ANNA UNIVERSITY CHENNAI

B.E./B.TECH. DEGREE EXAMINATION – January -2010.

MATHEMATICS – I

PART – A

1. Given $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$**. Find the eigenvalues of** A^2.

Since A **is a triangular matrix, the eigenvalues are** $-1, -3, 2$.

$\therefore$ The eigenvalues of A^2 are $1, 9, 4$.

2. Can $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **be diagonalized? Why?**

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ can be diagonalized, since A is symmetric and non-singular matrix.

3. Find the equation of the sphere concentric with $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ **and passing through the point** $(1, 2, 3)$.

The equation of the concentric circle is $x^2 + y^2 + z^2 - 4x + 6y - 8z + k = 0$

Since it passes through $(1,2,3)$, $1 + 4 + 9 - 4 + 12 - 24 + k = 0 \quad \Longrightarrow k = 2$

$\therefore x^2 + y^2 + z^2 - 4x + 6y - 8z + k = 0$

4. Find the equation of the cone whose vertex is the origin and guiding curve is

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1 \, , x + y + z = 1$.

The vertex of the cone is origin.

$\therefore$ The equation of the cone will be homogeneous of second degree in x, y, z.

Given $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1 \, , x + y + z = 1$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1(x + y + z)^2 \quad \Longrightarrow \mathbf{27x^2 + 32y^2 + 72xy + 72yz + 72xz = 0}$$

5. Find the curvature of the curve $2x^2 + 2y^2 + 5x - 2y + 1 = 0$.

$$2x^2 + 2y^2 + 5x - 2y + 1 = 0 \quad \Longrightarrow x^2 + y^2 + \frac{5}{2}x - y + \frac{1}{2} = 0$$

$$\therefore centre \left(-\frac{5}{4}, \frac{1}{2}\right) \quad and \; radius = \frac{\sqrt{21}}{4} \; and \; \therefore curvature = \frac{4}{\sqrt{21}}$$

6. **Find the envelope of the family of straight line $y = mx + \frac{a}{m}$, m being the parameter.**

$$y = mx + \frac{a}{m} \implies y = \frac{m^2x + a}{m} \implies m^2x - my + a = 0$$

$\therefore$ The envelope is $B^2 - 4AC = 0 \implies y^2 - 4ax = 0 \implies y^2 = 4ax.$

7. **Find $\frac{du}{dt}$ if $u = \sin\left(\frac{x}{y}\right)$, where $x = e^t, y = t^2$.**

$$Given\, u = \sin\left(\frac{x}{y}\right) x = e^t, y = t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)e^t + \cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t = \frac{1}{y}\cos\left(\frac{x}{y}\right) e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$$

8. **If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$.**

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2 - y^2}{2x^2} & \frac{y}{x} \end{vmatrix} = -\frac{y}{2x}.$$

9. **Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\, dr\, d\theta$.**

$$\int_0^{\pi}\int_0^{\sin\theta} r\, dr\, d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\pi}\frac{\sin^2\theta}{2}\, d\theta = \int_0^{\pi}\frac{1-\cos 2\theta}{4}\, d\theta = \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{\pi}{4}$$

10. **Change the order of integration for the double integral $\int_0^1\int_0^x f(x,y)\, dx\, dy$**

$$\int_0^1\int_0^x f(x,y)\, dx\, dy = \int_0^1\int_0^x f(x,y)\, dy\, dx = \int_0^1\int_y^1 f(x,y)\, dx\, dy$$

PART – B

11. **(a). (i). Verify Cayley Hamilton theorem and hence find A^{-1} for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$**

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Here $a_1 = 6 \quad a_2 = 9 \quad a_3 = 4.$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Cayley Hamilton theorem , $A^3 - 6A^2 + 9A - 4I = 0$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -1 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \quad , A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore$ $A^3 - 6A^2 + 9A - 4I = 0$ and hence Cayley Hamilton theorem is verified.

<u>To find A^{-1}:</u>

$$A^3 - 6A^2 + 9A - 4I = 0 \implies A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

(ii). Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Here $a_1 = -1 \quad a_2 = -21 \quad a_3 = 45 \quad and\ \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\therefore \lambda = -3$ is a root. And $\lambda^2 - 2\lambda - 15 = 0$

The eigenvalues are $-3, -3, 5$.

<u>To find eigen vectors:</u>

$$(A - \lambda I)X = 0 \quad \implies \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies (-2-\lambda)x + 2y - 3z = 0,\ \ 2x + (1-\lambda)y - 6z = 0\ \ , -x - 2y - \lambda z = 0\ \ --- (1)$

Case(i): When $\lambda = -3$

$x + 2y - 3z = 0,\ \ 2x + 4y - 6z = 0\ \ , -x - 2y + 3z = 0$

On solving, $X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ $and\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Case(ii): When $\lambda = 5$

$-7x + 2y - 3z = 0,\ \ 2x - 4y - 6z = 0\ \ , -x - 2y - 5z = 0$

On solving, $X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

11.(b). Reduce the Quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_1x_3 - 4x_1x_2$ to a canonical form through an orthogonal transformation and hence find rank, index, signature and also no-zero set of values for x_1, x_2, x_3(if they exist), that will make the quadratic form zero.

The matrix of the quadratic form is $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Here $a_1 = 17 \quad a_2 = 42 \quad a_3 = 0.$

The characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

The eigenvalues are $\lambda = 0, \; 3, \; 14$

<u>To find eigen vectors:</u>

$$(A - \lambda I)X = 0 \quad \Rightarrow \begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(10-\lambda)x - 2y - 5z = 0, -2x + (2-\lambda)y + 3z = 0 \;, -5x + 3y + (5-\lambda)z = 0 \quad --- (1)$

Case(i): When $\lambda = 0$

$10x - 2y - 5z = 0, -2x + 2y + 3z = 0 \;, -5x + 3y + 5z = 0$

On solving, $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

Case(ii): When $\lambda = 3$

$7x - 2y - 5z = 0, -2x - y + 3z = 0 \;, -5x + 3y + 2z = 0$

On solving, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case(iii): When $\lambda = 14$

$-4x - 2y - 5z = 0, -2x - 12y + 3z = 0 \;, -5x + 3y - 9z = 0$

On solving, $X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

The modal matrix is $M = \begin{bmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{bmatrix}$

The normalized modal matrix is $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$ $and \; N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

The quadratic form is $Q = Y(N^T AN)Y^T = (y_1 \;\; y_2 \;\; y_3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0Y_1^2 + 3y_2^2 + 14y_3^2$

$Rank = 2, \;\; Index = 2, \;\; Signature = 2, \;\; Nature = Positive\ definite$

12.(a) (i). Find the centre, radius and area of the circle given by

$$\boldsymbol{x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0, \qquad x + 2y + 2z + 7 = 0}$$

The centre of the sphere is $C(-1,1,2)$ $\;and\ radius = \sqrt{1+1+4+19} = 5$

Let $Q(x,y,z)$ be the centre of the circle. Then Q is the foot of the perpendicular from $C(-1,1,2)$ to the plane $x + 2y + 2z + 7 = 0$

The d.r's of CQ are $(x+1, y-1, z-2)$

The d.r's of the normal to the plane are 1,2,2.

Then $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$

Any point on CQ is $(r-1, 2r+1, 2r+2)$

If this point lies on $x + 2y + 2z + 7 = 0$, then

$$(r-1) + 2(2r+1) + 2(2r+2) + 7 = 0 \;\Rightarrow r = -\frac{4}{3}$$

∴ The coordinates of Q are $\left(\frac{-7}{3}, \frac{-5}{3}, \frac{-2}{3}\right)$ and length of $CQ = 4$

Radius of the circle is $QP = \sqrt{CP^2 - CQ^2} = 1 \quad and\ Area = \pi(1^2) = \pi$

(ii). Find the equation of the cone formed by rotating the line $\boldsymbol{2x + 3y = 6, z = 0}$ *about* $\boldsymbol{y - axis}$.

Given $2x + 3y = 6, \; z = 0$. And on $y - axis, x = 0\ and\ z = 0$.

The vertex is at the point $(0,2,0)$.

Then $\frac{x-0}{3} = \frac{y-2}{-2} = \frac{z-0}{0}$

Let $P(x,y,z)$ be any point on the cone. Then the d.c's of AP are $\frac{x}{\sqrt{x^2+(y-2)^2+z^2}}, \frac{y-2}{\sqrt{x^2+(y-2)^2+z^2}}, \frac{z}{\sqrt{x^2+(y-2)^2+z^2}}$

The direction cosines of y-axis are 0,1,0 .

$\cos\theta = \frac{y-2}{\sqrt{x^2+(y-2)^2+z^2}}$. But $\cos\theta = \frac{3}{\sqrt{13}} \times 0 + \frac{-2}{\sqrt{13}} \times 1 + 0 = \frac{-2}{\sqrt{13}}$

$$\therefore \frac{y-2}{\sqrt{x^2+(y-2)^2+z^2}} = \frac{-2}{\sqrt{13}} \implies 4x^2 - 9y^2 + 4z^2 + 36y - 36 = 0.$$

12 (b)(i). Find the two tangent planes to the sphere $x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0$, which are parallel to the plane $x + 4y + 8z = 0$. Find their point of contact.

Centre of the sphere is $(2,1,3)$ and radius $= 3$

The equation of the plane parallel it $x + 4y + 8z = 0$ is $x + 4y + 8z + k = 0$

Length of the $\perp^r$ from $(2,1,3)$ to the plane $x + 4y + 8z + k = 0$ is

Length of the $\perp^r = \pm\frac{2+4+24+k}{\sqrt{1+16+64}} = \pm\frac{30k}{9}$

The plane will touch the sphere is $Radius = Length\ of\ \perp^r \implies 3 = \pm\frac{30k}{9} \implies k = -3, -57$

The equation of two tangent planes: $x + 4y + 8z - 3 = 0$, $x + 4y + 8z - 57 = 0$

(ii). Find the equation of the right circular cylinder of radius 3 and axis $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$

Given radius $= 3$ and axis $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$

$PN^2 = AP^2 - AN^2 \implies 3^2 = [(x-1)^2 + (y-3)^2 + (z-5)^2$

$\implies 5x^2 + 5y^2 + 8z^2 - 8xy + 4xz + 4yz - 6x = 42y - 96z + 225 = 0$

13.(a)(i). Find the radius of curvature at $(a, 0)$ on $y^2 = \frac{a^3-x^3}{x}$.

Given $y^2 = a^3 - x^3$

Differentiating with respect to x,

$x.2y.\frac{dy}{dx} + y^2 = -3x^2 \implies \frac{dy}{dx} = -\frac{3x^2+y^2}{2xy} \implies \left(\frac{dy}{dx}\right)_{(a,0)} = \infty \therefore \left(\frac{dx}{dy}\right)_{(a,0)} = 0$

$$\frac{d^2x}{dy^2} = \frac{-2(3x^2+y^2)\left(x + y\frac{dy}{dx}\right) + 2xy(6x\frac{dy}{dx} + 2y)}{(3x^2+y^2)^2}, \quad \left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{2}{3a}$$

$$\boldsymbol{\rho = \frac{1}{\left(\frac{d^2x}{dy^2}\right)} = -\frac{3a}{2} = \frac{3a}{2}}$$

(ii). Find the equation of circle of curvature of the rectangular hyperbola

$\boldsymbol{xy = 12\ at\ (3,4)}$.

Given $xy = 12 \Rightarrow x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ and $\left(\frac{dy}{dx}\right)_{(3,4)} = -\frac{4}{3}$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y}{x^2} \Rightarrow \left(\frac{d^2x}{dy^2}\right)_{(3,4)} = \frac{8}{9}$$

$$\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''^2} = \frac{\left(1+\frac{16}{9}\right)^{\frac{3}{2}}}{\frac{8}{9}} = \frac{125}{24}$$

$$\bar{X} = x - \frac{y_1}{y_2}(1+y_1^2) = 3 - \frac{\left(-\frac{4}{3}\right)\left(1+\frac{16}{9}\right)}{\frac{8}{9}} = \frac{43}{6}$$

$$\bar{Y} = x + \frac{1}{y_2}(1+y_1^2) = 4 + \frac{\left(1+\frac{16}{9}\right)}{\frac{8}{9}} = \frac{57}{8}$$

The circle of curvature is $(x-\bar{X})^2 + (y-\bar{Y})^2 = \rho^2$

$$\left(x - \frac{43}{6}\right)^2 + \left(y - \frac{57}{8}\right)^2 = \left(\frac{125}{24}\right)^2$$

13.(b)(i). Show that the evolute of the parabola $y^2 = 4ax$ is the curve $27y^2 = 4(x-2a)^3$.

The parametric equations of the parabola $y^2 = 4ax$ are $x = at^2, y = 2at$.

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}\frac{1}{2at} = -\frac{1}{2at^3}$$

$$\bar{X} = at^2 - \frac{1/t}{-\frac{1}{2at^3}}(1+1/t^2) = at^2 + 2at^2\left(1+\frac{1}{t^2}\right) = 3at^2 + 2a - - - (1)$$

$$\bar{Y} = x + \frac{1}{-\frac{1}{2at^3}}(1+1/t^2) = 2at - 2at^3\left(1+\frac{1}{t^2}\right) = -2at^3 - - - (2)$$

The centre of curvature is $(3at^2 + 2a, -2at^3)$

Evolute of the curve is the locus of its centre of curvature.

$$From\,(1),\;\; t^2 = \frac{X-2a}{3a} \;\; => t^6 = \left(\frac{X-2a}{3a}\right)^3$$

$$From\,(2),\;\; t^3 = -\frac{Y}{2a} \;\; => t^6 = \left(-\frac{Y}{2a}\right)^2$$

$$\left(\frac{X-2a}{3a}\right)^3 = \left(-\frac{Y}{2a}\right)^2 \;\; => \;\; 4(X-2a)^3 = 27aY^2$$

$4(x-2a)^3 = 27ay^2$ is the required evolute of the parabola.

(ii) Find the envelope of the straight line $\frac{x}{a}+\frac{y}{b}=1$, where a and b are connected by the relation $ab = c^2$, c is a constant.

Given $\frac{x}{a}+\frac{y}{b} = 1$ and $ab = c^2 \;\Rightarrow b = \frac{c^2}{a}$

$\frac{x}{a}+\frac{c^2}{ab} = 1 \;\Rightarrow a^2y - ac^2 + xc^2 = 0,$ which is a quadratic equation in a.

$B^2 - 4AC = 0 \;\Rightarrow c^4 - 4yc^2x = 0 \;\Rightarrow\; 4xy = c^2,$ which is the envelope of the given curve.

14.(a)(i). ***If $z = f(x,y)$. where $x = u^2 - v^2, y = 2uv,$ prove that***

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

Solution:

Given $z = f(x,y)$

$$x = u^2 - v^2 \;\;\Rightarrow \frac{\partial x}{\partial u} = 2u \;\; and \;\; \frac{\partial x}{\partial v} = -2v$$

$$y = 2uv \;\;\Rightarrow \frac{\partial y}{\partial u} = 2v \;\; and \;\; \frac{\partial y}{\partial v} = 2u$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\,2u + \frac{\partial z}{\partial y}\,2v \quad and \; \frac{\partial}{\partial u} = \frac{\partial}{\partial x}\,2u + \frac{\partial}{\partial y}\,2v$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial}{\partial x}\,2u + \frac{\partial}{\partial y}\,2v\right)\left(\frac{\partial z}{\partial x}\,2u + \frac{\partial z}{\partial y}\,2v\right) = 4u^2\frac{\partial^2 z}{\partial x^2} + 8uv\frac{\partial^2 z}{\partial xy} + 4v^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\,(-2v) + \frac{\partial z}{\partial y}\,2u \quad and \; \frac{\partial}{\partial v} = \frac{\partial}{\partial x}\,(-2v) + \frac{\partial}{\partial y}\,2u$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(\frac{\partial}{\partial x}\,(-2v) + \frac{\partial}{\partial y}\,2u\right)\left(\frac{\partial z}{\partial x}\,(-2v) + \frac{\partial z}{\partial y}\,2u\right)$$

$$\frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial xy} + 4u^2\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \ \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (4u^2 + 4v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

(ii). Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ up to third degree terms.

Solution: Here $x = -2$, $y = 1$.

Let $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$

$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$	$f(-2,1) = 6$
$f_x = 2xy^2 + 4xy + 3y^2$	$f_x(-2,1) = -9$
$f_y = 2x^2y + 2x^2 + 6xy$	$f_y(-2,1) = 4$
$f_{xx} = 2y^2 + 4y$	$f_{xx}(-2,1) = 6$
$f_{xy} = 4x + 4x + 6y$	$f_{xy}(-2,1) = -10$
$f_{yy} = 2x^2 + 6x$	$f_{yy}(-2,1) = -4$
$f_{xxx} = 0$	
$f_{yyy} = 0$	

By Taylor's series

$$f(x,y) = f(a,b) + (x-a)f_x + (y-b)f_y + \frac{1}{2!}[(x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy} + \cdots]$$

$$f(x,y) = 6 - 9(x+2) + (y-1)4 + \frac{1}{2!}[(x+2)^2 6 - 20(x+2)(y-1) - 4(y-1)^2$$

14(b)(i). *If* $x + y + z = u, y + z = uv, z = uvw$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Given $z = uvw$, $y = uv - z = uv - uvw$, $x = u - z - y = u - uv + uvw - uvw = u - uv$

$\therefore \ z = uvw \qquad y = uv - uvw \qquad x = u - uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)(u^2v) + u(uv^2) = u^2 v$$

(ii). Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$$\frac{\partial f}{\partial x} = 3x^2 - 3 \ ; \ \frac{\partial f}{\partial y} = 3y^2 - 12 \ ; \ A = \frac{\partial^2 f}{\partial x^2} = 6x \ ; \ C = \frac{\partial^2 f}{\partial y^2} = 6y \ ; B = \frac{\partial^2 f}{\partial x \partial y} = 0$$

For maxima and minima,

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \implies x = \pm 1 \quad and \quad \frac{\partial f}{\partial y} = 3y^2 - 12 = 0 \implies y = \pm 2$$

The points at which the maximum or minimum occurs is $(1,2), (1,-2), (-1,2), (-1,-2)$.

$At\ (1,2),\quad AC - B^2 = 72 > 0 \ and\ A = 6 > 0$. ∴ The point $(1,2)$ is a minimum point.

And the minimum value $= 1$

$At\ (1,-2)$ & $(-1,2),\ AC - B^2 = -72 < 0$. ∴ The points $(-1,2)$& $(1,-2)$ are saddle points.

$At\ (-1,-2),\quad AC - B^2 = 72 > 0 \ and\ A = -6 < 0$. ∴ The point $(-1,-2)$ is a maximum point. And the maximum value $= 38$.

15(a)(i). Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dx\, dy$ using polar coordinates.

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ $and\ dx\, dy = r\, dr\, d\theta$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ r\, dr\, d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ (-2r)\, dr\, d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[e^{-r^2}\right]_0^\infty\ d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} (0-1)\, d\theta = \frac{\pi}{4}$$

(ii) Change the order of integration in the integral $\int_0^1 \int_{x^2}^{2-x} xy\ dx\, dy$.

$$\int_0^1 \int_{x^2}^{2-x} xy\ dx\, dy = \iint_{R_1} xy\ dx\, dy + \iint_{R_2} xy\ dx\, dy = \int_0^1 \int_0^{\sqrt{y}} xy\ dx\, dy + \int_1^2 \int_0^{2-y} xy\ dx\, dy$$

$$\int_0^1 \int_0^{\sqrt{y}} xy\ dx\, dy = \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy = \int_0^1 \frac{y^2}{2} dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{6}$$

$$\int_1^2\int_0^{2-y} xy\ dx\,dy = \int_1^2 \left[\frac{x^2y}{2}\right]_0^{2-y} dy = \int_1^2 \frac{(2-y)^2y}{2}\ dy = \frac{1}{2}\left[4.\frac{y^2}{2}+\frac{y^4}{4}-4\frac{y^3}{3}\right]_1^2 = \frac{5}{24}$$

$$\therefore \int_0^1\int_{x^2}^{2-x} xy\ dx\,dy = \int_0^1\int_0^{\sqrt{y}} xy\ dx\,dy + \int_1^2\int_0^{2-y} xy\ dx\,dy = \frac{1}{6}+\frac{5}{24} = \frac{3}{8}$$

15(b)(i). Find, by double integration, the area enclosed by the curves

$$\boldsymbol{y^2 = 4ax \text{ and } x^2 = 4ay}.$$

Given $y^2 = 4ax\,,\ x^2 = 4ay \ \Rightarrow\ y^4 = 16\,a^2x^2 = 16\,a^2(4ay)\ \Rightarrow y = 4a \text{ and } x = 4a$

<u>Limits:</u> $x: \frac{y^2}{4a}\ \ to\ \ 2\sqrt{ay}\qquad y: \ 0\ \ to\ 4a$

$$Area = \iint dx\ dy = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\ dy = \int_0^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\ dy = \int_0^{4a} 2\sqrt{ay}-\frac{y^2}{4a}\ dy$$

$$= \left[\frac{2ay^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\frac{y^3}{3}\right]_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

(ii). Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Volume of the ellipsoid = 8 × Volume of the positive octant.

<u>Limits:</u> $x: 0\ to\ a\quad y: 0\ \ to\ b\sqrt{1-\frac{x^2}{a^2}}\qquad z: 0\ \ to\ c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$V = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ dy\ dx = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ dy\ dx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\ dy\,dx = \ 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\ dy\,dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{y\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}}{2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}}\right)\right)_0^a dx$$

$$= \frac{8c}{b}\int_0^a \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\,\frac{\pi}{2}\,dx = 2\pi bc \int_0^a \left(1-\frac{x^2}{a^2}\right) dx = 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a = \frac{4\pi abc}{3}$$