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Anna University - Chennai

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MA 2161 - Mathematics II

Part- A

1) Solve the eqn $(D^2-6D+13)y=0$

Soln: A.E is $m^2-6m+13=0$

$$m = 3 \pm 2i$$

$$\boxed{y = e^{3x}[C_1\cos 2x + C_2 \sin 2x]}$$

2) Find the particular integral of $(D+1)^2 y = e^{-x}\cos x$

Soln:

$$P.I = \frac{1}{(D+1)^2} e^{-x}\cos x = e^{-x}\cdot\frac{1}{(D-1+1)^2}\cos x = e^{-x}\cdot\frac{1}{D^2}\cos x$$

$$\boxed{P.I = -e^{-x}\cos x}$$

3) Find $\text{grad}(r^n)$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.

Soln: $$\nabla(r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}(\nabla r) = nr^{n-1}\cdot\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

4) Find the unit normal to the surface $x^2+xy+z^2=4$ at $(1,-1,2)$.

Soln:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}, \quad \phi = x^2+xy+z^2-4$$

$$= (2x+y)\vec{i} + x\vec{j} + 2z\vec{k}$$

$$(\nabla\phi)_{(1,-1,2)} = \vec{i}+\vec{j}+4\vec{k}$$

$$|\nabla\phi| = \sqrt{18}$$

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$

real variable and that of a complex variable.

Soln:

Real Variable	Complex Variable
Limit takes along X axis and Y axis (or) Parallel to both axis	Limit takes along any path (straight or curved)

b) Prove that a bilinear transformation has atmost two fixed points.

Soln: The fixed points of the bilinear transformation

$w = \frac{az+b}{cz+d}$ are given by $z = \frac{az+b}{cz+d}$

$\Rightarrow z(cz+d) = az+b \Rightarrow cz^2 + dz - az - b = 0$

which is a quadratic eqn in z

$\therefore$ we get two fixed points for the bilinear transformation.

7) Identify the type of singularities of the foll fn $f(z) = e^{\frac{1}{z-1}}$

Ans:

$f(z) = e^{\frac{1}{z-1}}$

Here $z=1$ is a singular point.

Also $z=1$ is not a pole or removable singularity

$\therefore z=1$ is an essential singularity.

8) Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its poles.

Ans.

The poles are $z=-1$ (order 2)

$$\operatorname{Res} f(z)\Big/_{z=-1} = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{e^{2z}}{(z+1)^2}\right]$$

$$= \lim_{z\to -1} 2e^{2z}$$

$$= 2e^{-2}.$$

9] Find the Laplace transform of $t\cos at$

Soln:

$$L[t\cos at] = -\frac{d}{ds} L(\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left[\frac{(s^2+a^2) - s\,2s}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

10] Verify initial value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$.

Soln:

$$L[f(t)] = L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

L.H.S $\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[1 + e^{-t}(\sin t + \cos t)\right] = 2$

R.H.S $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right] = \lim_{s\to\infty} \frac{2s^2+4s+2}{(s+1)^2+1} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{s\to\infty} \frac{4s+4}{2(s+1)} \left(\frac{\infty}{\infty}\right) = \frac{4}{2} = 2.$$

Part-B

1) Solve the eqn $(D^2-3D+2)y = 2\cos(2x+3) + 2e^x$

Soln:

AE is $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$$C.F = C_1 e^{x} + C_2 e^{2x}.$$

$$P.I = \frac{1}{D^2-3D+2} 2\cos(2x+3) + \frac{1}{D^2-3D+2} 2e^x$$

$$= 2\left[\frac{1}{-4-3D+2}\cos(2x+3) + \frac{1}{1-3+2} e^x\right]$$

$$= 2\left[\frac{-1 \,(3D-2)}{(3D+2)(3D-2)}\cos(2x+3) + \frac{x}{2D-3} e^x\right]$$

$$= 2\left[\frac{2-3D}{9D^2-4}\cos(2x+3) - xe^x\right]$$

$$= 2\left[\frac{2\cos(2x+3)}{-40} + \frac{6\sin(2x+3)}{-40} - xe^x\right]$$

$$= \frac{-1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3) - 2xe^x$$

$$y = C.F + P.I$$

$$= c_1 e^x + c_2 e^{2x} - \frac{1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3) - 2xe^x$$

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a) (ii) Apply the method of Variation of parameters to solve $(D^2+4)y = \cot 2x$.

Soln: A.E is $m^2+4 = 0 \Rightarrow m = \pm 2i$

$$\boxed{C.F = c_1\cos 2x + c_2\sin 2x}$$

$$P.I = Pf_1 + Qf_2$$

$f_1 = \cos 2x$ | $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ | $f_2' = 2\cos 2x$

$$f_1f_2' - f_1'f_2 = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$P = -\int \frac{f_2 X}{f_1f_2' - f_1'f_2}\,dx = -\int \frac{\sin 2x \cot 2x}{2} = -\frac{1}{2}\int \cos 2x\,dx$$

$$\boxed{P = -\frac{\sin 2x}{4}}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\, dx = \int \frac{\cos 2x \cdot \cot 2x}{2}\, dx$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx = \frac{1}{2}\left[\int \frac{dx}{\sin 2x}\, dx - \int \sin 2x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{2}\right]$$

$$P.I = Pf_1 + Qf_2$$

$$= -\frac{1}{4}\sin 2x \cos 2x + \left[\frac{1}{4}\log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{4}\right]\sin 2x$$

$$y = P.I + C.F$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\sin 2x \cos 2x$$

$$+ \left[\frac{1}{4}\log(\operatorname{cosec} 2x - \cot 2x) + \frac{\cos 2x}{4}\right]\sin 2x.$$

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11 b) (i) Solve the diff eqn $(x^2D^2 - xD + 4)y = x^2 \sin(\log x)$

Soln:

Put $x = e^z$ (or) $z = \log x$

$xD = D'$, $x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - D' + 4]y = e^{2z}\sin z.$$

$$(D'^2 - 2D' + 4)y = e^{2z}\sin z$$

A.E is $m^2 - 2m + 4 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2}$

$$m = 1 \pm i\sqrt{3}$$

$$C.F = e^z(C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z)$$

$$P.I = \frac{1}{D'^2 - 2D' + 4} e^{2z}\sin z$$

$$= e^{2z} \cdot \frac{1}{(D'+2)^2 - 2(D'+2) + 4}\sin z$$

$$= e^{2z} \cdot \frac{1}{D'^2 + 2D' + 4} \sin z$$

$$= e^{2z} \cdot \frac{1}{-1 + 2D' + 4} \sin z$$

$$= e^{2z} \cdot \frac{1}{2D' + 3} \sin z = e^{2z} \cdot \frac{2D' - 3}{4D'^2 - 9} \sin z$$

$$= e^{2z} \cdot \frac{2\cos z - 3\sin z}{-4-9} = \frac{e^{2z}}{-13} (2\cos z - 3\sin z)$$

$$y = e^{z}(C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z) - \frac{e^{2z}}{13}(2\cos z - 3\sin z)$$

where $z = \log x$.

11) b (ii) Solve the simultaneous diff eqn $\frac{dx}{dt} + 2y = \sin 2t$, $\frac{dy}{dt} - 2x = \cos 2t$.

Soln:

$$Dx + 2y = \sin 2t \quad \text{—— (1)}$$
$$-2x + Dy = \cos 2t \quad \text{—— (2)}$$

$$(1) \times 2 \Rightarrow 2Dx + 4y = 2\sin 2t$$
$$(2) \times D \Rightarrow -2Dx + D^2y = -2\sin 2t$$

$$(D^2 + 4)y = 0$$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$\boxed{C.F = c_1 \cos 2t + c_2 \sin 2t}$$

$$P.I \Rightarrow y = C_1 \cos 2t + C_2 \sin 2t$$

$$\frac{dy}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t$$

Sub in eqn (2)

(4)

$$2x = \frac{dy}{dt} - \cos 2t$$

$$= -2A\sin 2t + 2B\cos 2t - \cos 2t$$

$$x = -C_1\sin 2t + C_2\cos 2t - \frac{\cos 2t}{2}$$

$$y = A\cos 2t + B\sin 2t$$

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12) a) (i) Prove that $\text{curl}(\vec{u}\times\vec{v}) = (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} + \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u}$.

Soln:

$$\text{Curl}(\vec{u}\times\vec{v}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \sum \vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v} + \vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \sum \vec{i}\times\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right) + \sum\vec{i}\times\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \sum(\vec{i}\cdot\vec{v})\frac{\partial\vec{u}}{\partial x} - \sum\left(\vec{i}\cdot\frac{\partial\vec{u}}{\partial x}\right)\vec{v}$$

$$+ \sum\left(\vec{i}\,\frac{\partial\vec{v}}{\partial x}\right)\vec{u} - \sum(\vec{i}\cdot\vec{u})\frac{\partial\vec{v}}{\partial x}$$

$$= \left(\sum\vec{i}\,\frac{\partial\vec{v}}{\partial x}\right)\vec{u} - \left(\sum\vec{i}\cdot\frac{\partial\vec{u}}{\partial x}\right)\vec{v}$$

$$+ \vec{v}\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{u}$$

$$- \vec{u}\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{v}$$

$$= \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u} + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v}.$$

(ii) Evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square bounded by the lines $x=0$, $x=1$, $y=0$ and $y=1$.

Soln: By Green's theorem

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

$u = xy + x^2$ | $v = x^2 + y^2$

$\frac{\partial u}{\partial y} = x$ | $\frac{\partial v}{\partial x} = 2x$

$$\int_C (x^2 + xy)dx + (x^2+y^2)dx = \int_0^1\int_0^1 x\,dx\,dy = \int_0^1 \left(\frac{x^2}{2}\right)_0^1 dy$$

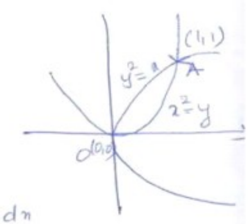
$$= \frac{1}{2}.$$

(or)

12) b) (i) Verify stoke's theorem when $\vec{F} = (2xy - x^2)\vec{i} - (x^2 - y^2)\vec{j}$ and c is the boundary of the region enclosed by the parabola $y^2 = x$ and $x^2 = y$.

Soln:

Stoke's theorem $\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$



L.H.S

$$\int_c \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = (2xy - x^2)dx - (x^2-y^2)dy$$

$$\int_{OA}\vec{F}\cdot d\vec{r} = \int_0^1 (2x(x^2) - x^2)dx - (x^2 - x^4)\,2x\,dx$$

$$= \int_0^1 (-x^2 + 2x^5)dx = 0$$

$$\int_{AO}\vec{F}\cdot d\vec{r} = \int_1^0 [(2y^3 - y^4)\,2y\,dy - (y^4 - y^2)dy]$$

$$= \int_1^0 [3y^4 - 2y^5 + y^2\,dy] = -3/5$$

$$\int_c \vec{F}\cdot d\vec{r} = -\frac{3}{5} \quad \text{--- (1)}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \\ 2xy - x^2 & -(x^2-y^2) & \end{vmatrix}$$

$$= -4x\vec{k}$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^1\int_{y^2}^{\sqrt{y}} (-4x\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= -4\int_0^1 \left(\frac{x^2}{2}\right)_{y^2}^{\sqrt{y}} dy$$

$$= -2\int_0^1 (y-y^4)\,dy = -2\left[\frac{y^2}{2}-\frac{y^5}{5}\right]_0^1$$

$$= -\frac{3}{5} \quad \text{--- (2)}$$

(1) = (2)

Hence stoke's theorem is verified.

b) (ii) Evaluate $\int_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ by using stoke's theorem, where c is the boundary of the rectangle defined by $0 < x < \pi,\ 0 \le y \le 1,\ z = 3$.

Soln:

Stoke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\vec{F} = \sin z\,\vec{i} - \cos x\,\vec{j} + \sin y\,\vec{k}.$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\,\vec{i} + \cos z\,\vec{j} + \sin x\,\vec{k}$$

$$\hat{n} = \vec{k}.$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^\pi \sin x\,dx\,dy = \int_0^1 (-\cos x)_0^\pi\,dy$$

$$= 2.$$

12) a) (i) Verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally when $u+iv = z^3$.

Soln: $u+iv = z^3 = (x+iy)^3 = (x^3-3xy^2) + i(3x^2y - y^3)$

$$u = x^3 - 3xy^2 = c_1$$

$$3x^2 - 3\left[x\,2y\,\frac{dy}{dx} + y^2\right] = 0 \implies 2xy\frac{dy}{dx} = x^2 - y^2$$

$$m_1 = \frac{dy}{dx} = \frac{x^2-y^2}{2xy}$$

$v = 3x^2y - y^3 = c_2$

$$3\left[x^2\frac{dy}{dx} + y\,2x\right] - 3y^2\frac{dy}{dx} = 0.$$

$$m_2 = \frac{dy}{dx} = -\frac{2xy}{x^2-y^2}$$

$$m_1 \times m_2 = \frac{x^2-y^2}{2xy} \times -\frac{2xy}{x^2-y^2} = -1$$

$\therefore$ $u = c_1$ and $v = c_2$ cut orthogonally if $u+iv = z^2$.

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a) (ii) Find the analytic fn $u+iv$, if $u = (x-y)(x^2+4xy+y^2)$. Also find the conjucate harmonic fn v.

Soln:

$u = (x-y)(x^2+4xy+y^2)$

$$\frac{\partial u}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)(1)$$

$$\varphi_1(z,0) = \frac{\partial u(z,0)}{\partial x} = 2z^2 + z^2 = 3z^2$$

$$\frac{\partial u}{\partial y} = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$$

$$\varphi_2(z,0) = \frac{\partial u(z,0)}{\partial y} = 4z^2 - z^2 = 3z^2$$

By Milne's method $f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$

$$= 3z^2 - i3z^2 = (1-i)3z^2$$

$$f(z) = (1-i)3\int z^2 dz = 3(1-i)\,z^3/3$$

$$f(z) = (1-i)z^3$$

$$= (1-i)(x+iy)^3$$

$$= (1-i)(x^3 - iy^3 + 3ix^2y - 3y^2x)$$

$$= (x^3 - 3xy^2 + 3x^2y - y^3) + i(3x^2y - y^3 - x^3 + 3y^2x)$$

$v = 3x^2y - y^3 - x^3 + 3xy^2$

3) b)(i) Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$.

Soln:

$$w=\frac{1}{z} \Rightarrow z=\frac{1}{w}.$$

The eqn of the circle $|z-1|=1 \Rightarrow |x+iy-1|=1$

$$(x-1)^2+y^2=1$$

$$x^2-2x+y^2=0 \quad \text{——} (1)$$

Now $w=u+iv$, $z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$\therefore x=\frac{u}{u^2+v^2}, \quad y=\frac{-v}{u^2+v^2}$$

$$(1) \Rightarrow \left(\frac{u}{u^2+v^2}\right)^2-2\left(\frac{u}{u^2+v^2}\right)+\left(\frac{v}{u^2+v^2}\right)^2=0$$

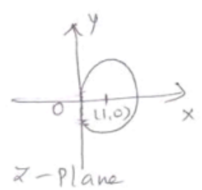
$$u^2-2u(u^2+v^2)+v^2=0$$

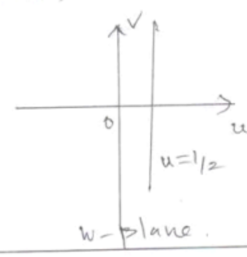
$$(u^2+v^2)(1-2u)=0 \qquad [u^2+v^2\neq 0]$$

$$1-2u=0$$

$$u=1/2$$

which is a straight line in w-plane.





b) (ii) when the fn $f(z)=u+iv$ is analytic, Prove that the curves $u=$ constant and $v=$ constant are orthogonal.

Proof:

when $u_x=v_y$ and $u_y=-v_x$, $f(z)$ is analytic

Consider $u(x,y) = c_1$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

The ~~slope~~ slope of the first curve

$$m_1 = \frac{-\partial u/\partial x}{\partial u/\partial y} = -\frac{u_x}{u_y}$$

Consider $v(x,y) = c_2$

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\partial v/\partial y}$$

The slope of the second curve is

$$m_2 = \frac{-\partial v/\partial x}{\partial v/\partial y} = -\frac{v_x}{v_y}$$

Since $f(z) = u+iv$ is analytic, $u_x = v_y$, $u_y = -v_x$

$$\therefore m_2 = -\frac{v_x}{v_y} = \frac{u_y}{u_x}.$$

$$\therefore m_1 m_2 = -\frac{u_x}{u_y} \cdot \frac{u_y}{u_x} = -1 = \text{Product of the slopes}$$

Hence the curves cut each orthogonally. The two families are said to be orthogonal trajectories of one another.

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4) a) (i) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where C is $|z-2| = \frac{1}{2}$ by using Cauchy's integral formula.

Soln: $f(z) = \frac{z}{(z-1)(z-2)^2}$

Here $z=1$ is a pole of order 1 (outside $|z-2| = 1/2$)

$z=2$ is a pole of order 2 (Inside $|z-2| = 1/2$)

$$\frac{z}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

Put $z=1$, $\boxed{A=1}$

$z=2$, $\boxed{C=2}$

Equating coeff of z^2, $0=A+B$

$\boxed{B=-1}$

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2}\,dz = \int \frac{dz}{z-1} - \int \frac{dz}{z-2} + \int \frac{2}{(z-2)^2}\,dz$$

$$= 0 - 2\pi i\, f(2) + 2 \times 2\pi i\, f'(2)$$

$$= -2\pi i.$$

Q) (ii) Evaluate $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for the regions $|z|>3$ and $1<|z|<3$.

Soln:

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

Put $z=-1$, $\boxed{A = 1/2}$

$z=-3$, $\boxed{B=-1/2}$

(i) Given $|z|>3$ (ie) $3<|z| \Rightarrow \left|\frac{3}{z}\right|<1$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2z(1+1/z)} - \frac{1}{2z(1+3/z)}$$

$$= \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{2z}\left[1+\frac{3}{z}\right]^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right] - \frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2+\cdots\right]$$

(ii) when $1<|z|<3$

$1<|z|$ and $|z|<3$

$\frac{1}{|z|}<1$ and $\left|\frac{z}{3}\right|<1$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2z(1+1/z)} - \frac{1}{6}\cdot\frac{1}{(1+z/3)}$$

$$= \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1+z/3\right]^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

(OR)

14) b (i) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Soln: Refer to Avichenna – May/June 2010 Q.No 14.b.(ii).

14) b (ii) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^3}$, $a>0$ using contour integration.

Soln: By Cauchy's residue theorem

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz.$$

The poles of $\phi(z)$ are the soln of $(z^2+a^2)^3 = 0$

$$[(z+ai)(z-ai)]^3 = 0$$

$z = ai$ and $z = -ai$ are the poles of order 3.

The pole $z = ai$ lies in the upper half plane

$$\operatorname{Res} \phi(z)\Big|_{z=ai} = \lim_{z\to ai} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-ai)^3 \cdot \frac{1}{(z-ai)^3(z+ai)^3}\right]$$

$$= \frac{1}{2}\lim_{z\to ai}\frac{d}{dz}\left[\frac{-3}{(z+ai)^4}\right]$$

$$= -\frac{3}{2}\lim_{z\to ai}\frac{d}{dz}\left[(z+ai)^{-4}\right]$$

$$= \frac{-3}{2}\lim_{z\to ai}(-4)(z+ai)^{-5}$$

$$= 6(2ai)^{-5} = \frac{6}{2^5 a^5 i^5} = \frac{3}{16a^5 i}$$

(8)

In $R \to \infty$, $\int_\Gamma \phi(z)\,dz = 0$

$$\int_C \phi(z)\,dz = 2\pi i\left[\frac{3}{16a^5 i}\right] = \frac{3\pi}{8a^5}.$$

$$\therefore \int_{-\infty}^{\infty} \phi(x)\,dx = \frac{3\pi}{8a^5} \Rightarrow 2\int_0^{\infty} \phi(x)\,dx = \frac{3\pi}{8a^5}$$

$$\int_0^{\infty} \phi(x)\,dx = \int_0^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{16a^5}.$$

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15) a (i) Using convolution theorem find the inverse Laplace transform of $\frac{1}{(s^2+1)(s+1)}$.

Soln:

$$L^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right] = L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{t} * \sin t = \int_0^t e^{-u}\sin(t-u)\,du$$

$$= \left[\frac{e^{-u}}{2}\left((-1)\sin(t-u) + \cos(t-u)\right)\right]_0^t$$

$$= \frac{e^{-t}}{2} + \frac{\sin t - \cos t}{2} = \frac{1}{2}\left[e^{-t} + \sin t - \cos t\right)$$

(ii) Find the Laplace transformation of $f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$ with $f(t+2a) = f(t)$.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\,dt + \int_a^{2a} (2a-t)\,e^{-st}\,dt\right)$$

$$= \frac{1}{1-e^{-2as}}\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) + \left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right] = \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right).$$

15] b)

(i) Find the Laplace transform of square wave fn defined by

$$f(t) = \begin{cases} 1, & 0<t<a \\ -1, & a<t<2a \end{cases}$$ with period $2a$.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right)$$

$$= \frac{1}{1-e^{-2as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{1}{s}\left[\frac{1-2e^{-as}+e^{-2as}}{1-e^{-2as}}\right] = \frac{1}{s}\frac{(1-e^{-sa})^2}{(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s}\cdot\frac{1-e^{-sa}}{1+e^{-as}} = \frac{1}{s}\tanh\left(\frac{sa}{2}\right)$$

15] b

(ii) Solve the diff eqn $\frac{d^2y}{dt^2}+y = \sin 2t$, $y(0)=0$, $y'(0)$ by using Laplace transform method.

Soln:

$$L\left[\frac{d^2y}{dt^2}\right] + L[y] = L[\sin 2t]$$

$$s^2L[y(t)] - sy(0) - y'(0) + L[y(t)] = \frac{2}{s^2+4}$$

(9)

$$s^2 L[y(t)] + L[y(t)] = \frac{2}{s^2+4}$$

$$L[y(t)]\,(s^2+1) = \frac{2}{(s^2+4)}$$

$$y(t) = L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right]$$

Consider $\dfrac{2}{(s^2+1)(s^2+4)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4}$.

$$2 = A(s^2+4) + B(s^2+1)$$

Put $s^2 = -4 \Rightarrow B = -2/3$

$s^2 = -1 \Rightarrow A = 2/3$

$$L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{\frac{2}{3}}{s^2+1}\right] - L^{-1}\left[\frac{2/3}{s^2+4}\right]$$

$$= \frac{2}{3}\sin t - \frac{2}{3}\cdot\frac{\sin 2t}{2}$$

$$= \frac{1}{3}[2\sin t - \sin 2t].$$