Anna University May / June 2009

(Regulation 2008)

MA 2161 – MATHEMATICS II

Part A

1) Find the P.I of $(D^2+2D+1)y = e^{-x}\cos x$

Soln. $P.I = \frac{1}{D^2+2D+1} e^{-x}\cos x = \frac{1}{(D+1)^2} e^{-x}\cos x = e^{-x}.\frac{1}{D^2}\cos x$

$= e^{-x}.\frac{1}{D}\sin x = -e^{-x}\cos x$.

2) Solve the eqn $x^2y'' - xy' + y = 0$.

Soln. Put $x = e^z$, $z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$

$\therefore (D'^2 - 2D' + 1)y = 0$.

A.E is $(m-1)^2 = 0$.

$\therefore y = (Az+B)e^z = (A\log x + B)x$.

3) Find the values of a, b, c so that the vector $\vec{F} = (x+y+az)\vec{i}$ $+ (bx+2y-z)\vec{j} + (-x+cy+2z)\vec{k}$ may be irrotational.

soln: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix} = 0$

$\vec{i}(c+1) - \vec{j}(-1-a) + \vec{k}(b-1) = 0$.

$a = -1, b = 1, c = -1$.

4) State Green's theorem in a plane.

If C is a simple closed curve enclosing a region R in the xy plane and P(x, y), Q(x, y) and their first order partial derivatives are continuous in R, then

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

5) state the C-R eqn in polar coordinates satisfied by an analytic fn:

If the fn $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z plane then (i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exists and satisfy the C-R eqns

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6) Find the invariant pts of the transformation $w = \frac{2z+6}{z+7}$

Soln:

$$z = \frac{2z+6}{z+7}$$

$$z^2 + 7z = 2z + 6$$

$$(z+6)(z-1) = 0.$$

$\therefore$ The invariant pts are $z = -6$ & $z = 1$.

7) Evaluate $\int_C \tan z \, dz$ where c is $|z| = 2$.

$$\int_{|z|=2} \tan z \, dz = \int_{|z|=2} \frac{\sin z}{\cos z} dz$$

The singularities of $\tan z$ that lie inside $|z| = 2$ are $z = \pm \pi/2$

$$\text{Res at } z = \pm \pi/2 = \lim_{z \to \pm\pi/2} \left[\frac{\sin z}{-\sin z} \right] = -1$$

By Cauchy's residue thm

$$\int_{|z|=2} \tan z \, dz = 2\pi i (R_1 + R_2) = 2\pi i(-1-1) = -4\pi i)$$

8) Find the Taylor series for $f(z) = \sin z$ about $z = \pi/4$.

Soln:

$f(z) = \sin z$ | $f(\pi/4) = 1/\sqrt{2}$

$f'(z) = \cos z$ | $f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$ | $f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$f'''(\pi/4) = -\frac{1}{\sqrt{2}}$

Taylor series of $f(z)$ about $z = \pi/4$ is

$$\sin z = \frac{1}{\sqrt{2}}\left[1 + \frac{1}{1!}(z - \pi/4) - \frac{1}{2!}(z - \pi/4)^2 - \frac{1}{3!}(z - \pi/4)^3 + \cdots\right]$$

9) Find the $L\left[\frac{1-\cos t}{t}\right]$.

Soln:

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L(1-\cos t)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty = \frac{1}{2}\log\frac{s^2+1}{s^2}.$$

10) Find $L^{-1}\left[\cot^{-1}(k/s)\right]$.

Soln:

$$L^{-1}\left[\cot^{-1}(k/s)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(k/s)\right] = -\frac{1}{t}\sin kt.$$

Part - B

1) Solve the eqn $(D^2+4)y = x^2\cos 2x$.

Soln: $(D^2+4)y = 0$.

Put $D = m$, The A.E is $m^2+4 = 0 \Rightarrow m = \pm 2i$

$$C.F = C_1\cos 2x + C_2\sin 2x.$$

To find P.I

$$P.I = \frac{1}{D^2+4}x^2\cos 2x.$$

$= R.P \text{ of } \frac{1}{D^2+4} x^2 e^{i2x}$

$= R.P \text{ of } e^{i2x} \cdot \frac{1}{(D+2i)^2+4} x^2$

$= R.P \text{ of } e^{i2x} \cdot \frac{1}{D^2+4Di-4+4} x^2$

$= R.P \text{ of } e^{i2x} \frac{1}{4Di\left[1+D^2/4iD\right]} x^2$

$= R.P \text{ of } e^{i2x} \frac{1}{4Di}\left[1-\frac{iD}{4}\right]^{-1} x^2$

$= R.P \text{ of } e^{i2x} \frac{1}{4i}\left[\frac{1}{D}+\frac{i}{4}-\frac{D}{16}-\frac{iD^2}{64}\right] x^2$

$= R.P \text{ of } \left(-\frac{ie^{i2x}}{4}\right)\left[\frac{1}{D}(x^2)+\frac{i}{4}(x^2)-\frac{D}{16}(x^2)-\frac{i}{64}D^2(x^2)\right]$

$= R.P \text{ of } \left(\frac{\sin 2x - i\cos 2x}{4}\right)\left(\frac{x^3}{3}-\frac{ix^2}{4}-\frac{2x}{16}-\frac{ix}{64}\right)$

$= \frac{\sin 2x}{4}\left(\frac{x^3}{3}-\frac{x}{8}\right)+\frac{\cos 2x}{16}\left(x^2-\frac{1}{8}\right)$

$y = C_1\cos 2x + C_2\sin 2x + \frac{\sin 2x}{4}\left(\frac{x^3}{3}-\frac{x}{8}\right)+\frac{\cos 2x}{16}\left(x^2-\frac{1}{8}\right)$

(ii) Solve the eqn $(D^2+a^2)y = \tan ax$ by the method of Variation of parameter.

Soln:

$(D^2+a^2)y = 0$

Put $D=m$, The A.E is $m^2+a^2=0 \Rightarrow m=\pm ai$

$C.F = C_1\cos ax + C_2\sin ax$

To find P.I

$P.I = Pf_1 + Qf_2$

③

$f_1 = \cos ax$ | $f_2 = \sin ax$

$f_1' = -a\sin ax$ | $f_2' = a\cos ax$

$f_1 f_2' - f_2 f_1' = a$.

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_2 f_1'}\,dx = -\int \frac{\sin ax \tan ax}{a}\,dx = -\int \frac{\sin^2 ax}{a\cos ax}\,dx$$

$$= \frac{-1}{a}\int \frac{(1-\cos^2 ax)}{\cos ax}\,dx = -\frac{1}{a}\int(\sec ax - \cos ax)\,dx$$

$$= \frac{1}{a^2}[\sin ax - \log(\sec ax + \tan ax)]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx = \int \frac{\cos ax \cdot \tan ax}{a}\,dx = \frac{1}{a}\int \sin ax\,dx$$

$$= -\frac{\cos ax}{a^2}$$

$P.I = Pf_1 + Qf_2$

$\therefore y = C.F + P.I$

$$= c_1\cos ax + c_2 \sin ax + \frac{1}{a^2}(\sin ax - \log(\sec ax + \tan ax))\cos ax$$

$$-\frac{1}{a^2}\cos ax \sin ax.$$

b) (i) Solve the eqn $(x^2D^2 + 3xD + 5)y = x\cos(\log x)$.

Soln:

Put $x = e^z$ (or) $z = \log x$

$xD = D'$, $x^2D^2 = D'(D'-1)$

$[D'(D'-1)]y + 3D'y + 5y = e^z\cos z$.

$(D'^2 + 2D' + 5)y = e^z\cos z$.

Put $D' = m$. $m^2 + 2m + 5 = 0$.

$m = -1 \pm 2i$

$C.F = e^{-z}[c_1\cos 2z + c_2\sin 2z]$

$$P.I = \frac{1}{D'^2+2D'+5} e^{z} \cos z$$

$$= e^{z} . \frac{1}{(D'+1)^2+2(D'+1)+5} \cos z$$

$$= e^{z} . \frac{1}{D'^2+4D'+8} \cos z = e^{z} . \frac{4D'-7}{(4D'+7)(4D'-7)} \cos z$$

$$= e^{z} \frac{4D'-7}{16(-1)-49} \cos z = -\frac{e^{z}}{65}\left[4 \frac{d}{dz}(\cos z) - 7\cos z\right]$$

$$= \frac{e^{z}}{65}\left[4\sin z + 7\cos z\right]$$

$$y = e^{-z}\left[c_1 \cos 2z + c_2 \sin 2z\right] + \frac{e^{z}}{65}(4\sin z + 7\cos z)$$

(ii) Solve $\frac{dx}{dt} + y = \sin t$, $x + \frac{dy}{dt} = \cos t$ given that $x = 2$ and $y = 0$ at t=

Soln:

$Dx + y = \sin t$ —(1), $x + Dy = \cos t$ —(2)

(1) $\times D \Rightarrow D^2 x + Dy = \cos t$

(2) $\Rightarrow x + Dy = \cos t$

$(D^2 - 1)x = 0$.

To find C.F: put $D = m$.

A.E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$C.F = c_1 e^{t} + c_2 e^{-t}$

$P.I = 0$. $\therefore x = c_1 e^{t} + c_2 e^{-t}$

$\frac{dx}{dt} = c_1 e^{t} - c_2 e^{-t}$

(1) $\Rightarrow c_1 e^{t} - c_2 e^{-t} + y = \sin t \Rightarrow y = \sin t - c_1 e^{t} + c_2 e^{-t}$

At $t = 0$, $x = 2$, $y = 0$.

(3) $\Rightarrow c_1 + c_2 = 2$ | (3) $\Rightarrow c_1 + 1 = 2$

$-c_1 + c_2 = 0$ $\boxed{c_2 = 1}$ | $c_1 = 1$.

$2c_2 = 2$

$\therefore x = e^t + e^{-t}, \; y = \sin t - e^t + e^{-t}$

12(a) (i) Find the angle between the normals to the surface $xy^3z^2 = 4$ at the points $(-1,-1,2)$ and $(4,1,-1)$.

Soln:

$\phi = xy^3z^2 - 4$

$\nabla\phi = \vec{i}\, y^3z^2 + \vec{j}\,(3xy^2z^2) + \vec{k}\,(2xy^3z)$

At $(-1,-1,2)$: $\nabla\phi_1 = -4\vec{i} - 12\vec{j} + 4\vec{k}$

At $(4,1,-1)$: $\nabla\phi_2 = \vec{i} + 12\vec{j} - 8\vec{k}$

Angle b/w the normals $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$

$= \dfrac{(-4\vec{i} - 12\vec{j} + 4\vec{k}) \cdot (\vec{i} + 12\vec{j} - 8\vec{k})}{\sqrt{16+144+16}\;\sqrt{1+144+64}}$

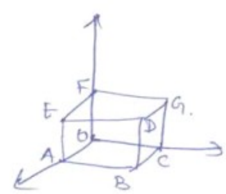
$= \dfrac{-180}{\sqrt{176}\,\sqrt{209}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-180}{\sqrt{176}\,\sqrt{209}}\right)$.

(ii) Verify stokes theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$ where S is open surface of the rectangular parallelopiped formed by the plane $x=0, x=1, y=0, y=2, z=0$ and $z=3$ above the xy plane.

Soln:

Stoke's theorem states that

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla\times\vec{F} \cdot \hat{n}\, ds$$



L.H.S

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

Along OA, $y=0$, $dy=0$, $x=0$, $dz=0$.

$$\int_{OA} \vec{F}\cdot d\vec{r} = 0$$

Along AB, $z=0$, $x=1$, $dz=0$, $dx=0$, $\int_{AB} \vec{F}\cdot d\vec{r} = 0$

Along BC, $y=2$, $z=0$, $\int_{BC} \vec{F}\cdot d\vec{r} = \int_1^0 2x\,dx = \left(2x^2/2\right)_1^0 = -1$

Along CO, $x=0, z=0$, $\int_{CO} \vec{F}\cdot d\vec{r} = 0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -1 \quad \text{--- (1)}$$

R.H.S

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\ \hat{n}=-i}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{x=1\\ \hat{n}=\vec{i}}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \nabla\times\vec{F}\cdot\hat{n}\,ds + \iint_{\substack{y=2\\ \hat{n}=\vec{j}}} \nabla\times\vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{\substack{z=3\\ \hat{n}=\vec{k}}} \nabla\times\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^3\int_0^2 -2y\,dy\,dz + \int_0^3\int_0^2 2y\,dy\,dz + \int_0^3\int_0^1 -z\,dx\,dz + \int_0^3\int_0^1 z\,dx\,dz + \int_0^2\int_0^1 -x\,dx\,d$$

$$= -1 \quad \text{--- (2)}$$

(1) = (2)

Hence stoke's theorem is verified.

b)

(i) Find the directional derivative of $\varphi = 2xy + z^2$ at the point $(1,-1,3)$ in the direction of $\vec{i}+2\vec{j}+2\vec{k}$.

Soln:

$$D.D\ \hat{n} = \nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|}$$

$\nabla\phi = 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$, $(\nabla\phi)_{(1,-1,3)} = -2\vec{i}+2\vec{j}+6\vec{k}$

$|\vec{a}| = \sqrt{1+4+4} = 3$

$$D.D\ \hat{n} = (-2\vec{i}+2\vec{j}+6\vec{k})\cdot\left(\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right) = \frac{14}{3}$$

(5)

(ii) Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ where S is the surface of the cuboid formed by the plane $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

Soln:

Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

L.H.S

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{\substack{x=0\\ \hat{n}=-\vec{i}}} \vec{F}\cdot\hat{n}\,dydz + \iint_{\substack{x=a\\ \hat{n}=\vec{i}}} \vec{F}\cdot\hat{n}\,dydz + \iint_{\substack{y=0\\ \hat{n}=-\vec{j}}} \vec{F}\cdot\hat{n}\,dxdz$$

$$+\iint_{\substack{y=b\\ \hat{n}=\vec{j}}} \vec{F}\cdot\hat{n}\,dxdz + \iint_{\substack{z=0\\ \hat{n}=-\vec{k}}} \vec{F}\cdot\hat{n}\,dxdy + \iint_{\substack{z=c\\ \hat{n}=\vec{k}}} \vec{F}\cdot\hat{n}\,dxdy$$

$$= 0 + \int_0^c\int_0^b a^2\,dydz + 0 + \int_0^c\int_0^a b^2\,dxdz + 0 + \int_0^b\int_0^a c^2\,dxdy$$

$$= a^2bc + ab^2c + ac^2b$$

$$= abc(a+b+c) \quad \text{——} \ (1)$$

R.H.S

$$\nabla\cdot\vec{F} = 2x+2y+2z$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^a\int_0^b\int_0^c (2x+2y+2z)\,dz\,dy\,dx$$

$$= \int_0^a\int_0^b \left(2xz + 2yz + 2z^2/2\right)_0^c dy\,dx$$

$$= \int_0^a\int_0^b (2xc + 2yc + c^2)\,dy\,dx$$

$$= \int_0^a \left(2xyc + \frac{2y^2}{2}c + c^2y\right)_0^b dx = \int_0^a (2xbc + b^2c + c^2b)\,dx$$

$$= abc(a+b+c) \quad \text{——} \ (2)$$

$(1) = (2)$

Hence Gauss divergence theorem is verified.

13) a)
(i) Find the analytic fn $f(z) = P+iQ$ if $P-Q = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Soln:

$$f(z) = P+iQ \quad \text{---} \; ①$$
$$if(z) = iP - Q \quad \text{---} \; ②$$
$$① + ② \Rightarrow (P-Q) + i(P+Q) = (1+i) f(z)$$
$$U + iV = F(z)$$

$U = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$U_x = \frac{(2\cos 2x)(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$U_y = \frac{-\sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$U_x(z,0) = \frac{(2\cos 2z)(1-\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}, \quad U_y(z,0) = 0.$$

$$\therefore f'(z) = U_x(z,0) - iU_y(z,0)$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2} = \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$f'(z) = \frac{-2}{1-\cos 2z} = \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z.$$

$$f(z) = -\int \operatorname{cosec}^2 z \, dz \Rightarrow \boxed{f(z) = \cot z + C}$$

13) a)
(ii) Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Soln: Bilinear transformation

$$\frac{(w-w_1)}{(w_1-w_2)} \cdot \frac{(w_2-w_3)}{(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)} \cdot \frac{(z_2-z_3)}{(z_3-z)}$$

$$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)\ (-i+1)}{(0-i)\ (1+z)}$$

$$\frac{w-i}{w} = \frac{z(1-i)^2}{(z+1)(-i)} = \frac{z-z-2iz}{-i(z+1)} = \frac{2z}{z+1}$$

$$(w-i)(z+1) = 2wz.$$

$$\therefore \boxed{w = \frac{i(1+z)}{1-z}}$$

13) b)

(i) If $f(z)$ is a regular fn of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

Soln:

Let $f(z) = u+iv$, $f'(z) = u_x + iv_x$

$|f(z)| = \sqrt{u^2+v^2}$ & $|f'(z)| = \sqrt{u_x^2+iv_x^2}$

$|f(z)|^2 = u^2+v^2$ & $|f'(z)|^2 = u_x^2+v_x^2$

L.Hs

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f'(z)|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial x^2}(v^2)+\frac{\partial^2}{\partial y^2}(u^2)+\frac{\partial^2}{\partial y^2}(v^2) \quad \text{(1)}$$

Consider $\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]$.

IIIly $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$.

$\frac{\partial^2}{\partial x^2}(v^2) = 2\left[v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$, $\frac{\partial^2}{\partial y^2}(v^2) = 2\left[v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$

(1) $\Rightarrow$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left[u\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^2 u}{\partial y^2}+v\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= 2\left[u_x^2 + u_y^2 + v_x^2 + v_y^2\right]$$

$$= 2\left[2u_x^2 + 2v_x^2\right] = 4\left[u_x^2 + v_x^2\right] = 4\,|f'(z)|^2.$$

Hence proved.

13) b
(ii) Find the image of the half plane $x > c$, when $c > 0$ under the transformation $w = \frac{1}{z}$, show the region graphically.

Soln:

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

Equating real & img parts

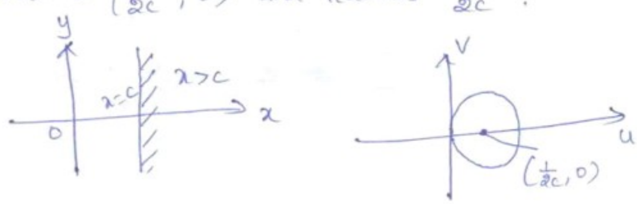
$$x = \frac{u}{u^2+v^2}$$

$\therefore$ The map of $x > c$ is $\frac{u}{u^2+v^2} > c$

$$u^2+v^2 < u/c$$

$$\left(u - \frac{1}{2c}\right)^2 + (v-0)^2 < \left(\frac{1}{2c}\right)^2$$

(ii) The interior of the circle in the uv-plane, whose centre is $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2c}$.



14) a)
(i) Evaluate $\int_c \frac{z\,dz}{(z-1)(z-2)^2}$ where c is the circle $|z-2| = \frac{1}{2}$

Using Cauchy's integral formula.

Soln:

Consider $f(z) = \frac{z}{(z-1)(z-2)^2}$

The point $z=1$ lies outside circle $|z-2| = 1/2$ and the point $z=2$ lies inside circle $|z-2| = 1/2$.

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{f(z)}{(z-2)^2}\,dz$$

Using Cauchy's integral formula, Here $a=2$

$$f(z) = \frac{z}{z-1}, \quad f'(z) = \frac{-1}{(z-1)^2}$$

$$f'(a) = f'(2) = -1$$

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{f(z)}{(z-2)^2}\,dz = 2\pi i\, f'(a) = 2\pi i\, f'(2)$$

$$= -2\pi i$$

14) a)

(ii) Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2}$ $(0<x<1)$ using contour integration.

Soln:

$z = e^{i\theta}$, $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2-1}{2iz}$.

$$I = \int_C \frac{dz/iz}{1-2x\left(\frac{z^2-1}{2iz}\right)+x^2} = \int_C \frac{dz}{iz - xz^2 + x + ix^2 z}.$$

$$= -\frac{1}{x}\int_C \frac{dz}{(z-ix)(z-i/x)}$$

The singularity of $f(z) = \frac{1}{(z-ix)(z-i/x)}$

$z = ix$ lies inside C and $z = i/x$ lies outside C.

$$\text{Res } f(z)\big|_{z=ix} = \frac{1}{i(x-1/x)} = \frac{ix}{1-x^2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2} = \frac{-1}{x} \times 2\pi i \times \frac{ix}{1-x^2} = \frac{2\pi}{1-x^2}$$

14) b)

(i) Find the Laurent's series of $f(z) = \frac{z^2-1}{z^2+5z+6}$ valid in the region $2<|z|<3$.

Soln:

$$f(z) = \frac{z^2-1}{z^2+5z+6} = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$ (by partial fraction)

$$f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1+2/z\right)^{-1} - \frac{8}{3}\left(1+z/3\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^n} - \frac{8}{3}\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^n}.$$

14) b)

(ii) Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ using contour integration where $a>b>0$.

Soln:

The singularities are $(z^2+a^2)(z^2+b^2)=0$.

(ie) $z = \pm ai$, $z = \pm bi$ which are simple poles.

$z=ai$ and $z=bi$ lie inside C.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz.$$

When $R \to \infty$, $\int_{\Gamma} f(z)\,dz \to 0$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,(\text{sum of the residue}) = \int_{-\infty}^{\infty} f(z)\,dz.$$

$$\text{Res}\, f(z)\big|_{z=ai} = \lim_{z\to ai} (z-ai)\cdot\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

|||ly

$$\text{Res}\, f(z)\big|_{z=bi} = \frac{-b}{2i(a^2-b^2)}.$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

$$= \frac{\pi}{a+b}.$$

15] a

(i) Find the Laplace Transform of $te^{-2t}\cos 3t$.

Soln:

$$L[te^{-2t}\cos 3t] = -\frac{d}{ds}L(\cos 3t)\Big|_{s\to s+2} = -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)\Big|_{s\to s+2} = \frac{s^2-9}{(s^2+9)^2}\Big|_{s\to s+2}$$

$$L[te^{-2t}\cos 3t] = \frac{(s+2)^2-9}{((s+2)^2+9)^2} = \frac{s^2+4s-5}{(s^2+4s+13)^2}.$$

15] a)

(ii) Find $L^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right]$.

Soln:

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Solving this we get $A = 1/5$, $B = 1/5$, $C = 1/5$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right] = \frac{1}{5}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{1}{s^2+4}\right)\right]$$

$$= \frac{1}{5}\left[e^{-t} - \cos 2t + \frac{\sin 2t}{2}\right)$$

15] a

(iii) Solve the eqn $y'' + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = -1$ using Laplace Transform.

Soln:

$$y'' + 9y = \cos 2t$$

$$L[y''] + 9L[y] = L(\cos 2t)$$

$s^2 L(y) - sy(0) - y'(0) + 9L(y) = \frac{s}{s^2+4}$

(ie) $(s^2+9)\bar{y} = \frac{s}{s^2+4} + s + k$ where $y'(0) = k$.

$$\bar{y} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}.$$

$$= \frac{1}{5}\left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$ by partial fractions.

$$y(t) = \frac{1}{5}(\cos 2t - \cos 3t) + \cos 3t + \sin 3t\,(k/3)$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{k}{3}\sin 3t$$

Using $y(\pi/2) = -1$, we have $-\frac{1}{5} - \frac{k}{3} = -1$

(ie) $\frac{k}{3} = \frac{4}{5}$ (or) $k = \frac{12}{15}$

$\therefore y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$.

15] b(i) Find the Laplace transform of $f(t) = \begin{cases} t & \text{in } 0 \le t \le a \\ 2a-t & \text{in } a \le t \le 2a \end{cases}$

and $f(t+2a) = f(t)$.

Soln: Refer AU Nov/Dev 2009.
Q.No. 15(b)(i).

15] b(ii) Find $L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right]$

Soln:

$$L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \left[L\int_0^t t\sin 3t\,dt\right]_{s\to s+4}.$$

Now $L\left[\int_0^t t\sin 3t\,dt\right] = \frac{1}{s}L(t\sin 3t) = \frac{1}{s}\left(-\frac{d}{ds}L(\sin 3t)\right) = \frac{-1}{s}\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6}{(s^2+9)^2}$

$$\therefore L\left[e^{-4t}\int_0^t t\sin 3t\,dt\right] = \left[\frac{6}{(s^2+9)^2}\right]_{s\to s+4} = \frac{6}{((s+4)^2+9)^2}.$$

$$= \frac{6}{(s^2+8s+25)^2} //.$$