B.E./B.Tech. Degree Examination, Chennai - April/May 2009.

Mathematics - II

Part - A

1. Solve $(D^4-1)y=0$.

Solution:

A.E. is $m^4-1=0$

$(m^2)^2-1^2=0$

$(m^2+1)(m^2-1)=0$

$m^2+1=0$, $m^2-1=0$

$\Rightarrow m^2=-1$, $m^2=1$

$m=\pm i$, $m=\pm 1$

$\therefore CF=(A\cos x+B\sin x)+Ce^{-x}+De^{x}$

& $PI=0$

$\therefore$ The solution is $y=CF+PI$

$\therefore y=A\cos x+B\sin x+Ce^{-x}+De^{x}$

2. Solve $x^2y''-20y=0$

Solution.

Given $(x^2D^2-20)y=0$

Let $x=e^z$. Then $z=\log x$

$xD=D'$ where $D'=\frac{d}{dz}$

$x^2D^2=D'(D'-1)=D'^2-D'$

$\therefore (D'^2-D'-20)y=0$.

A.E. is $m^2-m-20=0$

$(m-5)(m+4)=0$

$m=5, -4$.

$\therefore CF=Ae^{-4z}+Be^{5z}$ & $PI=0$.

$\therefore$ The solution is $y=Ae^{-4z}+Be^{5z}$

$=A(e^z)^{-4}+B(e^z)^5=Ax^{-4}+Bx^5$

$\therefore y=\frac{A}{x^4}+Bx^5$

3. If $\bar{F} = xz^3\bar{i} - 2xy\bar{j} + xz\bar{k}$, find $\text{div}\bar{F}$ and $\text{curl}\bar{F}$ at $(1,2,0)$

Solution:

Given $\bar{F} = xz^3\bar{i} - 2xy\bar{j} + xz\bar{k}$

$$\text{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(xz^3\bar{i} - 2xy\bar{j} + xz\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(xz)$$

$$= z^3 - 2x + x = z^3 - x.$$

$$\text{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2xy & xz \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(x - 3xz^2) + \bar{k}(-2y - 0)$$

$$\nabla\times\bar{F}_{\underset{x\ y\ z}{(1,2,0)}} = -4\bar{k}$$

$$\& \quad \nabla\cdot\bar{F}_{(1,2,0)} = -1.$$

4. If ϕ is a scalar point function, prove that $\text{curl}(\text{grad}\,\phi) = 0$.

Solution:

$$\text{curl}(\text{grad}\,\phi) = \nabla\times(\nabla\phi)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right)$$

$$+ \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= \bar{0}$$

5: Show that $\frac{x}{x^2+y^2}$ is harmonic.

Solution:

Let $u = \frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)^{\cancel{2}}(-2x) - (y^2-x^2)\,2(x^2+y^2)(2x)}{(x^2+y^2)^{\cancel{4}\,3}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{(x^2+y^2)^2} \times 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^{\cancel{2}}(-2x) - (-2xy)\,2(x^2+y^2) \times 2y}{(x^2+y^2)^{\cancel{4}\,3}}$$

$$= \frac{-2x(x^2+y^2) + 8xy^2}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-2x^3 - 2xy^2 - 4xy^2 + 4x^3 - 2x^3 - 2xy^2 + 8xy^2}{(x^2+y^2)^3}$$

$$= 0.$$

$\therefore u$ satisfies Laplace equation. $\therefore u$ is harmonic.

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6. Is $f(z) = z^3$ analytic?

Solution:

$f(z) = z^3 = (x+iy)^3 = x^3 + (iy)^3 + 3x^2 iy + 3x(iy)^2$

$f(z) = x^3 - iy^3 + i3x^2y - 3xy^2$

$\therefore u = x^3 - 3xy^2 \qquad v = -y^3 + 3x^2y$

$u_x = 3x^2 - 3y^2$ $\quad\quad$ $V_x = 6xy$

$u_y = -6xy$ $\quad\quad$ $V_y = 3x^2 - 3y^2$

$\therefore$ $u_x = V_y$ & $u_y = -V_x$.

$\therefore$ C-R equations are satisfied.

$\therefore$ $f(z) = z^3$ is analytic.

7. Evaluate $\int_C \sec z \, dz$ where C is the unit circle. $|z|=1$

Solution:

$$\int_C \sec z \, dz = \int_{C:|z|=1} \frac{1}{\cos z} dz$$

$\cos z = 0$

$z = (2n+1)\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$

$z = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\therefore$ All the poles lie outside $|z|=1$

$\therefore$ $\sec z$ is analytic with in $|z|=1$

$\therefore$ By Cauchy's theorem $\int_C \sec z \, dz = 0$.

8. Find the residue of $f(z) = \frac{50z}{(z+4)(z-1)^2}$ at $z=1$

Solution:

$z=1$ is a pole of order 2.

$z=-4$ is a pole of order 1.

$$\therefore \operatorname{Res} f(z)\Big]_{z=1} = \lim_{z \to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \times \frac{50z}{(z+4)\cancel{(z-1)^2}}\right]$$

$$= \lim_{z \to 1} \frac{(z+4)(50) - 50z(1)}{(z+4)^2}$$

$$= \frac{5 \times 50 - 50}{5^2} = \frac{200}{25} = 8.$$

9. State the condition for the existence of laplace transform of a function.

Solution:

The sufficient condition for the existence of the laplace transfor:

(i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$ where $a > 0$

ii) $f(t)$ should be of exponential order.

10. Find $L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+w^2}\right)\right]$

Solution:

$$L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+w^2}\right)\right] = \int_0^t L^{-1}\left(\frac{1}{s^2+w^2}\right) dt$$

$$= \int_0^t \frac{1}{w} \sin wt \, dt$$

$$= \frac{1}{w}\left[-\frac{\cos wt}{w}\right]_0^t$$

$$= \frac{1}{w^2}(1 - \cos wt)$$

Part - B

11. (a) (i) Solve the differential equation $y'' + y = \sec x$ by the method of variation of parameters.

Solution;

A.E is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$CF = A\cos x + B\sin x$, $f_1 = \cos x$, $f_2 = \sin x$

$PI = Pf_1 + Qf_2$

where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$

& $Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx.$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$

$$P = -\int \frac{\frac{\sin x}{\cancel{\cos x}} \sec x}{1} dx = -\int \frac{\sin x}{\cos x} dx = -\int \tan x \, dx$$

$$= \log(\cos x).$$

$$Q = \int \frac{\cos x \sec x}{1} dx = \int \frac{\cos x}{\cos x} dx = \int dx = x$$

$\therefore PI = \cos x . \log(\cos x) + x\sin x$

$\therefore$ The solution is

$$y = A\cos x + B\sin x + \cos x . \log(\cos x) + x\sin x.$$

(ii) Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Solution:

Given $(x^2D^2 + 4xD + 2)y = e^x$

Put $x = e^z \Rightarrow z = \log x$

$xD = D' \quad , x^2D^2 = D'(D'-1) = D'^2 - D'$

$(D'^2 - D' + 4D' + 2)y = e^{e^z}$

$(D'^2 + 3D' + 2)y = e^{e^z}$

The auxiliary eqn is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2.$

$\therefore CF = Ae^{-2z} + Be^{-z}$

$$PI = \frac{1}{D'^2 + 3D' + 2} e^{e^z} = \frac{1}{(D'+1)(D'+2)} e^{e^z}$$

$$= \frac{1}{D'+2}\left[\frac{1}{D'+1} e^{-z} . e^z e^{e^z}\right]$$

$$= \frac{1}{D'+2}\left[e^{-x} \frac{1}{D'-1+1} e^{x} e^{e^{x}} \right]$$

$$= \frac{1}{D'+2} e^{-x} \frac{1}{D'} \left(e^{x} e^{e^{x}} \right) = \frac{1}{D'+1} e^{-x} \int e^{x} e^{e^{x}} dx$$

Put $e^{x} = u$

$e^{x} dx = du$

$$= \frac{1}{D'+2} e^{-x} \int e^{u}\, du$$

$$= \frac{1}{D'+2} e^{-x} \left[e^{u} \right] = \frac{1}{D'+2} e^{-x} e^{e^{x}}$$

$$= \frac{1}{D'+2} e^{-2x} e^{x} e^{e^{x}}$$

$$= e^{-2x} \frac{1}{D'-\cancel{2}+\cancel{2}} e^{x} e^{e^{x}}$$

$$= e^{-2x} \int e^{x} e^{e^{x}} dx$$

Put $u = e^{x}$ $du = e^{x} dx$

$$= e^{-2x} \int e^{u} du = e^{-2x} e^{u} = e^{-2x} e^{e^{x}}$$

$$PI = e^{-2x} e^{e^{x}}$$

The solution is $y = CF + PI$

$$y = A e^{-2x} + B e^{-x} + e^{-2x} e^{e^{x}}$$

$$= A (e^{z})^{-2} + B (e^{x})^{-1} + (e^{z})^{-2} \cdot e^{e^{x}}$$

$$= A x^{-2} + B x^{-1} + x^{-2} \cdot e^{x}$$

$$y = \frac{A}{x^{2}} + \frac{B}{x} + \frac{e^{x}}{x^{2}}$$

(OR)

14.(b)(i) Solve the simultaneous equation

$$\frac{dx}{dt}+2x-3y=5t \quad ; \quad \frac{dy}{dt}-3x+2y=2e^{2t}$$

Solution:

Given $Dx+2x-3y=5t$ & $Dy-3x+2y=2e^{2t}$

$(D+2)x-3y=5t$ —— ① $\quad -3x+(D+2)y=2e^{2t}$ —— ②

$-3\times$ ① $\Rightarrow$ $\cancel{-3(D+2)x}+9y=-15t$

$(D+2)\times$ ② $\Rightarrow$ $\cancel{-3(D+2)x}+(D+2)^2y=(D+2)2e^{2t}$

$(+) \quad (-) \quad (-)$

$$9y-(D+2)^2y=-15t-2(D+2)e^{2t}$$

$$9y-D^2y-4Dy-4y=-15t-2\times2e^{2t}-4e^{2t}$$

$$-D^2y-4Dy+5y=-15t-8e^{2t}$$

ie, $(D^2+4D-5)y=15t+8e^{2t}$

The auxiliary eqn is $m^2+4m-5=0$.

$(m-1)(m+5)=0$

$m=1,-5$

$\therefore$ $CF=Ae^{-5t}+Be^{t}$

$$PI_1=\frac{1}{D^2+4D-5}15t$$

$$=\frac{1}{-5\left(1-\frac{4}{5}D\right)}15t$$

$$=-3\left(1-\frac{4}{5}D\right)^{-1}t \quad =-3\left(1+\frac{4}{5}D+\cdots\right)t$$

$$=-3\left(t+4/5\right).$$

$$PI_2 = \frac{1}{D^2+4D-5} 8e^{2t} = 8\frac{1}{4+8-5}e^{2t} = \frac{8}{7}e^{2t}$$

$\therefore$ The solution is $y = Ae^{-5t} + Be^{t} - 3\left(t+\frac{4}{5}\right) + \frac{8}{7}e^{2t}$

(ii) Solve $(D^4+D^3+D^2)\,y = 5x^2+\cos x$

Solution:

The auxiliary eqn is $m^4+m^3+m^2=0$

$$m^2(m^2+m+1)=0$$

$m^2=0$, $m^2+m+1=0$

$m=0,0$ $\quad m = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\therefore CF = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + (Cx+D)e^{0x}$$

$$PI_1 = \frac{1}{D^4+D^3+D^2}5x^2 = \frac{1}{D^2(1+D+D^2)}5x^2$$

$$= \frac{1}{D^2}(1+D+D^2)^{-1}5x^2$$

$$= \frac{1}{D^2}\left(1-(D+D^2)+(D+D^2)^2+\dots\right)5x^2$$

$$= \frac{1}{D^2}\left(1-D-\cancel{D^2}+\cancel{D^2}\right)5x^2$$

$$= \frac{1}{D^2}(5x^2-10x) = \frac{1}{D}\left(\frac{5x^3}{3} - \cancel{10}^{5}\frac{x^2}{\cancel{2}}\right)$$

$$PI_1 = \frac{5}{12}x^4 - \frac{5}{3}x^3$$

$$PI_2 = \frac{1}{D^4+D^3+D^2}(\cos x) \qquad D^2=-1$$

$$= \frac{1}{(-1)^2+(-1)D-1}\cos x = \frac{-1}{D}\cos x = -\sin x$$

$$\therefore y = e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + (Cx+D) + \frac{5}{12}x^4 - \frac{5}{3}x^3 - \sin x$$

12. (a) (i) Verify Green's Theorem in the xy plane for $\int_C \{(xy+y^2)\,dx + x^2\,dy\}$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

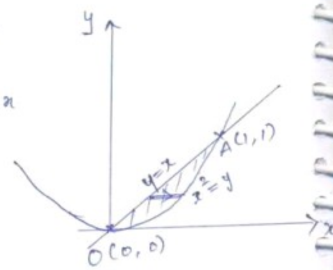
Solution:

By Green's Theorem, $\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

where $P = xy + y^2$ $\qquad Q = x^2$

$$\frac{\partial P}{\partial y} = x + 2y \qquad \frac{\partial Q}{\partial x} = 2x$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x - 2y.$$



$$RHS = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_S (x - 2y)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\int_{x=y}^{\sqrt{y}} (x - 2y)\,dx\right] dy$$

$$= \int_{y=0}^{1} \left[\frac{x^2}{2} - 2xy\right]_{x=y}^{\sqrt{y}} dy$$

$$= \int_{y=0}^{1} \left(\frac{y}{2} - 2y^{3/2}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_{y=0}^{1} \left(\frac{y}{2} - 2y^{3/2} + \frac{3y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{4} - 2\frac{y^{5/2}}{5/2} + \frac{3y^3}{3 \times 2}\right]_{y=0}^{1}$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

$$LHS = \int_C P\,dx + Q\,dy = \int_{OA} + \int_{AO} (P\,dx + Q\,dy)$$

Along OA, $y = x^2$, $dy = 2x\,dx$, $x = 0$ to 1

Along AO, $y = x$, $dy = dx$, $x = 1$ to 0.

$$\int_{OA} (xy + y^2)\,dx + x^2\,dy = \int_0^1 (x^3 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_0^1 (x^4 + 3x^3)\,dx = \left[\frac{x^5}{5} + \frac{3x^4}{4}\right]_0^1 = \frac{1}{5} + \frac{3}{4} = \frac{19}{20}$$

$$\int_{AO} (xy + y^2)\,dx + x^2\,dy = \int_{-1}^{0} 3x^2\,dx = \left(\frac{\cancel{3}x^3}{\cancel{3}}\right)_{-1}^{0} = -1$$

$$LHS = \int_C P\,dx + Q\,dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$\therefore$ LHS = RHS. & hence Green's theorem is verified.

(ii) Find $\nabla^2(r^n)$ and hence deduce $\nabla^2\left(\frac{1}{r}\right)$ where $r = |\bar{r}|$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Solution:

$$\nabla^2(r^n) = \frac{\partial^2}{\partial x^2}(r^n) + \frac{\partial^2}{\partial y^2}(r^n) + \frac{\partial^2}{\partial z^2}(r^n)$$

$$\frac{\partial}{\partial x}(r^n) = n r^{n-1} \cdot \frac{\partial r}{\partial x} = n \cdot r^{n-1} \cdot \frac{x}{r} = n r^{n-2} x$$

$$\frac{\partial^2}{\partial x^2}(r^n) = n\left((n-2) r^{n-3} \frac{\partial r}{\partial x} x + r^{n-2}(1)\right)$$

$$= n(n-2) r^{n-4} x^2 + n r^{n-2}$$

$$\frac{\partial^2}{\partial y^2}(r^n) = n(n-2) r^{n-4} y^2 + n r^{n-2}$$

$$\frac{\partial^2}{\partial z^2}(r^n) = n(n-2) r^{n-4} z^2 + n r^{n-2}$$

$\therefore \nabla^2(r^n) = n(n-2)r^{n-4}(x^2+y^2+z^2) + 3nr^{n-2}$

$= n(n-2)r^{n-4}r^2 + 3nr^{n-2}$

$= n(n-2)r^{n-2} + 3nr^{n-2}$

$= nr^{n-2}(n-2+3)$

$\nabla^2(r^n) = n(n+1)r^{n-2}$

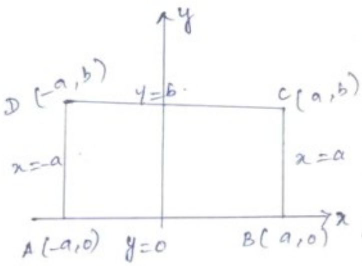
Putting $n=-1$

$\nabla^2(r^{-1}) = -1(-1+1)r^{-1-2} = 0.$

$\therefore \nabla^2\left(\frac{1}{r}\right) = 0.$

(OR)

12.(b)(i) Verify Stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.



Solution:

By Stoke's theorem,

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds.$$

$$RHS = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}\frac{\partial}{\partial z}(-2y-2y)$$

$$= -4y\bar{k}$$

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b}(-4y\bar{k}\cdot\bar{k})\,dx\,dy$$

$$= -4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx.$$

$$= -4\int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx = -\cancel{4}^{\,2}\frac{b^2}{\cancel{2}}\int_{-a}^{a} dx$$

$$= -2b^2[x]_{-a}^{a} = -2b^2(a-(-a))$$

$$RHS = -4ab^2$$

$$LHS = \int_C \bar{F}\cdot d\bar{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}(\bar{F}\cdot d\bar{r})$$

Along AB, $y=0$, $dy=0$, $x=-a \to a$

Along BC, $x=a$, $dx=0$, $y=0\to b$

Along CD, $y=b$, $dy=0$, $x=a\to -a$

Along DA, $x=-a$, $dx=0$, $y=b$ to 0.

$$\therefore \int_{AB}\bar{F}\cdot d\bar{r} = \int_{AB}(x^2+\cancel{y^2}^{\,0})\,dx - 2\cancel{xy}^{\,0}\,dy$$

$$= \int_{x=-a}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} - \frac{-a^3}{3} = \frac{2}{3}a^3.$$

$$\int_{BC}\bar{F}\cdot d\bar{r} = \int_{AB}\cancel{(x^2+y^2)\,dx}^{\,0} - 2xy\,dy = \int_{y=0}^{b} -2a\,y\,dy$$

$$= -\cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_0^b = -ab^2$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} (x^2+y^2)\,dx - 2xy\,\cancel{dy}^{\,0}$$

$$= \int_a^{-a} (x^2+b^2)\,dx$$

$$= \left(\frac{x^3}{3}+b^2x\right)_a^{-a} = \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3}+ab^2\right)$$

$$= -\frac{2}{3}a^3 - 2ab^2$$

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{DA} \cancel{(x^2+y^2)\,dx}^{\,0} - 2xy\,dy$$

$$= 2a\int_b^0 y\,dy = \cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_b^0 = -ab^2$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \cancel{\frac{2}{3}a^3} - ab^2 - \cancel{\frac{2}{3}a^3} - 2ab^2 - ab^2$$

$$= -4ab^2$$

$\therefore$ LHS = RHS.

$\therefore$ Stokes' theorem is verified.

(ii) A vector field is given by $\bar{F} = (x^2-y^2+x)\bar{i} - (2xy+y)\bar{j}$. Show that the field is irrotational and find its scalar potential. Hence evaluate the line integral from $(1,2)$ to $(2,1)$.

Solution:

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ x^2-y^2+x & -(2xy+y) & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(-2y+2y)$$

$$\nabla \times \bar{F} = \bar{0}$$

$\therefore \bar{F}$ is irrotational.

<u>To find Scalar potential</u>:

Since $\bar{F}$ is irrotational, $\bar{F} = \nabla\phi$.

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2-y^2+x)\,\bar{i} - (2xy+y)\,\bar{j}$$

$$\frac{\partial\phi}{\partial x} = x^2-y^2+x \quad\Bigg|\quad \frac{\partial\phi}{\partial y} = -2xy-y \quad\Bigg|\quad \frac{\partial\phi}{\partial z} = 0$$

$$\phi_1 = \frac{x^3}{3} - y^2x + \frac{x^2}{2} + c_1 \quad\Bigg|,\quad \phi_2 = -\frac{\cancel{2}xy^2}{\cancel{2}} - \frac{y^2}{2} + c_2 \quad\Bigg|\quad \phi_3 = c_3$$

$$\phi = \phi_1 + \phi_2 + \phi_3$$

$$\therefore \phi = \frac{x^3}{3} - y^2x + \frac{x^2}{2} - \frac{y^2}{2} + c.$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{(1,2)}^{(2,1)} \bar{F}\cdot d\bar{r} = \int_{(1,2)}^{(2,1)} \nabla\phi\cdot d\bar{r} = \int_{(1,2)}^{(2,1)} d\phi$$

$$= [\phi]_{(1,2)}^{(2,1)} = \left[\frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c\right]_{(1,2)}^{(2,1)}$$

$$= \left(\frac{8}{3} - \cancel{2} + \frac{\cancel{4}^{2}}{\cancel{2}} - \frac{1}{2} + c\right)$$

$$- \left(\frac{1}{3} - 4 + \frac{1}{2} - \frac{\cancel{4}^{2}}{\cancel{2}} + c\right)$$

$$= \frac{22}{3}$$

13. (a)(i) Find the analytic function, whose real part is $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$

Solution:

Let $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

By Milne's method, $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)\,2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$\phi_1(z,0) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z}.$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = \frac{-2\sin 2z(0)}{(1-\cos 2z)^2} = 0.$$

$$f(z) = \int \frac{-2}{1-\cos 2z}\,dz - i\int 0\,dz$$

$$= \int \frac{2}{2\sin^2 z}\,dz = \int(-\operatorname{cosec}^2 z)\,dz$$

$$f(z) = \cot z + C.$$

(iii) Find the image of the infinite strips (i) $\frac{1}{4} < y < \frac{1}{2}$, (ii) $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

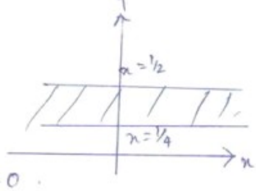
Solution:

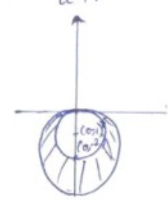
Let $z = x + iy$, $w = u + iv$.

$$w = \frac{1}{z} = \frac{1}{x+iy}$$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2} \quad \text{and} \quad x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

(i) $\frac{1}{4} < y < \frac{1}{2}$





$y = \frac{1}{4}$

$$\frac{-v}{u^2+v^2} = \frac{1}{4} \Rightarrow u^2+v^2+4v = 0.$$

centre $(0, -2)$

$r = 2$

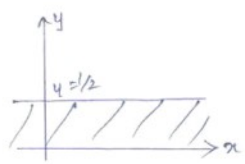
$$y = \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} = \frac{1}{2} \Rightarrow u^2+v^2+2v = 0.$$

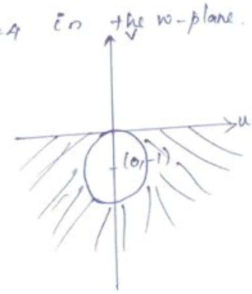
centre $(0,-1)$, $r = 1$.

$\therefore$ The infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region common to the circles

$$u^2 + (v+1)^2 = 1 \quad \text{and} \quad u^2 + (v+2)^2 = 4 \quad \text{in the } w\text{-plane}.$$

(ii) Consider $0 < y < \frac{1}{2}$





$y > 0$

$$\frac{-v}{u^2+v^2} > 0 \Rightarrow v < 0.$$

$$y = \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} = \frac{1}{2} \Rightarrow u^2+v^2+2v = 0$$

centre $(0,-1)$

$r = 1$

The infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + v^2 + 2v = 0$ in the lower half plane.

(OR)

(b) (i) Find the bilinear transformation which maps $-1, 0, 1$ of the z-plane into $-1, -i, 1$ of the w-plane. Show that under this transformation the upper half of the z-plane maps into the interior of the unit circle $|w| = 1$.

Solution:

Given $z_1 = -1$, $w_1 = -1$

$z_2 = 0$, $w_2 = -i$

$z_3 = 1$, $w_3 = 1$

The Bilinear Transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+1)(-i-1)}{(-1+i)(1-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)} = \frac{1+z}{1-z}$$

$$\frac{(w+1)}{(w-1)} \frac{(i+1)(i+1)}{-2} = \frac{1+z}{1-z}$$

$$\frac{(w+1)}{(w-1)} \frac{(\cancel{-1}+2i\cancel{+1})}{-2} = \frac{1+z}{1-z}$$

$$\frac{(w+1)}{w-1} = \frac{i+iz}{1-z}$$

$$(w+1)(1-z) = (w-1)(i+iz)$$

$$w - zw + 1 - z - wi - iwz = -i - iz$$

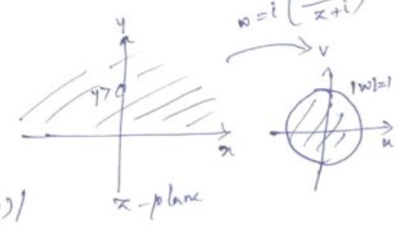
$$w(1 - z - i - iz) = -i - iz - 1 + z$$

$$w = \frac{z(1-i)-(1+i)}{-z(1+i)+(1-i)} = \frac{(1-i)\left(z-\frac{1+i}{1-i}\right)}{-(1+i)\left(z-\frac{1-i}{1+i}\right)}$$

$$w = i\left(\frac{z-i}{z+i}\right) \qquad \left(\because \frac{1+i}{1-i} = -i,\ \frac{1-i}{1+i} = i\right)$$

If $|w|<1$

then $\left|i\left(\frac{z-i}{z+i}\right)\right| < 1$



$\Rightarrow |z-i| < |z+i|$

$\Rightarrow |x+i(y-1)| < |x+i(y+1)|$

$\Rightarrow \sqrt{x^2+(y-1)^2} < \sqrt{x^2+(y+1)^2}$

$\cancel{x^2}+\cancel{y^2}+\cancel{1}-2y < \cancel{x^2}+\cancel{y^2}+2y+\cancel{1}$

$4y>0 \Rightarrow y>0.$

Thus $y>0$, ie, All the points above the real axis of z-plane are mapped onto the interior of the circle $|w|=1$.

(ii) If $f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

Solution:

Let $f(z) = u+iv$ Then $|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2$

$|f'(z)| = |u_x+iv_x| = \sqrt{u_x^2+v_x^2}$

$|f'(z)|^2 = u_x^2+v_x^2.$

$$\frac{\partial}{\partial x}(u^2+v^2) = 2uu_x + 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(u^2+v^2) = \frac{\partial}{\partial x}(2uu_x+2vv_x)$$

$$= 2u_x^2 + 2uu_{xx} + 2v_x^2 + 2vv_{xx}$$

$$\frac{\partial}{\partial y}(u^2+v^2) = 2uu_y + 2vv_y$$

$$\frac{\partial^2}{\partial y^2}(u^2+v^2) = 2u_y^2 + 2uu_{yy} + 2v_y^2 + 2vv_{yy}$$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 2\left(u_x^2 + u_y^2 + v_x^2 + v_y^2 + uu_{xx} + uu_{yy} + vv_{xx} + vv_{yy}\right)$$

$$= 2\left(u_x^2 + (-v_x)^2 + v_x^2 + (u_x)^2 + u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy})\right)$$

(from C-R eqns & ∵ u & v are harmonic)

$$= 2(2u_x^2 + 2v_x^2) + u(0) + v(0)$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

---

14. (a) (i) Show that $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

<u>Solution:</u>

Let $\displaystyle\int_0^\infty \frac{dz}{z^4+1} = \int_C \phi(z)\,dz$.

where C consists of the semicircle Γ above the real axis and the bounding diameter $[-R, R]$.

i.e, $\displaystyle\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz$.

To find the poles:

$$z^4+1=0$$
$$z^4=-1$$
$$z=(-1)^{1/4}$$
$$= \left(e^{i\pi}\right)^{\frac{1}{4}} = e^{\frac{i(2n+1)\pi}{4}} = \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}.$$

$n=0$, $z=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \frac{1}{\sqrt2}+i\frac{1}{\sqrt2} = e^{i\pi/4}$

$n=1$, $z=e^{i\frac{3\pi}{4}}$

$n=2$, $z=e^{i\frac{5\pi}{4}}$

$n=3$, $z=e^{i\frac{7\pi}{4}}$

$\therefore$ The poles are $z=e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

& The poles $z=e^{\frac{i\pi}{4}}$, $z=e^{\frac{i3\pi}{4}}$ lie inside the upper half plane

& the poles $z=e^{\frac{i5\pi}{4}}$, $z=e^{i\frac{7\pi}{4}}$ lie out side

$$\therefore \operatorname{Res}\,\phi(z)\Big]_{z=e^{i\pi/4}} = \lim_{z\to e^{i\pi/4}} \left(z-e^{\frac{i\pi}{4}}\right)\times\frac{1}{z^4+1} \quad \left(\tfrac{0}{0}\text{ form}\right.$$

$$= \lim_{z\to e^{i\pi/4}}\left[\frac{1}{4z^3}\right] \text{ by L' Hospitals Rule}$$

$$= \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}}$$

$$\operatorname{Res}\,\phi(z)\Big]_{z=i\frac{3\pi}{4}} = \lim_{z\to e^{i3\pi/4}} \frac{\left(z-e^{i\frac{3\pi}{4}}\right)}{z^4+1} \quad \left(\tfrac{0}{0}\text{ form}\right)$$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{4z^3} \quad \text{(by L'Hospitals Rule)}$$

$$= \frac{1}{4\left(e^{i\frac{3\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{9\pi}{4}}}$$

$$\therefore \int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_{\Gamma} \phi(z)\,dz.$$

when $R \to \infty$, $|z| \to \infty$ & $\phi(z) \to 0$.

$$\therefore \int_{\Gamma} \phi(z)\,dz = 0.$$

and $\int_C \phi(z)\,dz = \int_{-\infty}^{\infty} \phi(x)\,dx.$

$$\int_C \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

$$\therefore 2\int_0^{\infty} \frac{dx}{x^4+1} = \int_C \frac{dz}{z^4+1} = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i \left(\text{residue at } z = e^{i\pi/4} + \text{residue at } z = e^{i3\pi/4} \right)$$

$$\cancel{2}\int_0^{\infty} \frac{dx}{x^4+1} = \cancel{2}\pi i \left(\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right)$$

$$= \frac{\pi i}{4}\left(e^{-i3\pi/4} + e^{-i9\pi/4} \right)$$

$$= \frac{\pi i}{4}\left(\cos\frac{3\pi}{4} - i\sin 3\pi/4 + \cos 9\pi/4 - i\sin 9\pi/4 \right)$$

$$= \frac{\pi i}{4}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{\cancel{1}}{\cancel{\sqrt{2}}} - \frac{i}{\sqrt{2}}\right) = \frac{\pi i}{\cancel{4}\,2}\left(\frac{-\cancel{2}i}{\sqrt{2}}\right)$$

$$\therefore \int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

(ii) Find all possible Laurent's expansion of $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ about $z=0$. Indicate the region of convergence in each case.

Solution:

Given $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$

$$\frac{4-3z}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}.$$

$$\frac{4-3z}{\cancel{z(1-z)(2-z)}} = \frac{A(1-z)(2-z) + Bz(2-z) + C(1-z)z}{\cancel{z(1-z)(2-z)}}$$

$$4-3z = A(1-z)(2-z) + Bz(2-z) + Cz(1-z)$$

when $z=0$, $4 = 2A \Rightarrow \boxed{A=2}$

when $z=1$, $\boxed{1 = B}$.

when $z=2$, $-2 = -2C \Rightarrow \boxed{C=1}$

$$\therefore f(z) = \frac{4-3z}{z(1-z)(2-z)} = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}.$$

Case (i) $f(z) = \frac{4-3z}{z(1-z)(2-z)}$

$$= \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2(1-z/2)}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}(1-z/2)^{-1}, \ |z|<1, |z/2|<1$$

$$= \frac{2}{z} + 1 + z + z^2 + \cdots + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$$

The region of ~~integration~~ convergence is $0 < |z| < 1$.

(ii) Case (iii)

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}.$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{-z\left(1-\frac{2}{z}\right)}$$

$$= \frac{2}{z} - (1-z)^{-1} - \frac{1}{z}(1 - 2/z)^{-1}, \ \left|\frac{1}{z}\right| < 1, \ |2/z| < 1$$

$$= \frac{2}{z} - \left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \cdots\right)$$

Region of convergence

$\left|\frac{1}{z}\right| < 1$, $|2/z| < 1 \Rightarrow 2 < |z|$, $|z| > 2$.

ie, $|z| > 1$ & $|z| > 2$.

ie, $2 < |z| < \infty$.

Case (ii):

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{2}{z} + \frac{1}{-z(1-\frac{1}{z})} + \frac{1}{2(1-z/2)}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}, \quad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$$

Region of Convergence

$\left|\frac{1}{z}\right| < 1$ or $|z| > 1$

& $\left|\frac{z}{2}\right| < 1$ or $|z| < 2$

ie, $1 < |z| < 2$.

(OR)

(b) (i) Evaluate $\int_0^{2\pi} \frac{d\theta}{2-\cos\theta}$

Solution:

Put $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left[e^{i\theta}+\frac{1}{e^{i\theta}}\right] = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}.$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \int_C \frac{(dz/iz)}{2-\frac{1}{2}(z+1/z)}, \quad |z|=1$$

$$= \int_C \frac{(dz/iz)}{2-\frac{1}{2}\left(\frac{z^2+1}{z}\right)} = \int_C \frac{dz}{iz\left(\frac{4z-z^2-1}{2z}\right)}$$

$$= \frac{-2}{i}\int_C \frac{dz}{z^2-4z+1} = \frac{-2}{i}\int_C f(z)\,dz,$$

where $f(z) = \frac{1}{z^2-4z+1}$

$$\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \frac{-2}{i}\left(2\pi i \times \text{sum of the residues of } f(z)\right)$$

$z^2 - 4z + 1 = 0$.

$z = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$.

$\therefore$ $z = 2+\sqrt{3}$ lies outside $|z| = 1$

$z = 2-\sqrt{3}$ lies inside $|z| = 1$.

$\therefore$ Res of $f(z)$ at $z = 2-\sqrt{3} = \lim\limits_{z \to (2-\sqrt{3})} \cancel{z-(2-\sqrt{3})} \times \frac{1}{\cancel{z-(2-\sqrt{3})}\, z-(2+\sqrt{3})}$

$= \frac{1}{2-\sqrt{3}-2-\sqrt{3}} = \frac{1}{-2\sqrt{3}}$.

Res of $f(z)$ at $z = 2+\sqrt{3} = 0$.

$\therefore \int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \frac{-\cancel{2}}{\cancel{i}} \times 2\pi \cancel{i} \times \frac{1}{-\cancel{2}\sqrt{3}} = \frac{2}{\sqrt{3}}\pi$.

(ii) Evaluate using Cauchy's integral formula, $\int_c \frac{z^2+1}{z^2-1}\,dz$ where C is a circle of unit radius on centre at (1) $z=1$, (2) $z=-1$, (3) $z=i$

Solution:

$f(z) = \frac{z^2+1}{z^2-1}$

The poles are the roots of $z^2-1=0$.

ie, $(z+1)(z-1)=0$

$\therefore$ The poles are $z=-1$, $z=1$.

(i) when $|z-1| = 1$, the pole $z=1$ lies inside c.

$$\therefore \int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{z^2+1}{(z+1)(z-1)}\,dz$$

$$= \int_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz \quad , \quad f(z) = \frac{z^2+1}{z+1}$$

$= 2\pi i\, f(1)$ By Cauchy's integral formula.

$$= 2\pi i \times \frac{2}{2} = 2\pi i$$

(ii) when $|z+1| = 1$,

the pole $z=-1$ lies inside C.

$$\therefore \int_C \frac{z^2+1}{(z+1)(z-1)}\,dz = \int_C \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1}\,dz, \quad f(z) = \frac{z^2+1}{z-1}$$

$$\therefore f(-1) = \frac{1+1}{-1-1} = -1$$

$$= 2\pi i\, f(-1),$$

$$= 2\pi i(-1) = -2\pi i$$

(iii) when $|z-i| = 1$

Both the poles $z=1$, $z=-i$ lies outside C.

$\therefore$ By Cauchy's theorem,

$$\int_C \frac{z^2+1}{(z+1)(z-1)}\,dz = 0.$$

15.(a)(i) Show that the Laplace transformation of

$f(t) = \begin{cases} t, & 0<t<1 \\ 2-t, & 1<t<2 \end{cases}$ and $f(t+2) = f(t)$ is $\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$

Solution: The given function is periodic with period 2

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st}(2-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^1 + \left[(2-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\cancel{\frac{e^{-s}}{-s}} - \frac{e^{-s}}{s^2} + \frac{e^{-0s}}{s^2} + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \frac{(1-2e^{-s}+e^{-2s})}{s^2} = \frac{(1-e^{-s})^{\cancel{2}}}{(1+e^{-s})\cancel{(1-e^{-s})}\, s^2}$$

$$= \frac{1}{s^2} \left[\frac{\cancel{e^{-s/2}}(e^{s/2} - e^{-s/2})}{\cancel{e^{-s/2}}(e^{s/2} + e^{-s/2})} \right]$$

$$\therefore L[f(t)] = \frac{1}{s^2} \tanh(s/2)$$

(ii) Using convolution theorem, find

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$$

Solution.

By Convolution theorem,

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{s}{s^2+1}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \cos t * \frac{\sin 2t}{2}$$

$$= \frac{1}{2}\int_0^t \cos u \cdot \sin 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2t-2u+u)+\sin(2t-2u-u)}{2}\, du$$

$$= \frac{1}{4}\int_0^t \left(\sin(2t-u) + \sin(2t-3u)\right) du$$

$$= \frac{1}{4}\left[\frac{-\cos(2t-u)}{-1} + \frac{-\cos(2t-3u)}{-3}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t + \frac{\cos t}{3} - \cos 2t - \frac{\cos 2t}{3}\right.$$

$$= \frac{1}{4}\left[\frac{4}{3}\cos t - \frac{4}{3}\cos 2t\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t).$$

(OR)

(b)(i) Solve the initial value problem,

$$y'' - 3y' + 2y = 4t \quad , y(0)=1 \ , y'(0) = -1.$$

Solution:

Given $y'' - 3y' + 2y = At$

$$L[y'' - 3y' + 2y] = L[At]$$

$$L(y'') - 3L(y') + 2L(y) = A\frac{1}{s^2}.$$

$$s^2 L(y(t)) - s y(0) - y'(0) - 3(s L(y(t)) - y(0)) + 2L(y(t)) = \frac{4}{s^2}.$$

$$L(y(t))[s^2 - 3s + 2] - s + 1 + 3 = \frac{4}{s^2}$$

$$L(y(t))(s^2 - 3s + 2) = \frac{4}{s^2} + s - 4 = \frac{s^3 - 4s^2 + 4}{s^2}$$

$$L(y(t)) = \frac{s^3 - 4s^2 + 4}{s^2(s^2 - 3s + 2)}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^3 - 4s^2 + 4}{s^2(s^2 - 3s + 2)}\right]$$

Consider $\dfrac{s^3 - 4s^2 + 4}{s^2(s^2 - 3s + 2)} = \dfrac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)}$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s-1}$$

$$\Rightarrow s^3 - 4s^2 + 4 = A s(s-2)(s-1) + B(s-2)(s-1) + C s^2(s-1) + D(s-2)s^2$$

$s = 0 \Rightarrow 4 = 2B \Rightarrow B = 2$

$s = 1 \Rightarrow 1 = -D \Rightarrow D = -1$

$s = 2 \Rightarrow -4 = 4C \Rightarrow C = -1$

Equating the coefficients of s^3,

$$1 = A + C + D \Rightarrow 1 = A - 2$$

$$\Rightarrow A = 3.$$

$$\therefore y(t) = L^{-1}\left[\frac{3}{s} + \frac{2}{s^2} - \frac{1}{s-2} - \frac{1}{s-1}\right]$$

$$= L^{-1}\left[\frac{3}{s}\right] + L^{-1}\left[\frac{2}{s^2}\right] - L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{s-1}\right]$$

$$= 3(1) + 2(t) - e^{2t} - e^{t}$$

$$y(t) = 3 + 2t - e^{t} - e^{2t}$$

(i) Find $L\left[e^{-2t} + \sin 2t\right]$

Solution:

$$L\left[t\left(e^{-2t}\sin 2t\right)\right] = \frac{-d}{ds} L\left(e^{-2t}\sin 2t\right)$$

$$= \frac{-d}{ds} L(\sin 2t)_{s \to s+2}$$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+4}\right)_{s \to s+2}$$

$$= \frac{-d}{ds}\left(\frac{2}{(s+2)^2+4}\right)$$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+4s+8}\right)$$

$$= -\left(\frac{-2}{(s^2+4s+8)^2}\right) \times (2s+4)$$

$$= \frac{4(s+2)}{(s^2+4s+8)^2}$$

(iii) Find $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$

<u>Solution:</u>

Let $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = f(t)$

Then $\therefore$ $L(f(t)) = \tan^{-1}\left(\frac{2}{s^2}\right)$.

But $L(t.f(t)) = \frac{-d}{ds} L(f(t))$

$$\therefore L(t.f(t)) = -\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)$$

$$= -\cancel{\frac{d}{d}}\left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2} \times \left(\frac{-4}{s^3}\right)\right.$$

$$L(t f(t)) = \frac{4}{\cancel{s}} \times \frac{1}{\frac{s^4+4}{s^{\cancel{4}}}} = \frac{4s}{s^4+4}$$

$$t f(t) = L^{-1}\left(\frac{4s}{s^4+4}\right)$$

But $\frac{s}{s^4+4} = \frac{1}{4}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}\right]$

$$\therefore t f(t) = \frac{4}{4}\left[L^{-1}\left[\frac{1}{s^2+2s+2}\right] - L^{-1}\left[\frac{1}{s^2-2s+2}\right]\right]$$

$$= \frac{4}{4}\left\{L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s-1)^2+1}\right]\right\}$$

$$= \frac{4}{4}\left[e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t} L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$t f(t) = e^{-t}\sin t - e^{t}\sin t$$

$$f(t) = \frac{\sin t}{t}\left(e^{-t} - e^{t}\right) = -\frac{\sin t}{t}\left(e^{t} - e^{-t}\right) = -\frac{2}{t}\sin t \sinh t$$