INTEGER PROGRAMMING AND GAME THEORY

INTEGER PROGRAMMING

1. Solve the following Integer Programming Problem.

$$Max\ Z = 7x_1 + 9x_2$$

Subject to

$$-x_1 + 3x_2 \leq 6$$

$$7x_1 + x_2 \leq 35$$

$$x_1, x_2 \geq 0\ \ and\ are\ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - 7x_1 - 9x_2 + 0s_1 + 0s_2 = 0$$

Subject to

$$-x_1 + 3x_2 + s_1 = 6$$

$$7x_1 + x_2 + s_2 = 35$$

$$x_1, x_2, s_1, s_2 \geq 0\ \ and\ are\ integers.$$

Basis	z	x_1	x_2	s_1	s_2	**Solution**	**Ratio**
s_1	0	-1	3	1	0	6	2
s_2	0	7	1	0	1	35	35
$z_j - c_j$	1	-7	-9	0	0	0	

Basis	z	x_1	x_2	s_1	s_2	**Solution**	**Ratio**
x_2	0	-1/3	1	1/3	0	2	
s_2	0	22/3	0	-1/3	1	33	4.5
$z_j - c_j$	1	-10	0	3	0	18	

Basis	z	x_1	x_2	s_1	s_2	**Solution**
x_2	0	0	1	7/22	1/22	7/2
x_1	0	1	0	-1/22	3/22	9/2
$z_j - c_j$	1	0	0	28/11	15/11	63

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the values of x_1 and x_2 are not integers the solution is infeasible.

$$x_1 = 4 + \frac{1}{2}, x_2 = 3 + \frac{1}{2}$$

Since both the decimal values are equal either x_1 or x_2 row is taken for further process

Here x_2^{th} row is taken for further process

$$\frac{7}{2} = x_2 + \frac{7}{22}s_1 + \frac{1}{22}s_2$$

$$3 + \frac{1}{2} = (1+0)x_2 + \left(0 + \frac{7}{22}\right)s_1 + \left(0 + \frac{1}{22}\right)s_2$$

$$-\frac{1}{2} = -\frac{7}{22}s_1 - \frac{1}{22}s_2 + s_3$$

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution
x_2	0	0	1	7/22	1/22	0	7/2
x_1	0	1	0	-1/22	3/22	0	9/2
s_3	0	0	0	-7/22	-1/22	1	-1/2
$z_j - c_j$	1	0	0	28/11	15/11	0	63
Ratio				8	30		

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution
x_2	0	0	1	0	0	1	3
x_1	0	1	0	0	1/7	-1/7	32/7
s_1	0	0	0	1	1/7	-22/7	11/7
$z_j - c_j$	1	0	0	0	1	8	59

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the value of x_1 is not an integer the solution is infeasible.

$$x_1 = 4 + \frac{4}{7},$$

x_1^{th} row is taken for further process

$$\frac{32}{7} = x_1 + \frac{1}{7}s_2 - \frac{1}{7}s_3$$

$$4 + \frac{4}{7} = (1 + 0)x_1 + \left(0 + \frac{1}{7}\right)s_2 + \left(-1 + \frac{6}{7}\right)s_3$$

$$-\frac{4}{7} = -\frac{1}{7}s_2 - \frac{6}{7}s_3 + s_4$$

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
x_2	0	0	1	0	0	1	0	3
x_1	0	1	0	0	1/7	- 1/7	0	32/7
s_1	0	0	0	1	1/7	- 22/7	0	11/7
s_4	0	0	0	0	- 1/7	- 6/7	1	- 4/7
$z_j - c_j$	1	0	0	0	1	8	0	59
Ratio					7	9.33		

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
x_2	0	0	1	0	0	1	0	3
x_1	0	1	0	0	0	- 1	1	4
s_1	0	0	0	1	0	-4	1	1
s_2	0	0	0	0	1	6	- 7	4
$z_j - c_j$	1	0	0	0	0	2	7	55

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1 and\ x_2$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 4, x_2 = 3, Max\ Z = 55$$

2. Solve $\quad Max\ Z = x + 4y$

Subject to

$$2x + 4y \leq 7$$

$$5x + 3y \leq 15$$

where x and y are positive integers.

Solution:

The problem is rearranged as follows

$$Max\ Z - x - 4y + 0s_1 + 0s_2 = 0$$

Subject to

$$2x + 4y + s_1 = 7$$

$$5x + 3y + s_2 = 15$$

$$x, y, s_1, s_2 \geq 0 \ \ and\ are\ integers.$$

Basis	z	x	y	s_1	s_2	Solution	Ratio
s_1	0	2	4	1	0	7	1.75
s_2	0	5	3	0	1	15	5
$z_j - c_j$	1	-1	- 4	0	0	0	

Basis	z	x	y	s_1	s_2	Solution
y	0	1/2	1	1/4	0	7/4
s_2	0	7/2	0	- 3/4	1	39/4
$z_j - c_j$	1	1	0	1	0	7

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the value of y is not an integer the solution is infeasible.

$$y = 1 + \frac{3}{4}, s_2 = 9 + \frac{3}{4}$$

y^{th} row is taken for further process

$$\frac{3}{4} = \frac{1}{2}x + y + \frac{1}{4}s_1$$

$$\frac{3}{4} = \left(0 + \frac{1}{2}\right)x + (1 + 0)y + \left(0 + \frac{1}{4}\right)s_1$$

$$-\frac{3}{4} = -\frac{1}{2}x - \frac{1}{4}s_1 + s_3$$

Basis	z	x	y	s_1	s_2	s_3	Solution
y	0	1/2	1	1/4	0	0	7/4
s_2	0	7/2	0	- 3/4	1	0	39/4
s_3	0	-1/2	0	-1/4	0	1	- 3/4
$z_j - c_j$	1	1	0	1	0	0	7
Ratio		2		4			

Now solving the problem by dual simplex method, we get

Basis	z	x	y	s_1	s_2	s_3	Solution
y	0	0	1	0	0	1	1
s_2	0	0	0	-5/2	1	7	9/2
x	0	1	0	1/2	0	-2	3/2
$z_j - c_j$	1	0	0	1/2	0	3/2	11/2

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the values of x is not an integer the solution is infeasible.

$$x = 1 + \frac{1}{2}$$

$$\frac{3}{2} = x + \frac{1}{2}s_1 - 2s_3$$

$$1 + \frac{1}{2} = (1+0)x + \left(0 + \frac{1}{2}\right)s_1 + (-2+0)s_3$$

$$-\frac{1}{2} = -\frac{1}{2}s_1 + s_4$$

Basis	z	x	y	s_1	s_2	s_3	s_4	**Solution**
y	0	0	1	0	0	1	0	1
s_2	0	0	0	-5/2	1	7	0	9/2
x	0	1	0	1/2	0	-2	0	3/2
s_4	0	0	0	- 1/2	0	0	1	- 1/2
$z_j - c_j$	1	0	0	1/2	0	3/2	0	11/2
Ratio				1				

Now solving the problem by dual simplex method, we get

Basis	z	x	y	s_1	s_2	s_3	s_4	**Solution**
y	0	0	1	0	0	1	0	1
s_2	0	0	0	0	1	7	- 5/4	13/4
x	0	1	0	0	0	- 2	1	1
s_1	0	0	0	1	0	0	- 1/2	1
$z_j - c_j$	1	0	0	0	0	3/2	1	5

Since all the values in the $z_j - c_j$ row is ≥ 0 and x *and* y are integers. Therefore optimum solution is reached.

$$\therefore x = 1, y = 1, Max\ Z = 5$$

3. Solve $Max\ Z = x_1 + 2x_2$

Subject to

$$x_1 + x_2 \leq 7$$

$$2x_1 \leq 11$$

$$2x_2 \leq 7$$

$$x_1, x_2 \geq 0 \ \ and\ are\ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$

Subject to

$$x_1 + x_2 + s_1 = 7$$

$$2x_1 + s_2 = 11$$

$$2x_2 + s_3 = 7$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0\ \ and\ x_1, x_2\ are\ integers.$$

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	1	1	1	0	0	7	7
s_2	0	2	0	0	1	0	11	
s_3	0	0	2	0	0	1	7	3.5
$z_j - c_j$	1	-1	-2	0	0	0	0	

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	1	0	1	0	-1/2	7/2	3.5
s_2	0	2	0	0	1	0	11	5.5
x_2	0	0	1	0	0	1/2	7/2	
$z_j - c_j$	1	-1	0	0	0	1	7	

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution
x_1	0	1	0	1	0	-1/2	7/2
s_2	0	0	0	-2	1	1	4
x_2	0	0	1	0	0	1/2	7/2
$z_j - c_j$	1	0	0	1	0	1/2	21/2

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the values of $x_1\ and\ x_2$ are not integers the solution is infeasible.

$$x_1 = 3 + \frac{1}{2}, x_2 = 3 + \frac{1}{2}$$

Here x_2^{th} row is taken for further process

$$\frac{7}{2} = x_2 + \frac{1}{2}s_3$$

$$3 + \frac{1}{2} = (1 + 0)x_2 + \left(0 + \frac{1}{2}\right)s_3$$

$$-\frac{1}{2} = -\frac{1}{2}s_3 + s_4$$

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
x_1	0	1	0	1	0	-1/2	0	7/2
s_2	0	0	0	-2	1	1	0	4
x_2	0	0	1	0	0	1/2	0	7/2
s_4	0	0	0	0	0	- 1/2	1	- 1/2
$z_j - c_j$	1	0	0	1	0	1/2	0	21/2
Ratio						0		

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
x_1	0	0	0	1	0	0	-1	4
s_2	0	1	0	-2	1	0	2	3
x_2	0	0	1	0	0	0	1	3
s_3	0	0	0	0	0	1	-2	1
$z_j - c_j$	1	0	0	1	0	0	0	10

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1 \, and \, x_2$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 4, x_2 = 3, Max\ Z = 10$$

4. Solve the following integer programming problem using the cutting plane algorithm.

$$Max\ Z = 2x_1 + 20x_2 - 10x_3$$

Subject to the constraints

$$2x_1 + 20x_2 + 4x_3 \leq 15$$

$$6x_1 + 20x_2 + 4x_3 = 20$$

$$x_1, x_2 \ and \ x_3 \ are \ non-negative \ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - 2x_1 - 20x_2 + 10x_3 + 0s_1 + MA_1 = 0$$

Subject to

$$2x_1 + 20x_2 + 4x_3 + s_1 = 15$$

$$6x_1 + 20x_2 + 4x_3 + A_1 = 20$$

$$x_1, x_2, x_3, s_1, A_1 \geq 0 \text{ and are integers.}$$

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution
s_1	0	2	20	4	1	0	15
A_1	0	6	20	4	0	1	20
$z_j - c_j$	1	-2	- 20	10	0	M	0

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution	Ratio
s_1	0	2	20	4	1	0	15	0.75
A_1	0	6	20	4	0	1	20	1
$z_j - c_j$	1	-2-6M	- 20-20M	10-4M	0	0	-20M	

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution	Ratio
x_2	0	1/10	1	1/5	1/20	0	3/4	7.5
A_1	0	4	0	0	-1	1	5	1.25
$z_j - c_j$	1	-4M	0	14	1+M	0	15-5M	

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution
x_2	0	0	1	1/5	3/40	- 1/40	5/8
x_1	0	1	0	0	- 1/4	1/4	5/4
$z_j - c_j$	1	0	0	14	1	M	15

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_1 *and* x_2 are not integers, the solution is infeasible.

$$x_1 = \frac{5}{4} = 1 + \frac{1}{4}, x_2 = \frac{5}{8}$$

Here x_2^{th} row is taken for further process since fractional part of x_2 is greater than the fractional part of x_1.

$$\frac{5}{8} = x_2 + \frac{1}{5}x_3 + \frac{3}{40}s_1 - \frac{1}{40}A_1$$

$$\frac{5}{8} = (1+0)\, x_2 + \left(0 + \frac{1}{5}\right)x_3 + \left(0 + \frac{3}{40}\right)s_1 + \left(-1 + \frac{39}{40}\right)A_1$$

$$-\frac{5}{8} = -\frac{1}{5}x_3 - \frac{3}{40}s_1 - \frac{39}{40}A_1 + s_2$$

Basis	z	x_1	x_2	x_3	s_1	s_2	A_1	Solution
x_2	0	0	1	1/5	3/40	0	- 1/40	5/8
x_1	0	1	0	0	- 1/4	0	1/4	5/4
s_2	0	0	0	- 1/5	- 3/40	1	- 39/40	- 5/8
$z_j - c_j$	1	0	0	14	1	0	M	15
Ratio				70	13.33			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	A_1	Solution
x_2	0	0	1	0	0	1	-1	0
x_1	0	1	0	2/3	0	- 10/3	7/2	10/3
s_1	0	0	0	8/3	1	- 40/3	13	25/3
$z_j - c_j$	1	0	0	34/3	0	40/3	-13+M	20/3

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_1 is not an integer, the solution is infeasible.

$$x_1 = \frac{10}{3} = 3 + \frac{1}{3}$$

Here x_1^{th} row is taken for further process

$$\frac{10}{3} = x_1 + \frac{2}{3}x_3 - \frac{10}{3}s_2 + \frac{7}{2}A_1$$

$$3 + \frac{1}{3} = (1+0)x_1 + \left(0 + \frac{2}{3}\right)x_3 + \left(-4 + \frac{2}{3}\right)\, s_2 + \left(3 + \frac{1}{2}\right)A_1$$

$$-\frac{1}{3} = -\frac{2}{3}x_3 - \frac{2}{3}s_2 - \frac{1}{2}A_1 + s_3$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Solution
x_2	0	0	1	0	0	1	0	-1	0
x_1	0	1	0	2/3	0	- 10/3	0	7/2	10/3
s_1	0	0	0	8/3	1	- 40/3	0	13	25/3
s_3	0	0	0	- 2/3	0	- 2/3	1	- 1/2	- 1/3
$z_j - c_j$	1	0	0	34/3	0	40/3	0	-13+M	20/3
Ratio				17		20			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Solution
x_2	0	0	1	0	0	1	0	-1	0
x_1	0	1	0	0	0	- 4	1	3	3
s_1	0	0	0	0	1	- 16	4	5	7
x_3	0	0	0	1	0	1	- 3/2	3	1/2
$z_j - c_j$	1	0	0	0	0	2	17	- 22/3 +M	1

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_3 is not an integer, the solution is infeasible.

$$x_3 = \frac{1}{2}$$

Here x_1^{th} row is taken for further process

$$\frac{1}{2} = x_3 + s_2 - \frac{3}{2}s_3 + 3A_1$$

$$\frac{1}{2} = (1+0)x_3 + (1+0)\,s_2 + \left(-2 + \frac{1}{2}\right)\,s_3 + (3+0)A_1$$

$$-\frac{1}{2} = -\frac{1}{2}s_3 + s_4$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	Solution
x_2	0	0	1	0	0	1	0	0	-1	0
x_1	0	1	0	0	0	- 4	1	0	3	3
s_1	0	0	0	0	1	- 16	4	0	5	7
x_3	0	0	0	1	0	1	- 3/2	0	3	1/2
s_4	0	0	0	0	0	0	- 1/2	1	0	- 1/2
$z_j - c_j$	1	0	0	0	0	2	17	0	- 22/3 +M	1
Ratio							34			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	Solution
x_2	0	0	1	0	0	1	0	0	-1	0
x_1	0	1	0	0	0	- 4	0	2	3	2
s_1	0	0	0	0	1	- 16	0	8	5	3
x_3	0	0	0	1	0	1	0	-3	3	2
s_3	0	0	0	0	0	0	1	-2	0	1
$z_j - c_j$	1	0	0	0	0	2	0	34	- 22/3+M	- 16

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1, x_2 and\ x_3$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 2, x_2 = 0, x_3 = 2, Max\ Z = -16$$

5. Solve the following integer programming problem by cutting plane algorithm.

$$Max\ Z = x_1 + x_2$$

Subject to the constraints

$$x_1 + 2x_2 \leq 12$$

$$4x_1 + 3x_2 \leq 14$$

$$x_1\ and\ \ x_2\ are\ non-negative\ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - x_1 - x_2 + 0s_1 + 0s_2 = 0$$

Subject to

$$x_1 + 2x_2 + s_1 = 12$$

$$4x_1 + 3x_2 + s_2 = 14$$

$$x_1, x_2, s_1, s_2 \geq 0\ \ and\ x_1, x_2\ are\ integers.$$

Basis	z	x_1	x_2	s_1	s_2	**Solution**	**Ratio**
s_1	0	1	2	1	0	12	6
s_2	0	4	3	0	1	14	4.667
$z_j - c_j$	1	-1	-1	0	0	0	

Basis	z	x_1	x_2	s_1	s_2	**Solution**
s_1	0	- 5/3	0	1	- 2/3	8/3
x_2	0	4/3	1	0	1/3	14/3
$z_j - c_j$	1	1/3	0	0	1/3	14/3

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the value of x_2 is not an integer the solution is infeasible.

$$x_2 = 4 + \frac{2}{3}$$

Here x_2^{th} row is taken for further process

$$\frac{14}{3} = \frac{4}{3}x_1 + x_2 + \frac{1}{3}s_2$$

$$4 + \frac{2}{3} = \left(1 + \frac{1}{3}\right)x_1 + (1 + 0)x_2 + \left(0 + \frac{1}{3}\right)s_2$$

$$-\frac{2}{3} = -\frac{1}{3}x_1 - \frac{1}{3}s_2 + s_3$$

Basis	z	x_1	x_2	s_1	s_2	s_3	**Solution**
s_1	0	- 5/3	0	1	- 2/3	0	8/3
x_2	0	4/3	1	0	1/3	0	14/3
s_3	0	- 1/3	0	0	- 1/3	1	- 2/3
$z_j - c_j$	1	1/3	0	0	1/3	0	14/3
Ratio		1			1		

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	s_1	s_2	s_3	**Solution**
s_1	0	-1	0	1	0	- 2/9	4
x_2	0	1	1	0	0	- 1/3	4
s_2	0	1	0	0	1	- 1/3	2
$z_j - c_j$	1	0	0	0	0	1	4

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1 \, and \, x_2$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 0, x_2 = 4, Max\ Z = 4$$

6. Solve the following integer programming problem by Gomory technique.

$$Max\ Z = 3x_2$$

Subject to the constraints

$$3x_1 + 2x_2 \geq 7$$

$$-x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0 \ and\ are\ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - 3x_2 + 0s_1 + 0s_2 + MA_1 = 0$$

Subject to

$$3x_1 + 2x_2 - s_1 + A_1 = 7$$

$$-x_1 + x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2, A_1 \geq 0\ \ and\ are\ integers.$$

Basis	z	x_1	x_2	s_1	s_2	A_1	**Solution**
A_1	0	3	2	- 1	0	1	7
s_2	0	- 1	1	0	1	0	2
$z_j - c_j$	1	0	- 3	0	0	M	0

Basis	z	x_1	x_2	s_1	s_2	A_1	**Solution**	**Ratio**
A_1	0	3	2	- 1	0	1	7	7/3
s_2	0	- 1	1	0	1	0	2	
$z_j - c_j$	1	-3M	- 3-2M	M	0	0	-7M	

Basis	z	x_1	x_2	s_1	s_2	A_1	**Solution**	**Ratio**
x_1	0	1	2/3	- 1/3	0	1/3	7/3	3.5
s_2	0	0	5/3	-1/3	1	1/3	13/3	2.6
$z_j - c_j$	1	0	-3	0	0	M	0	

Basis	z	x_1	x_2	s_1	s_2	A_1	**Solution**
x_1	0	1	0	-1/5	-2/5	1/5	3/5
x_2	0	0	1	-1/5	3/5	1/5	13/5
$z_j - c_j$	1	0	0	- 3/5	9/5	3/5+M	39/5

Since all the values in the pivot column is negative, the solution is unbounded.

7. Solve the following integer programming problem using the cutting plane algorithm.

$$Max\ Z = 2x_1 + 20x_2 + 4x_3$$

Subject to the constraints

$$2x_1 + 20x_2 + 4x_3 \leq 15$$

$$6x_1 + 20x_2 + 4x_3 = 20$$

$$x_1, x_2 \ and \ x_3 \ are \ non-negative \ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - 2x_1 - 20x_2 - 4x_3 + 0s_1 + MA_1 = 0$$

Subject to

$$2x_1 + 20x_2 + 4x_3 + s_1 = 15$$

$$6x_1 + 20x_2 + 4x_3 + A_1 = 20$$

$$x_1, x_2, x_3, s_1, A_1 \geq 0 \ and \ are \ integers.$$

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution
s_1	0	2	20	4	1	0	15
A_1	0	6	20	4	0	1	20
$z_j - c_j$	1	-2	- 20	- 4	0	M	0

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution	Ratio
s_1	0	2	20	4	1	0	15	0.75
A_1	0	6	20	4	0	1	20	1
$z_j - c_j$	1	-2-6M	- 20-20M	- 4 - 4M	0	0	-20M	

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution	Ratio
x_2	0	1/10	1	1/5	1/20	0	3/4	7.5
A_1	0	4	0	0	-1	1	5	1.25
$z_j - c_j$	1	-4M	0	0	1+M	0	15-5M	

Basis	z	x_1	x_2	x_3	s_1	A_1	Solution
x_2	0	0	1	1/5	3/40	- 1/40	5/8
x_1	0	1	0	0	- 1/4	1/4	5/4
$z_j - c_j$	1	0	0	0	1	M	15

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since $x_1 \ and \ x_2$ are not integers, the solution is infeasible.

$$x_1 = \frac{5}{4} = 1 + \frac{1}{4}, x_2 = \frac{5}{8}$$

Here x_2^{th} row is taken for further process since fractional part of x_2 is greater than the fractional part of x_1.

$$\frac{5}{8} = x_2 + \frac{1}{5}x_3 + \frac{3}{40}s_1 - \frac{1}{40}A_1$$

$$\frac{5}{8} = (1+0)\,x_2 + \left(0 + \frac{1}{5}\right)x_3 + \left(0 + \frac{3}{40}\right)s_1 + \left(-1 + \frac{39}{40}\right)A_1$$

$$-\frac{5}{8} = -\frac{1}{5}x_3 - \frac{3}{40}s_1 - \frac{39}{40}A_1 + s_2$$

Basis	z	x_1	x_2	x_3	s_1	s_2	A_1	Solution
x_2	0	0	1	1/5	3/40	0	- 1/40	5/8
x_1	0	1	0	0	- 1/4	0	1/4	5/4
s_2	0	0	0	- 1/5	- 3/40	1	- 39/40	- 5/8
$z_j - c_j$	1	0	0	0	1	0	M	15
Ratio				70	13.33			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	A_1	Solution	Ratio
x_2	0	0	1	0	0	1	-1	0	
x_1	0	1	0	2/3	0	- 10/3	7/2	10/3	5
s_1	0	0	0	8/3	1	- 40/3	13	25/3	3.125
$z_j - c_j$	1	0	0	- 8/3	0	40/3	-13+M	20/3	

Basis	z	x_1	x_2	x_3	s_1	s_2	A_1	Solution
x_2	0	0	1	0	0	1	-1	0
x_1	0	1	0	0	- 1/4	0	1/4	5/4
x_3	0	0	0	1	3/8	- 5	39/8	25/8
$z_j - c_j$	1	0	0	0	1	0	M	15

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_1 and x_3 are not integers, the solution is infeasible.

$$x_1 = \frac{5}{4} = 1 + \frac{1}{4}, x_3 = \frac{25}{8} = 3 + \frac{1}{8}$$

Here x_1^{th} row is taken for further process since fractional part of x_1 is greater than the fractional part of x_2.

$$\frac{5}{4} = x_1 - \frac{1}{4}s_1 + \frac{1}{4}A_1$$

$$1 + \frac{1}{4} = (1+0)x_1 + \left(-1 + \frac{3}{4}\right)s_1 + \left(0 + \frac{1}{4}\right)A_1$$

$$-\frac{1}{4} = -\frac{3}{4}s_1 - \frac{1}{4}A_1 + s_3$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	A_1	**Solution**
x_2	0	0	1	0	0	1	0	-1	0
x_1	0	1	0	0	- 1/4	0	0	1/4	5/4
x_3	0	0	0	1	3/8	- 5	0	39/8	25/8
s_3	0	0	0	0	- 3/4	0	1	- 1/4	- 1/4
$z_j - c_j$	1	0	0	0	1	0	0	M	15
Ratio					1.33				

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	A_1	**Solution**
x_2	0	0	1	0	0	1	0	-1	0
x_1	0	1	0	0	0	0	- 1/3	1/3	4/3
x_3	0	0	0	1	0	- 5	1/2	19/4	3
s_1	0	0	0	0	1	0	- 4/3	1/3	1/3
$z_j - c_j$	1	0	0	0	0	0	4/3	- 1/3 +M	44/3

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_1 is not an integer, the solution is infeasible.

$$x_3 = \frac{4}{3}$$

Here x_1^{th} row is taken for further process

$$\frac{4}{3} = x_1 - \frac{1}{3}s_3 + \frac{1}{3}A_1$$

$$1 + \frac{1}{3} = (1+0)x_1 + \left(-1 + \frac{2}{3}\right)s_3 + \left(0 + \frac{1}{3}\right)A_1$$

$$-\frac{1}{3} = -\frac{2}{3}s_3 - \frac{1}{3}A_1 + s_4$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	Solution
x_2	0	0	1	0	0	1	0	0	-1	0
x_1	0	1	0	0	0	0	- 1/3	0	1/3	4/3
x_3	0	0	0	1	0	- 5	1/2	0	19/4	3
s_1	0	0	0	0	1	0	- 4/3	0	1/3	1/3
s_4	0	0	0	0	0	0	- 2/3	1	- 1/3	- 1/3
$z_j - c_j$	1	0	0	0	0	0	4/3	0	- 1/3 +M	44/3
Ratio							2			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	A_1	Solution
x_2	0	0	1	0	0	1	0	0	-1	0
x_1	0	1	0	0	0	0	0	- 1/2	1/2	3/2
x_3	0	0	0	1	0	-5	0	3/4	9/2	11/4
s_1	0	0	0	0	1	0	0	-2	1	1
s_3	0	0	0	0	0	0	1	-3/2	1/2	1/2
$z_j - c_j$	1	0	0	0	0	0	0	2	-1+M	14

Since all the values in the row $z_j - c_{ij}$ are ≥ 0 and artificial variable is not present in the basis. Since x_1 and x_3 are not integers, the solution is infeasible.

$$x_1 = \frac{3}{2} = 1 + \frac{1}{2}, x_3 = \frac{11}{4} = 2 + \frac{3}{4}$$

Here x_1^{th} row is taken for further process since fractional part of x_3 is greater than the fractional part of x_1.

$$\frac{11}{4} = x_3 - 5s_2 + \frac{3}{4}s_4 + \frac{9}{2}A_1$$

$$2 + \frac{3}{4} = (1+0)x_3 + (-5+0)s_2 + \left(0 + \frac{3}{4}\right)s_4 + \left(4 + \frac{1}{2}\right)A_1$$

$$-\frac{3}{4} = -\frac{3}{4}s_4 - \frac{1}{2}A_1 + s_5$$

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	Solution
x_2	0	0	1	0	0	1	0	0	0	-1	0
x_1	0	1	0	0	0	0	0	- 1/2	0	1/2	3/2
x_3	0	0	0	1	0	-5	0	3/4	0	9/2	11/4
s_1	0	0	0	0	1	0	0	-2	0	1	1
s_3	0	0	0	0	0	0	1	-3/2	0	1/2	1/2
s_5	0	0	0	0	0	0	0	- 3 /4	1	-1/2	- 3/4
$z_j - c_j$	1	0	0	0	0	0	0	2	0	-1+M	14
Ratio								2.67			

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	A_1	Solution
x_2	0	0	1	0	0	1	0	0	0	-1	0
x_1	0	1	0	0	0	0	0	0	- 2/3	5/6	2
x_3	0	0	0	1	0	-5	0	0	1	4	2
s_1	0	0	0	0	1	0	0	0	- 8/3	7/3	3
s_3	0	0	0	0	0	0	1	0	-2	3/2	2
s_4	0	0	0	0	0	0	0	1	- 4/3	2/3	1
$z_j - c_j$	1	0	0	0	0	0	0	0	8/3	-7/3+M	12

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1, x_2 and\ x_3$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 2, x_2 = 0, x_3 = 2, Max\ Z = 12$$

8. Solve $Max\ Z = x_1 + 2x_2$

Subject to

$$x_1 + x_2 \leq 7$$

$$2x_1 \leq 4$$

$$2x_2 \leq 7$$

$$x_1, x_2 \geq 0\ \ and\ are\ integers.$$

Solution:

The problem is rearranged as follows

$$Max\ Z - x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$

Subject to

$$x_1 + x_2 + s_1 = 7$$

$$2x_1 + s_2 = 4$$

$$2x_2 + s_3 = 7$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0\ \ and\ x_1, x_2\ are\ integers.$$

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	1	1	1	0	0	7	7
s_2	0	2	0	0	1	0	4	
s_3	0	0	2	0	0	1	7	3.5
$z_j - c_j$	1	-1	-2	0	0	0	0	

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	1	0	1	0	-1/2	7/2	3.5
s_2	0	2	0	0	1	0	4	2
x_2	0	0	1	0	0	1/2	7/2	
$z_j - c_j$	1	-1	0	0	0	1	7	

Basis	z	x_1	x_2	s_1	s_2	s_3	Solution
s_1	0	0	0	1	- 1/2	- 1/2	3/2
x_1	0	1	0	0	1/2	0	2
x_2	0	0	1	0	0	1/2	7/2
$z_j - c_j$	1	0	0	0	1/2	1	9

Since all the values in the $z_j - c_j$ row is ≥ 0. Therefore solution is reached.

Since the value of x_2 is not an integer the solution is infeasible.

$$x_2 = 3 + \frac{1}{2}$$

Here x_2^{th} row is taken for further process

$$\frac{7}{2} = x_2 + \frac{1}{2}s_3$$

$$3 + \frac{1}{2} = (1+0)x_2 + \left(0 + \frac{1}{2}\right)s_3$$

$$-\frac{1}{2} = -\frac{1}{2}s_3 + s_4$$

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
s_1	0	0	0	1	- 1/2	- 1/2	0	3/2
x_1	0	1	0	0	1/2	0	0	2
x_2	0	0	1	0	0	1/2	0	7/2
s_4	0	0	0	0	0	- 1/2	1	- 1/2
$z_j - c_j$	1	0	0	0	1/2	1	0	9
Ratio						2		

Now solving the problem by dual simplex method, we get

Basis	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
s_1	0	0	0	1	- 1/2	0	-1	2
x_1	0	1	0	0	1/2	0	0	2
x_2	0	0	1	0	0	0	1	3
s_3	0	0	0	0	0	1	-2	1
$z_j - c_j$	1	0	0	0	1/2	0	0	8

Since all the values in the $z_j - c_j$ row is ≥ 0 and $x_1 and\ x_2$ are integers. Therefore optimum solution is reached.

$$\therefore x_1 = 2, x_2 = 3, Max\ Z = 8$$

GAME THEORY

1. Reduce the following game by dominance and find the game value:

Player B

Player A		**I**	**II**	**III**	**IV**
	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Solution:

Player B

Player A		**I**	**II**	**III**	**IV**	**Row Min**
	I	3	2	4	0	0
	II	3	4	2	4	2
	III	4	2	4	0	0
	IV	0	4	0	8	0
	Column Max	4	4	4	8	

$$Maximin = Max\ Row\ Min = 2$$

$$Minimax = Min\ Column\ Max = 4$$

$$Maximin \neq Minimax$$

$\therefore$ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Row III is greater than the values of Row I. $\therefore$ Row I is dominated by Row III, so eliminate Row I.

Player B (columns), **Player A** (rows)

	I	II	III	IV
II	3	4	2	4
III	4	2	4	0
IV	0	4	0	8

The values of Column III is lesser than the values of Column I. ∴ Column I is dominated by Column III, so eliminate Column I.

Player B (columns), **Player A** (rows)

	II	III	IV
II	4	2	4
III	2	4	0
IV	4	0	8

Now the average of Column III and Column IV is less than Column II. ∴ Column II is dominated by Columns III and IV respectively, so eliminate Column II.

Player B (columns), **Player A** (rows)

	III	IV
II	2	4
III	4	0
IV	0	8

Now the average of Row III and Row IV is equal to Row II. ∴ Row II is dominated by Rows III and IV respectively, so eliminate Row II.

Player B (columns), **Player A** (rows)

	III	IV
III	4	0
IV	0	8

Now we can solve this 2 × 2 by short cut method.

Player B (columns), **Player A** (rows)

	III	IV	
III	4	0	8
IV	0	8	4
	8	4	

$$p_3 = \frac{8}{12} = \frac{2}{3}, p_4 = \frac{4}{12} = \frac{1}{3}$$

$$q_3 = \frac{8}{12} = \frac{2}{3}, q_4 = \frac{4}{12} = \frac{1}{3}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3 \quad p_4) = \left(0 \quad 0 \quad \frac{2}{3} \quad \frac{1}{3}\right)$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3 \quad q_4) = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = 4 \times \frac{2}{3} + 0 \times \frac{1}{3} = \frac{8}{3}$$

2. Solve the following game graphically.

	Player B	
Player A	- 3	1
	5	3
	6	-1
	1	4
	2	2
	0	- 5

Solution:

		Player B 1	**2**	**Row Min**
Player A	**1**	- 3	1	-3
	2	5	3	3
	3	6	-1	-1
	4	1	4	1
	5	2	2	2
	6	0	- 5	0
Column Max		6	4	

$$Maximin = Max\ Row\ Min = 3$$

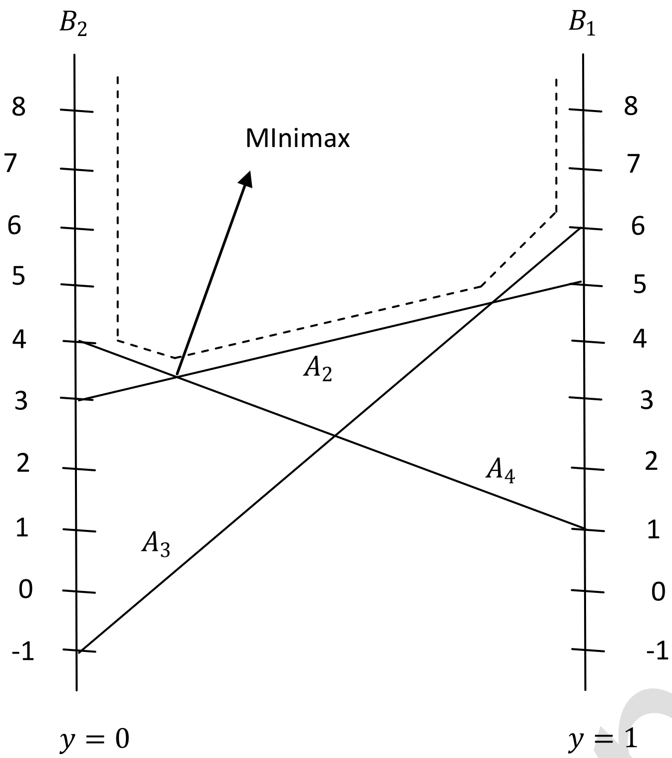
$$Minimax = Min\ Column\ Max = 4$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Row 2 are greater than the values of Rows 1, 5 and 6. ∴ Rows 1, 5 and 6 are dominated by Row 2, so eliminate Rows 1, 5 and 6.

		Player B 1	**2**
Player A	**2**	5	3
	3	6	-1
	4	1	4



∴ From the graph Minimax value involves strategies A_2 and A_4. Therefore eliminating Rows except strategies A_2 and A_4 to make it a 2×2 Game.

		Player B	
		1	**2**
Player A	**2**	5	3
	4	1	4

Now we can solve this 2×2 by short cut method.

		Player B		
		1	**2**	
Player A	**2**	5	3	3
	4	1	4	2
		1	4	

$$p_2 = \frac{3}{5}, p_4 = \frac{2}{5}$$

$$q_1 = \frac{1}{5}, q_2 = \frac{4}{5}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6) = \left(0 \quad \frac{3}{5} \quad 0 \quad \frac{2}{5} \quad 0 \quad 0\right)$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2) = \left(\frac{1}{5} \quad \frac{4}{5}\right)$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = 5 \times \frac{1}{5} + 3 \times \frac{4}{5} = \frac{17}{5}$$

3. Solve the following game whose payoff matrix is given below.

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	5	-10	9	0
	A_2	6	7	8	1
	A_3	8	7	15	2
	A_4	3	4	-1	4

Solution:

		Player B				
		B_1	B_2	B_3	B_4	Row Min
Player A	A_1	5	-10	9	0	0
	A_2	6	7	8	1	1
	A_3	8	7	15	2	2
	A_4	3	4	-1	4	- 1
	Column Max	8	7	15	4	

$$Maximin = Max\ Row\ Min = 2$$

$$Minimax = Min\ Column\ Max = 4$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

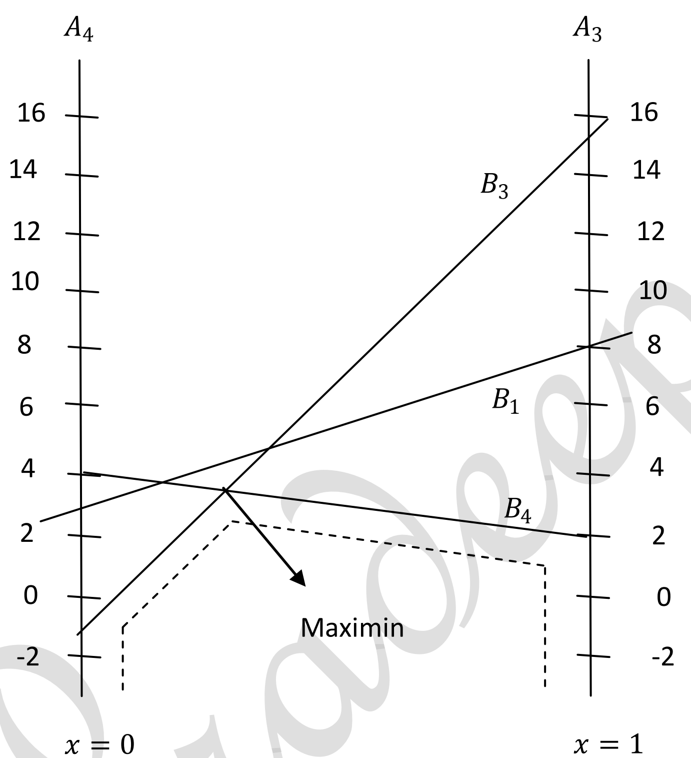
The values of Row A_3 is greater than the values of Rows A_1 and A_2 ∴ Rows A_1 and $A_2$are dominated by RowA_3, so eliminate the Rows A_1 and A_2.

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_3	8	7	15	2
	A_4	3	4	-1	4

The values of Column B_4 is lesser than the values of Column B_2. $\therefore$ Column B_2 is dominated by Column B_4, so eliminate Column B_2.

		Player B		
		B_1	B_3	B_4
Player A	A_3	8	15	2
	A_4	3	-1	4

Further we cannot reduce by using dominance rule. Since it is 2×3 Game we can solve using Graphical method to reduce it to 2×2 Game.



$\therefore$ From the graph Maximin value involves strategies B_3 and B_4. Therefore eliminating columns except strategies B_3 and B_4 to make it a 2×2 Game.

		Player B	
		B_3	B_4
Player A	A_3	15	2
	A_4	-1	4

Now we can solve this 2×2 by short cut method.

Player A / Player B

	B_3	B_4	
A_3	15	2	5
A_4	-1	4	13
	2	16	

$$p_3 = \frac{5}{18}, p_4 = \frac{13}{18}$$

$$q_3 = \frac{2}{18} = \frac{1}{9}, q_4 = \frac{16}{18} = \frac{8}{9}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3 \quad p_4) = \begin{pmatrix} 0 & 0 & \frac{5}{18} & \frac{13}{18} \end{pmatrix}$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3 \quad q_4) = \begin{pmatrix} 0 & 0 & \frac{1}{9} & \frac{8}{9} \end{pmatrix}$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = 15 \times \frac{1}{9} + 2 \times \frac{8}{9} = \frac{31}{9}$$

4. Use graphical method in solving the following game and find the optimal strategies of player A and Player B and the value of the game.

Player A / Player B

	B_1	B_2	B_3	B_4
A_1	2	2	3	- 2
A_2	4	3	2	6

Solution:

Player A / Player B

	B_1	B_2	B_3	B_4	Row Min
A_1	2	2	3	- 2	-2
A_2	4	3	2	6	2
Column Max	4	3	3	6	

$$Maximin = Max\ Row\ Min = 2$$

$$Minimax = Min\ Column\ Max = 3$$

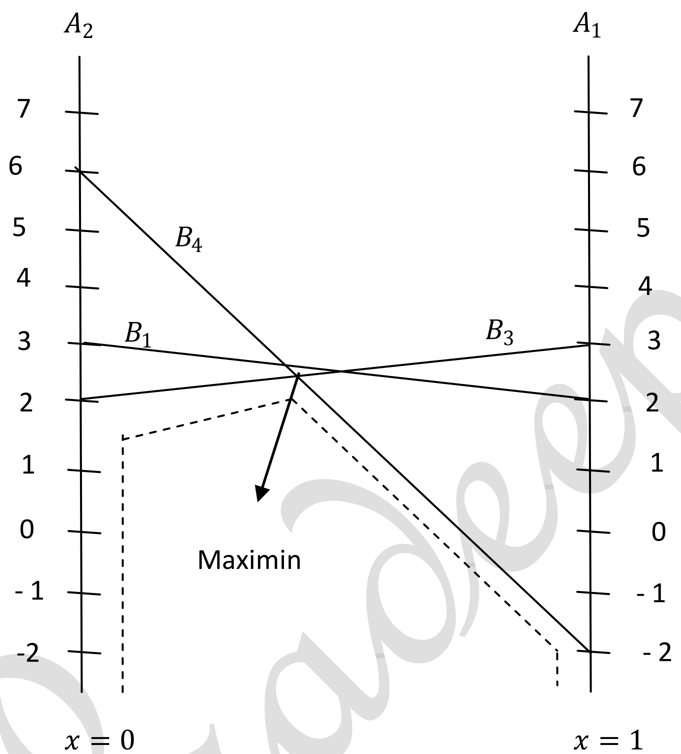
$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Column B_2 is lesser than the values of Column B_1 ∴ Column B_1 is dominated by Column B_2, so eliminate the Column B_1.

		Player B		
		B_1	B_3	B_4
Player A	A_1	2	3	- 2
	A_2	3	2	6

Further we cannot reduce by using dominance rule. Since it is 2×3 Game we can solve using Graphical method to reduce it to 2×2 Game.



∴ From the graph Maximin value involves strategies B_3 and B_4. Therefore eliminating columns except strategies B_3 and B_4 to make it a 2×2 Game.

		Player B	
		B_3	B_4
Player A	A_1	3	- 2
	A_2	2	6

Now we can solve this 2×2 by short cut method.

Player B

Player A	B_3	B_4	
A_1	3	- 2	4
A_2	2	6	5
	8	1	

$$p_1 = \frac{4}{9}, p_2 = \frac{5}{9}$$

$$q_3 = \frac{8}{9}, q_4 = \frac{1}{9}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2) = \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \end{pmatrix}$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3 \quad q_4) = \begin{pmatrix} 0 & 0 & \frac{8}{9} & \frac{1}{9} \end{pmatrix}$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = 3 \times \frac{8}{9} - 2 \times \frac{1}{9} = \frac{22}{9}$$

5. Two breakfast food manufactures, ABC and XYZ are competing for an increased market share. The payoff matrix, shown in the following table, describes the increase in market share for ABC and decrease in market share of XYZ.

		XYZ			
		GC	**DP**	**MPS**	**IA**
ABC	**Give Coupons (GC)**	2	- 2	4	1
	Decrease Price (DP)	6	1	12	3
	Maintain Present Strategy (MPS)	- 3	2	0	6
	Increase Advertising (IA)	2	- 3	7	11

Determine optimal strategies for both the manufacturing and the value of the game.

Solution:

		Player XYZ				
		GC	**DP**	**MPS**	**IA**	**Row Min**
Player ABC	**GC**	2	- 2	4	1	- 2
	DP	6	1	12	3	1
	MPS	- 3	2	0	6	- 3
	IA	2	- 3	7	11	- 3
	Column Max	6	2	12	11	

$$Maximin = Max\ Row\ Min = 1$$

$$Minimax = Min\ Column\ Max = 2$$

$$Maximin \neq Minimax$$

$\therefore$ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Row DP are greater than the values of Row GC. ∴ Row GC is dominated by Row DP, so eliminate Row GC.

		Player XYZ			
		GC	**DP**	**MPS**	**IA**
Player ABC	**DP**	6	1	12	3
	MPS	- 3	2	0	6
	IA	2	- 3	7	11

The values of Column GC are lesser than the values of Column MPS. ∴ Column MPS is dominated by Column GC, so eliminate Column MPS.

		Player XYZ		
		GC	**DP**	**IA**
Player ABC	**DP**	6	1	3
	MPS	- 3	2	6
	IA	2	- 3	11

The values of Column DP are lesser than the values of Column IA. ∴ Column IA is dominated by Column DP, so eliminate Column IA.

		Player XYZ	
		GC	**DP**
Player ABC	**DP**	6	1
	MPS	- 3	2
	IA	2	- 3

The values of Row DP are greater than the values of Row IA. ∴ Row IA is dominated by Row DP, so eliminate Row IA.

		Player XYZ	
		GC	**DP**
Player ABC	**DP**	6	1
	MPS	- 3	2

Now we can solve this 2 × 2 by short cut method.

		Player XYZ		
		GC	**DP**	
Player ABC	**DP**	6	1	5
	MPS	- 3	2	5
		1	9	

$$p_2 = \frac{5}{10} = \frac{1}{2}, p_3 = \frac{5}{10} = \frac{1}{2}$$

$$q_1 = \frac{1}{10}, q_2 = \frac{9}{10}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3 \quad p_4) = \left(0 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\right)$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\frac{1}{10} \quad \frac{9}{10} \quad 0 \quad 0\right)$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = 6 \times \frac{1}{10} + 1 \times \frac{9}{10} = \frac{15}{10}$$

6. Players A and B play a game in which each has three coins Re. 1, Rs. 2 and Rs. 5. Each select a coin without the knowledge of other's choice. If the sum of the coins is an odd amount, A wins B's coin, if the sum is even B wins A's coin. Find the best strategy for each player and value of the game.

Solution:

The payoff matrix is

		Player B		
		1	**2**	**5**
Player A	**1**	-1	2	-1
	2	1	- 2	5
	5	-5	2	-5

		Player B			
		1	**2**	**5**	**Row Min**
Player A	**1**	-1	2	-1	-1
	2	1	- 2	5	-2
	5	-5	2	-5	-5
Column Max		1	2	5	

$$Maximin = Max\ Row\ Min = -1$$

$$Minimax = Min\ Column\ Max = 1$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Column 1 is lesser than the values of Column 5 ∴ Column 5 is dominated by Column 1, so eliminate the Column 5.

		Player B	
		1	2
Player A	1	-1	2
	2	1	- 2
	5	-5	2

The values of Row 1 is greater than the values of Row 5 ∴ Row 5 is dominated by Row 1, so eliminate the Row 5.

		Player B	
		1	2
Player A	1	-1	2
	2	1	- 2

Now we can solve this 2×2 by short cut method.

		Player B		
		1	2	
Player A	1	-1	2	3
	2	1	- 2	3
		4	2	

$$p_1 = \frac{3}{6} = \frac{1}{2}, p_2 = \frac{3}{6} = \frac{1}{2}$$

$$q_1 = \frac{4}{6} = \frac{2}{3}, q_2 = \frac{2}{6} = \frac{1}{3}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = -1 \times \frac{2}{3} + 2 \times \frac{1}{3} = 0$$

7. Players A and B play a game in which each has three coins 5 paise, 10 paise and 20 paise. Each selects a coin without the knowledge of other's choice. If the sum of the coins is an odd amount, A wins B's coin, if the sum is even B wins A's coin. Find the best strategy for each player and value of the game.

Solution: The payoff matrix is

		Player B		
		5	10	20
Player A	5	-5	10	20
	10	5	- 10	-10
	20	-5	-20	-20

		Player B			
		5	**10**	**20**	**Row Min**
Player A	**5**	-5	10	20	-5
	10	5	- 10	-10	-10
	20	-5	-20	-20	-20
Column Max		5	10	20	

$$Maximin = Max\ Row\ Min = -5$$

$$Minimax = Min\ Column\ Max = 5$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

The values of Column 10 is lesser than the values of Column 20 ∴ Column 20 is dominated by Column10, so eliminate the Column 20

		Player B	
		5	**10**
Player A	**5**	-5	10
	10	5	- 10
	20	-5	-20

The values of Row 5 is greater than the values of Row 20 ∴ Row 20 is dominated by Row 5, so eliminate the Row 20.

		Player B	
		5	**10**
Player A	**5**	-5	10
	10	5	- 10

Now we can solve this 2×2 by short cut method.

		Player B		
		5	**10**	
Player A	**5**	-5	10	15
	10	5	- 10	15
		20	10	

$$p_1 = \frac{15}{30} = \frac{1}{2}, p_2 = \frac{15}{30} = \frac{1}{2}$$

$$q_1 = \frac{20}{30} = \frac{2}{3}, q_2 = \frac{10}{30} = \frac{1}{3}$$

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$Value\ of\ the\ game\ V = aq_1 + bq_2 = -5 \times \frac{2}{3} + 10 \times \frac{1}{3} = 0$$

8. Find the value of the game by using Linear Programming A_1, A_2, A_3 are A' s strategy, B_1, B_2, B_3 are B's strategy

	B_1	B_2	B_3
A_1	3	-1	-3
A_2	-2	4	-1
A_3	-5	-6	2

Solution:

	B_1	B_2	B_3	**Row Min**
A_1	3	-1	-3	-3
A_2	-2	4	-1	-2
A_3	-5	-6	2	-6
Column Max	3	4	2	

$$Maximin = Max\ Row\ Min = -2$$

$$Minimax = Min\ Column\ Max = 2$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

Here we cannot able to reduce using dominance rule. To make all the values of payoff matrix positive add all the values of payoff matrix with the absolute value of the most negative value plus one.

Here most negative value is -6. The absolute value of -6 is 6 so add all values with 6+1=7

	B_1	B_2	B_3
A_1	10	6	4
A_2	5	11	6
A_3	2	1	9

Now the linear programming problem for player B is given by

$$Max\ Z = y_1 + y_2 + y_3$$

Subject to

$$10y_1 + 6y_2 + 4y_3 \leq 1$$

$$5y_1 + 11y_2 + 6y_3 \leq 1$$

$$2y_1 + y_2 + 9y_3 \leq 1$$

$$y_1, y_2, y_3 \geq 0$$

$$where\ y_1 = \frac{q_1}{V}, y_2 = \frac{q_2}{V}, y_3 = \frac{q_3}{V}\ and\ Z = \frac{1}{V}$$

The above LPP is rearranged as follows

$$Max\ Z - y_1 - y_2 - y_3 = 0$$

Subject to

$$10y_1 + 6y_2 + 4y_3 + s_1 = 1$$

$$5y_1 + 11y_2 + 6y_3 + s_2 = 1$$

$$2y_1 + y_2 + 9y_3 + s_3 = 1$$

$$y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

Basis	Z	y_1	y_2	y_3	s_1	s_2	s_3	Solution	Ratio
s_1	0	10	6	4	1	0	0	1	0.1
s_2	0	5	11	6	0	1	0	1	0.2
s_3	0	2	1	9	0	0	1	1	0.5
$z_j - c_j$	1	-1	-1	-1	0	0	0	0	
y_1	0	1	0.6	0.4	0.1	0	0	0.1	0.25
s_2	0	0	8	4	-0.5	1	0	0.5	0.125
s_3	0	0	-0.2	8.2	-0.2	0	1	0.8	0.098
$z_j - c_j$	1	0	-0.4	-0.6	0.1	0	0	0.1	
y_1	0	1	0.6098	0	0.1098	0	-0.0488	0.0610	0.1
s_2	0	0	8.0976	0	-0.4024	1	-0.4878	0.1098	0.013554
y_3	0	0	-0.0244	1	-0.0244	0	0.1220	0.0976	
$z_j - c_j$	1	0	-0.4146	0	0.0854	0	0.0732	0.1585	
y_1	0	1	0	0	0.1401	-0.0753	-0.0120	0.0527	
y_2	0	0	1	0	-0.0497	0.1235	-0.0602	0.0136	
y_3	0	0	0	1	-0.0256	0.0030	0.1205	0.0979	
$z_j - c_j$	1	0	0	0	0.0648	0.0512	0.0482	0.1642	

Since all the values in the $z_j - c_j$ is ≥ 0, Therefore optimum solution is reached.

$$Max\ Z = \frac{1}{V} = 0.16 \Rightarrow V = \frac{1}{0.1642} = 6.09$$

$$y_1 = \frac{q_1}{V} \Rightarrow q_1 = y_1 V = 0.0527 \times 6.09 = 0.32$$

$$y_2 = \frac{q_2}{V} \Rightarrow q_2 = y_2 V = 0.0136 \times 6.09 = 0.08$$

$$y_3 = \frac{q_3}{V} \Rightarrow q_3 = y_3 V = 0.0979 \times 6.09 = 0.6$$

The values of $s_1, s_2\ and\ s_3$ in $z_j - c_j$ row are the values of $x_1, x_2 and\ x_3$ for Player A.

$$x_1 = 0.0648, x_2 = 0.0512, x_3 = 0.0482$$

$$x_1 = \frac{p_1}{V} \Rightarrow p_1 = x_1 V = 0.0648 \times 6.09 = 0.39$$

$$x_2 = \frac{p_2}{V} \Rightarrow p_2 = x_2 V = 0.0512 \times 6.09 = 0.31$$

$$x_3 = \frac{p_3}{V} \Rightarrow p_3 = x_3 V = 0.0482 \times 6.09 = 0.29$$

Optimal strategies for Player A is $(0.39 \quad 0.31 \quad 0.29)$

Optimal strategies for Player B is $(0.32 \quad 0.08 \quad 0.6)$

Value of the original game = $6.09 - 7 = -0.91$

9. Solve the game

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	19	6	7	5
	A_2	7	14	14	6
	A_3	12	8	18	4
	A_4	8	7	13	-1

Solution:

		Player B				
		B_1	B_2	B_3	B_4	**Row Min**
Player A	A_1	19	6	7	5	5
	A_2	7	14	14	6	6
	A_3	12	8	18	4	4
	A_4	8	7	13	-1	- 1
	Column Max	19	14	18	6	

$$Maximin = Max\ Row\ Min = 6$$

$$Minimax = Min\ Column\ Max = 6$$

$$Maximin = Minimax$$

∴ Saddle point exists. The game has pure strategy.

$$Strategy\ for\ game\ A\ is\ (p_1 \quad p_2 \quad p_3 \quad p_4) = (0 \quad 1 \quad 0 \quad 0) = A_2$$

$$Strategy\ for\ game\ B\ is\ (q_1 \quad q_2 \quad q_3 \quad q_4) = (0 \quad 0 \quad 0 \quad 1) = B_4$$

$$Value\ of\ the\ game\ V = 6$$

10. Find the value of the game by using Matrix method

	B_1	B_2	B_3
A_1	3	-1	-3
A_2	-2	4	-1
A_3	-5	-6	2

Solution:

	B_1	B_2	B_3	**Row Min**
A_1	3	-1	-3	-3
A_2	-2	4	-1	-2
A_3	-5	-6	2	-6
Column Max	3	4	2	

$$Maximin = Max\ Row\ Min = -2$$

$$Minimax = Min\ Column\ Max = 2$$

$$Maximin \neq Minimax$$

∴ No saddle point. The game has mixed strategy. So we apply dominance rule to minimize the problem.

Here we cannot able to reduce using dominance rule, so solve the problem by matrix method.

Subtract the values of B_2 from B_1 and the values of B_3 from B_2 and write it on the right side of the payoff matrix. Similarly subtract the values of A_2 from A_1 and the values of A_3 from A_2 and write it on the Bottom of the payoff matrix.

	B_1	B_2	B_3		
A_1	3	-1	-3	4	2
A_2	-2	4	-1	-6	5
A_3	-5	-6	2	1	-8
	5	-5	-2		
	3	10	-3		

Oddments of $A_1 = \begin{vmatrix} -6 & 5 \\ 1 & -8 \end{vmatrix} = 48 - 5 = 43$

Oddments of $A_2 = \begin{vmatrix} 4 & 2 \\ 1 & -8 \end{vmatrix} = -32 - 2 = -34$

Oddments of $A_3 = \begin{vmatrix} 4 & 2 \\ -6 & 5 \end{vmatrix} = 20 + 12 = 32$

Oddments of $B_1 = \begin{vmatrix} -5 & -2 \\ 10 & -3 \end{vmatrix} = 15 + 20 = 35$

Oddments of $B_2 = \begin{vmatrix} 5 & -2 \\ 3 & -3 \end{vmatrix} = -15 + 6 = -9$

Oddments of $B_3 = \begin{vmatrix} 5 & -5 \\ 3 & 10 \end{vmatrix} = 50 + 15 = 65$

Now consider only the value not the sign.

	B_1	B_2	B_3	
A_1	3	-1	-3	43
A_2	-2	4	-1	34
A_3	-5	-6	2	32
	35	9	65	

$p_1 = \frac{43}{109} = 0.39, p_2 = \frac{34}{109} = 0.31, p_3 = \frac{32}{109} = 0.29$

$q_1 = \frac{35}{109} = 0.32, q_2 = \frac{9}{109} = 0.08, q_3 = \frac{65}{109} = 0.6$

Optimal strategies for Player A is $(0.39 \quad 0.31 \quad 0.29)$

Optimal strategies for Player B is $(0.32 \quad 0.08 \quad 0.6)$

$$Value\ of\ the\ game = V = 3 \times 0.32 - 1 \times 0.08 - 3 \times 0.6 = -0.92$$