$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \cos y\,(xe^x + e^x) - y\sin y \cdot e^x$$

$$\phi_1(z,0) = ze^z + e^z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -xe^x\sin y - e^x(\sin y + y\cos y)$$

$$\phi_2(z,0) = 0.$$

By Milne's Thomson method,

$$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$

$$= ze^z + e^z$$

$$f(z) = \int e^z(z+1)\,dz = ze^z - e^z + e^z$$

$$\boxed{f(z) = ze^z + c}$$

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13) a)
(ii) Find the bilinear transformation that transforms 1, i and -1 of the z-plane onto 0, 1, and ∞ of the w-plane. Also show that the transformation maps interior of the unit circle of the z-plane onto upper half of the w-plane.

Soln:

$$\frac{w-w_1}{w_1-w_2} \cdot \frac{w_2-w_3}{w_3-w} = \frac{z-z_1}{z_1-z_2} \cdot \frac{z_2-z_3}{z_3-z}$$

$$\frac{w-w_1}{w_1-w_2} \cdot \frac{\left(\frac{w_2}{w_3}-1\right)}{\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)}{(z_1-z_2)} \cdot \frac{(z_2-z_3)}{(z_3-z)}$$

$$\frac{w-0}{0-1} \cdot \frac{0-1}{1-0} = \frac{z-1}{1-i} \cdot \frac{i+1}{-1-z}$$

$$w = \frac{1-z}{1+z} \cdot \frac{2i}{2} = i\left(\frac{1-z}{1+z}\right)$$

$$w + wz = i - iz$$

$$z = \frac{1-w}{i+w}$$

Unit circle in the z-plane is $|z| = 1$

$$\left|\frac{i-w}{i+w}\right| < 1 \Rightarrow |i-w| < |i+w|$$

$$|i-u-iv| < |i+u+iv|$$

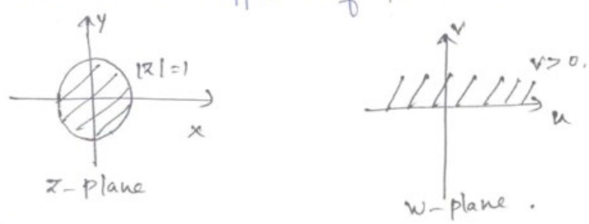
$$u^2+(1-v)^2 < u^2+(1+v)^2$$

$$-2v < 2v$$

$$4v > 0$$

$$v > 0$$

which is the upper half plane in the w-plane.



Thus $|z|<1$ is mapped into the upper half plane in the w-plane.

(3) b)

(i) Prove that $u = x^2-y^2$ and $v = \frac{-y}{x^2+y^2}$ are harmonic but $u+iv$ is not regular.

Soln:

$u = x^2-y^2$ and $v = \frac{-y}{x^2+y^2}$

$u_x = 2x$ —— (1) $\quad v_x = \frac{2xy}{(x^2+y^2)^2}$ —— (2)

$u_{xx} = 2$

$u_y = -2y$ —— (3) $\quad v_{xx} = \frac{(x^2+y^2)^2 2y - 2xy[2(x^2+y^2)(2x)]}{(x^2+y^2)^4} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}$

$u_{yy} = -2$

$v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$ —— (4)

$v_{yy} = \frac{6x^2y-2y^3}{(x^2+y^2)^2}$

$\therefore u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

v & u are harmonic fns.

From (1), (2), (3), (4)

$u_x \neq v_y$ & $u_y \neq -v_x$.

$\therefore$ u and v are not analytic.

(ii) Find the image of the half plane $x > c$, $c > 0$ under $w = \frac{1}{z}$. Sketch graphically.

Soln:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

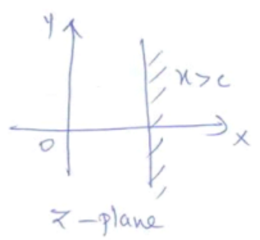
when $x = c$, $\quad c = \frac{u}{u^2+v^2} \Rightarrow c(u^2+v^2) = u \Rightarrow u^2+v^2 - \frac{u}{c} = 0$

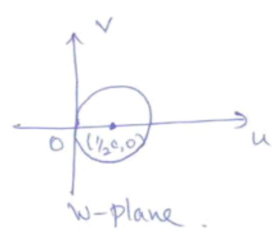
$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$$

which is the eqn of a circle whose centre is $(1/2c, 0)$ and radius is $\frac{1}{2c}$.

Thus the half plane $x > c$ in the z-plane is transformed into a circle

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2 \text{ in the } w\text{-plane}$$





14) a)

(i) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1+i|=2$, using Cauchy's integral formula.

Soln:

Given circle is $|z+1-i|=2$

$$|z-(-1+i)|=2$$

(ie) C is the circle with centre $-1+i$ and radius 2 units

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int_C \frac{\frac{z+4}{z+1+2i}}{z-(-1+2i)}\,dz.$$

where $f(z)=\frac{z+4}{z+1+2i} \Rightarrow z_0=-1+2i$

By Cauchy's integral formula, we have

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\int_C \frac{\frac{z+4}{z+1+2i}}{z-(-1+2i)}\,dz = 2\pi i\, f(-1+2i)$$

$\left[\because z_0=-1+2i = (-1,2) \text{ lies within the circle } |z+1-i|=2\right]$

$$= 2\pi i \cdot \frac{-1+2i+4}{-1+2i+1+2i}$$

$$= 2\pi i \cdot \frac{2i+3}{4i}$$

$$= \frac{\pi}{2}(2i+3)$$

(ii) Find the residues of $f(z)=\frac{z^2}{(z-1)(z+2)^2}$ at the isolated singularities using Laurent's series expansions. Also state the valid region.

Soln:

$$f(z) = \frac{z^2}{(z-1)(z+2)^2}$$

$$\frac{z^2}{(z-1)(z+2)^2} = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}$$

$$z^2 = A(z+2)^2 + B(z-1)(z+2) + C(z-1)$$

Put $z=1$, $1 = 9A \Rightarrow \boxed{A = \frac{1}{9}}$

$z = -2$, $4 = -3C \Rightarrow \boxed{C = -4/3}$

Coeff of z^2, $1 = A+B \Rightarrow \boxed{B = 8/9}$

$$f(z) = \frac{\frac{1}{9}}{z-1} + \frac{\frac{8}{9}}{z+2} - \frac{\frac{4}{3}}{(z+2)^2}$$

Here $z=1$ and $z=-2$ are isolated singularities

To find the residue of $f(z)$ at $z=1$, we have to expand $f(z)$ in series of power of $(z-1)$ which is valid in $0<|z-1|<r$

The coeff of $\frac{1}{z-1}$ gives residue.

$$f(z) = \frac{1}{9}\left(\frac{1}{z-1}\right) + \frac{8}{9}\left(\frac{1}{z+2}\right) - \frac{4}{3}\left(\frac{1}{(z+2)^2}\right)$$

$$= \frac{1}{9}\left(\frac{1}{z-1}\right) + \frac{8}{9}\,\frac{1}{3+(z-1)} - \frac{4}{3}\,\frac{1}{(3+(z-1))^2}$$

$$= \frac{1}{9}\cdot\frac{1}{z-1} + \frac{8}{27}\left[1 + \left(\frac{z-1}{3}\right)\right]^{-1} - \frac{4}{27}\left[1 + \frac{(z-1)}{3}\right]^{-2}$$

$$= \frac{1}{9}\cdot\frac{1}{z-1} + \frac{8}{27}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^n}$$

$$- \frac{4}{27}\sum_{n=0}^{\infty}(-1)^n(n+1)\,\frac{(-1)^n(n+1)(z-1)^n}{3^n}$$

(8)

This expansion is valid in $\left|\frac{z-1}{3}\right| < 1$ (ie) $0 < |z-1| < 3$

$$\text{Res } f(z)\Big|_{z=1} = \text{Coeff of } \frac{1}{z-1} = \frac{1}{9}.$$

To find the residue of $f(z)$ at $z=-2$ we have to expand $f(z)$ in series of power of $(z+2)$ which is valid in $0<|z+2|<r$. The coeff of $\frac{1}{z+2}$ gives the residue

$$f(z) = \frac{1}{9}\cdot\frac{1}{z-1} + \frac{8}{9}\cdot\frac{1}{z+2} - \frac{4}{3}\cdot\frac{1}{(z+2)^2}$$

$$= \frac{1}{9}\cdot\frac{1}{(z+2)-3} + \frac{8}{9}\cdot\frac{1}{z+2} - \frac{4}{3}\cdot\frac{1}{(z+2)^2}$$

$$= \frac{1}{9}\left[\frac{1}{-3\left[1-\left(\frac{z+2}{3}\right)\right]}\right] + \frac{8}{9}\,\frac{1}{z+2} - \frac{4}{3}\cdot\frac{1}{(z+2)^2}$$

$$= -\frac{1}{27}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n} + \frac{8}{9}\,\frac{1}{z+2} - \frac{4}{3}\,\frac{1}{(z+2)^2}$$

This expansion is valid in $\left|\frac{z+2}{3}\right| < 1$ (ie) $0<|z+2|<3$

$$\text{Res } f(z)\Big|_{z=-2} = \text{Coeff of } \frac{1}{z+2} = \frac{8}{9}.$$

(or)

9 (i) Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$

Soln:

Refer to AU-Chennai May/June 2010.

14-(b)-(iii).

(ii) Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}$ using Contour Integration.

Soln:

Transforming the given integral into the contour integral of the form $\int_C g(z)\,dz$.

$$\int_C \phi(z)\,dz = \int_C \frac{1}{(z^2+1)(z^2+4)}\,dz.$$

where c consists of the semicircle Γ above the real axis and the bounding diameter $[-R, R]$

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz.$$

$$\phi(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

Hence the poles are $z = i, -i, 2i, -2i$

Here $z = i$ and $z = 2i$ lies in the upper half plane while $z = -i$ and $z = -2i$ lie in the lower half plane.

$$\operatorname{Res} \phi(z)\big|_{z=i} = \lim_{z\to i} \cancel{(z-i)} \cdot \frac{1}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)}$$

$$= \frac{1}{6i}$$

$$\operatorname{Res} \phi(z)\big|_{z=2i} = \lim_{z\to 2i} (z-2i) \cdot \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= -\frac{1}{12i}$$

In $R\to\infty$, $\int_\Gamma \phi(z)\,dz$.

$$\int_C \phi(z)\,dz = \int_{-\infty}^{\infty} \phi(x)\,dx = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$$

$$= \int_C \frac{dz}{(z^2+1)(z^2+4)}$$

$$= 2\pi i \left[\frac{1}{6i} - \frac{1}{12i}\right] = \frac{\pi}{6}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$$

15) a)

(i) Find the Laplace transform of $f(t) = \begin{cases} \epsilon, & 0 \le t \le a \\ -\epsilon, & a \le t \le 2a \end{cases}$

and $f(t+2a) = f(t)$ for all t.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st}\, \epsilon\, dt + \int_a^{2a} e^{-st} (-\epsilon)\,dt \right]$$

$$= \frac{\epsilon}{1-e^{-2as}} \left\{ \left(\frac{e^{-st}}{-s} \right)_0^{a} - \left(-\frac{e^{-st}}{s} \right)_a^{2a} \right\}$$

$$= \frac{\epsilon}{s} \left[\frac{1 - 2e^{-as} + e^{-2as}}{1-e^{-2as}} \right] = \frac{\epsilon}{s} \left[\frac{(1-e^{-as})^2}{(1+e^{-as})(1-e^{-as})} \right]$$

$$= \frac{\epsilon}{s} \left[\frac{1-e^{-as}}{1+e^{-as}} \right] = \frac{\epsilon}{s} \tanh\left(\frac{as}{2}\right)$$

(ii) Find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.

Soln:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2} \right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2} \right] * L^{-1}\left[\frac{s}{s^2+b^2} \right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au \cdot \cos b(t-u)\,du$$

$$= \int_0^t \frac{\cos(au+bt-bu) + \cos(au-bt+bu)}{2}\,du$$

$$= \frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b} + \frac{\sin((a+b)u-bt)}{a+b} \right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right]$$

$$= \frac{a \sin at - b \sin bt}{a^2-b^2}$$

15] b]
(i) verify initial and final value theorems for the fn

$f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln:
Refer to AU chennai - May | June 2010
Q. No 15 (b) (i).

b]
(ii) Using Laplace transform solve the differential eqn

$y'' - 3y' - 4y = 2e^{-t}$ with $y(0) = 1 = y'(0)$

Soln:
Refer to AU chennai - May | June 2010
Q. No 15 (b) (ii).