B.E/B.Tech Degree Examinations April/May 2011
Regulation 2008
MA 2161 - MATHEMATICS-II

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Answer ALL Questions

Part A – (10 x 2 = 20 marks)

1. Transform the equation $x^2y'' + xy' = x$ into a linear differential equation with constant coefficients.

Ans: $\theta^2 y = e^z$. put $xD = D'$ or θ and $x^2D^2 = D'(D'-1)$ or $\theta(\theta-1)$, $x = e^z$ or $z = \log x$

$\therefore x^2y'' + xy' = x$ becomes, $x^2D^2y + xDy = x$ — ①

$\therefore$ ① becomes $\theta(\theta-1)y + \theta y = (x) e^z$

$(\theta^2 - \cancel{\theta} + \cancel{\theta})y = e^z$

$\Rightarrow \theta^2 y = e^z$

2. Find the particular integral of $(D^2-4)y = \cosh 2x$.

$$P.I = \frac{1}{D^2-4}\cosh 2x = \frac{1}{D^2-4}\left(\frac{e^{2x}+e^{-2x}}{2}\right)$$

$$= \frac{1}{D^2-4}\frac{e^{2x}}{2} + \frac{1}{D^2-4}\frac{e^{-2x}}{2} = \frac{1}{2}\cdot\frac{1}{D^2-4}e^{-2x} + \frac{1}{2}\frac{1}{D^2-4}e^{2x}$$

Put $D = a = -2$ $\quad D = a = 2$

$$= \frac{1}{2}\frac{e^{-2x}}{4-4} + \frac{1}{2}\frac{1}{4-4}e^{2x} = \frac{1}{2}\frac{e^{-2x}}{0} + \frac{1}{2}\frac{e^{2x}}{0}.$$

$$= \frac{1}{2}\frac{x}{2D}e^{-2x} + \frac{1}{2}\frac{x}{2D}e^{2x}$$

$$= \frac{1}{2}\frac{x}{2(-2)}e^{-2x} + \frac{1}{2}\frac{x}{2(+2)}e^{2x} = \frac{1}{2}\left[\frac{x}{-4}e^{-2x} + \frac{x}{4}e^{2x}\right]$$

$$= \frac{x}{4}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$P.I = \frac{x}{4}\sinh 2x$

3. Find the value of m so that the vector $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+mz)\vec{k}$ is solenoidal.

Given $\vec{F}$ is solenoidal i.e $\nabla\cdot\vec{F} = 0$

ie $\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+mz) = 0$

$1 + 1 + m = 0 \Rightarrow \boxed{m = -2}$

4. State the physical interpretation of the line integral $\int_A^B \vec{F}\cdot d\vec{r}$.

If $\vec{F}(x,y,z)$ is a force acting on a particle which moves along a given curve C, then $\int_C \vec{F}\cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacement along C. Thus Work done by force $\vec{F} = \int_C \vec{F}\cdot d\vec{r}$.

Suppose the work done by force $\vec{F}$ in moving a particle from one point to another point in a field is independent of the path joining the two points, then $\vec{F}$ is said to be conservative field.

i.e $\vec{F}$ is a conservative field, if $\int_A^B \vec{F}\cdot d\vec{r}$ is independent of the path joining A and B.

5. Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

$u = 2x - x^3 + 3xy^2$.

$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2\cdot 1$ | $\frac{\partial u}{\partial y} = 0 - 0 + 3x(2y)$

$\frac{\partial^2 u}{\partial x^2} = -6x$ | $\frac{\partial^2 u}{\partial y^2} = 6x(1) = 6x$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

Since $\nabla^2 u = 0$, u is harmonic.

6. Find the map of the circle $|z| = 3$ under the transformation $w = 2z$

$w = 2z$

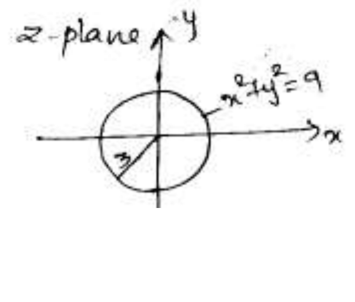
$u + iv = 2(x + iy)$

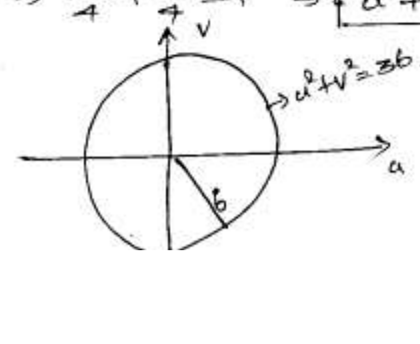
$u = 2x \Rightarrow x = \frac{u}{2}$

$v = 2y \Rightarrow y = \frac{v}{2}$

$|z| = 3 \Rightarrow x^2 + y^2 = 9$ i.e the circle with centre $(0,0)$ radius 3.

z-plane $\Rightarrow \left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 9 \Rightarrow \frac{u^2}{4} + \frac{v^2}{4} = 9 \Rightarrow \boxed{u^2 + v^2 = 36}$





7. Define Singular point.

If at a point $z=a$, the function $f(z)$ is not analytic, then point $z=a$ is called singular point of the function $f(z)$.

Example: $f(z) = \dfrac{z^2}{(z-1)(z+1)^2}$ has singular point $z = \pm 1$.

8. Expand $f(z) = \sin z$ as a Taylor series about origin.

$$\begin{array}{l|l} f(z) = \sin z & f(0) = 0 \\ f'(z) = \cos z & f'(0) = 1 \\ f''(z) = -\sin z & f''(0) = 0 \\ f'''(z) = -\cos z & f'''(0) = -1 \end{array}$$

$\therefore$ Taylor's series is $f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots$

$\therefore f(z) = \frac{z}{1!} - \frac{z^3}{3!} + \cdots$

9. State the condition under which Laplace transform of $f(t)$ exists.

If $f(t)$ is piecewise continuous on every finite interval in $(0,\infty)$ and is of exponential order 'a' for $t>0$, then the Laplace transform of $f(t)$ exists for all $s>a$, i.e $F(s)$ exists $\forall\, s>a$.

10. Find $L^{-1}[\cot^{-1}(s)]$.

$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$; $F(s) = \cot^{-1}s \Rightarrow F'(s) = \frac{-1}{1+s^2}$.

$\therefore L^{-1}[\cot^{-1}(s)] = \frac{-1}{t} L^{-1}\left[\frac{-1}{s^2+1}\right]$ $\quad \left(\text{since } L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at\right)$.

$L^{-1}[\cot^{-1}s] = -\frac{1}{t}(-\sin t) = \underline{\underline{\frac{\sin t}{t}}}$

Part B - $(5 \times 16 = 80$ marks$)$

11) a) i) Solve the equation $(D^2+5D+4)y = e^{-x}\sin 2x$.

Auxiliary equation is, $m^2+5m+4=0 \Rightarrow (m+1)(m+4)=0$. $\begin{array}{c}1 \\ 4 \\ \hline 5\end{array}\Big]4$

$\Rightarrow m = -1, -4$

$\therefore$ Complementary function $= Ae^{-x} + Be^{-4x}$.

$$P.I = \frac{1}{(D+1)(D+4)}(e^{-x}\sin 2x)$$

$$= \frac{1}{D^2+5D+4} e^{-x}\sin 2x \quad \text{Type IV} \left(e^{ax} \begin{cases}\sin ax \\ \text{or} \\ \cos ax\end{cases}\right)$$

Put $D = D+a = D-1$

$$= e^{-x}\frac{1}{(D-1)^2+5(D-1)+4}\sin 2x = e^{-x}\frac{1}{D^2+\underline{1}-2D+5D-\cancel{5}+\underline{4}}\sin 2x.$$

$$= e^{-x}\cdot\frac{1}{D^2+3D}\sin 2x \quad \text{Type II} \quad \therefore D^2 = -a^2 = -4.$$

$$= e^{-x}\frac{1}{-4+3D}\sin 2x = e^{-x}\frac{1}{3D-4}\sin 2x$$

$$= e^{-x}\cdot\frac{3D+4}{(3D)^2-4^2}\sin 2x = e^{-x}\cdot\frac{3D+4}{9D^2-16}\sin 2x \qquad 9(D^2) = 9(-4) = -36$$

$$-16$$

$$-52$$

$$= \frac{e^{-x}}{-52}\left[3D\sin 2x + 4\sin 2x\right] = -\frac{e^{-x}}{52}\left[6\cos 2x + 4\sin 2x\right]$$

$$PI = \frac{-e^{-x}}{26}(3\cos 2x + 2\sin 2x)$$

$\therefore$ The solution $y = Ae^{-x} + Be^{-4x} - \frac{e^{-x}}{26}(3\cos 2x + 2\sin 2x)$.

(ii) Solve the equation $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by the method of variation of parameters.

Given the differential equation $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

First we solve $\frac{d^2y}{dx^2} + y = 0$. Auxilliary equation is $m^2+1=0$

$\therefore m^2 = -1 \Rightarrow m = \pm i$.

$\therefore y_{C.F} = e^{0x}(A\cos x + B\sin x)$

$$f_1 = \cos x \qquad f_2 = \sin x \qquad \therefore f_1 f_2' - f_1' f_2 = \cos^2 x - (-\sin^2 x)$$

$$f' = -\sin x \qquad f_2' = \cos x \qquad = \cos^2 x + \sin^2 x$$

$$= 1.$$

$$P = -\int \frac{f(x)\cdot f_2}{f_1 f_2' - f_1' f_2}\,dx + C_1$$

$$= -\int \frac{\operatorname{cosec}x \cdot \sin x}{1}\,dx + C_1$$

$$= -\int \frac{1}{\cancel{\sin x}}\cdot \cancel{\sin x}\,dx + C_1$$

$$= -\int dx + C_1$$

$$P = -x + C_1$$

$$Q = \int \frac{f(x)\cdot f_1}{f_1 f_2' - f_1' f_2}\,dx + C_2.$$

$$= \int \frac{\operatorname{cosec}x \cdot \cos x}{1}\,dx + C_2$$

$$= \int \frac{\frac{1}{\sin x}\cos x}{1}\,dx + C_2$$

$$= \int \frac{\cos x}{\sin x}\,dx + C_2$$

$$= \int \cot x\,dx + C_2$$

$$Q = \log \sin x + C_2$$

$\therefore$ The solution is $y = A\cos x + B\sin x - x\cos x + \sin x \log(\sin x) + C.$

11 b) i) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\left(\log(1+x)\right)$ — ①

① can be reduced to linear differential equation with constant coefficients by taking $ax+b = e^z$ or $z = \log(ax+b)$.

Here $a=1$, $b=1$.

$$(1+x)\frac{dy}{dx} = aD'y = D'y$$

$$(x+1)^2\frac{d^2y}{dx^2} = a^2 D'(D'-1)y = D'(D'-1)y.$$

$x+1 = e^z$ and $z = \log(x+1)$.

$$\therefore ① \Rightarrow D'(D'-1)y + D'y + y = 2\sin z.$$

$$(D'^2 - \cancel{D'} + \cancel{D'} + 1)y = 2\sin z \Rightarrow \boxed{(D'^2+1)\,y = 2\sin z} \text{ — ②}.$$

A.E $m^2+1=0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i.$

C.F $= A\cos z + B\sin z.$

$$P.I = \frac{1}{D'^2+1}\,2\sin z = 2\cdot\frac{1}{D'^2+1}\sin z = 2\cdot\frac{1}{-1+1}\sin z = 2\frac{1}{0}\sin z$$

$D'^2 = -a^2 = -1.$

$$= \cancel{2}.\ \frac{z}{\cancel{2}D'}\sin z = z\cdot\frac{1}{D'}\sin z. = z\int \sin z\,dz = z(-\cos z) + C$$

$$P.I = -z\cos z + C$$

The solution is,

$$y = A\cos z + B\sin z - z\cos z + C. \quad , \quad z = \log(1+x).$$

ii) Solve $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} + x = t^2$.

Given: $Dx - y = t$ —— ①

$x + Dy = t^2$ —— ②

Solving ① and ② eliminate y,

① $\times D +$ ② $\Rightarrow D^2 x + x = D(t) + t^2$

$(D^2+1)x = 1 + t^2$

A.E: $m^2 + 1 = 0 \Rightarrow m = \pm i$

$C.F = A\cos t + B\sin t$

$P.I = \frac{1}{D^2+1}(1+t^2) = (1+D^2)^{-1}(1+t^2)$

$= (1-D^2)(1+t^2)$

$= 1+t^2 - D^2(1+t^2)$

$= t^2 + 1 - 2$

$= t^2 - 1$

$\therefore x(t) = A\cos t + B\sin t + t^2 - 1$

① $\Rightarrow y = Dx - t$.

$y = \frac{d}{dt}\left[A\cos t + B\sin t + t^2 - 1\right] - t$

$y = -A\sin t + B\cos t + 2t - t$

$\therefore$ Solution is $x = A\cos t + B\sin t + t^2 - 1$

$y = A\sin t + B\cos t + t$ //

12) a) i) Find the angle between the normals to the surface $xy = z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$.

$\phi = xy - z^2$

6 $\quad \nabla\phi = y\vec{i} + x\vec{j} + 2z\vec{k}$

$(\nabla\phi)_{(1,4,2)} = \nabla\phi_1 = 4\vec{i} + \vec{j} + 4\vec{k}$

$(\nabla\phi)_{(-3,-3,3)} = \nabla\phi_2 = -3\vec{i} - 3\vec{j} + 6\vec{k}$.

$$\cos\theta = \frac{\nabla\phi_1\cdot\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

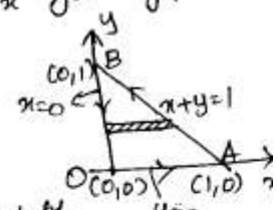
$$\therefore \cos\theta = \left(\frac{3}{\sqrt{33}\sqrt{6}}\right)$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{33}\sqrt{6}}\right).$$

ii) Verify Green's theorem in a plane for $\int_c [(3x^2-8y^2)dx + (4y-6xy)dy]$, where c is the boundary of the region defined by the lines $x=0, y=0$ and $x+y=1$.

Solution:-

Green's theorem is given by $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy$.



Here $P = 3x^2-8y^2$ | $Q = 4y-6xy$

$\frac{\partial P}{\partial y} = P_y = -16y$ | $Q_x = \frac{\partial Q}{\partial x} = -6y$

consider $\iint_R (Q_x-P_y)\,dx\,dy = \iint -6y-(-16y)\,dx\,dy = \int_0^1\int_0^{1-y} 10y\,dx\,dy$.

$= \int_0^1 10y\,(x)_0^{1-y}\,dy = \int_0^1 10y(1-y)\,dy = 10\int_0^1 (y-y^2)\,dy$

$= 10\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 = 10\left[\frac{1}{2}-\frac{1}{3}\right] = 10\left[\frac{3-2}{6}\right] = 10\left(\frac{1}{6}\right) = \frac{5}{3}.$

consider $\int_C P\,dx+Q\,dy = \left(\int_{OA}+\int_{AB}+\int_{BO}\right)(P\,dx+Q\,dy)$.

$I_1 = \int_{OA}(3x^2-8y^2)dx+(4y-6xy)dy = \int_0^1 3x^2\,dx = 3\left(\frac{x^3}{3}\right)_0^1 = 1$ $\quad (1-y)^2 = 1+y^2-2y$

$y=0$

$dy=0$

$x:$ 0 to 1

$I_2 = \int_{AB}\{(3x^2-8y^2)dx+(4y-6xy)dy\} = \int_{AB}[3(1-y)^2-8y^2](-dy) + [4y-6y(1-y)]dy$

$x+y=1$

$x=1-y$

$dx=-dy$

$= \int_{AB}[(6y-1-y^2)^{\times 3}+8y^2+4y-6y+6y^2]\,dy$

$y:$ 0 to 1

$$= ~~\int_0^1 (y^2+1)\,dy = \left[\frac{y^3}{3}+y\right]_0^1 = \frac{13}{3}+1 = \frac{16}{3}~~.$$

$$I_3 = \int_{\substack{BO \\ x=0 \\ dx=0}} (3x^2-8y^2)dx + (4y-6xy)dy = \int_1^0 4y\,dy = 4\left[y^2/2\right]_1^0 = 0-4(1/2) = -2.$$

$$I_2 = \int_0^1 [-3-3y^2+6y+8y^2+4y-6y+6y^2]dy = \int_0^1 (11y^2+4y-3)dy$$

$$I_2 = \left(\frac{11y^3}{3}+\frac{4y^2}{2}-3y\right)_0^1 = \frac{11}{3}+2-3 = \frac{11}{3}-1 = \frac{8}{3}.$$

$$\therefore \int_C P\,dx + Q\,dy = 1+8/3-2 = \frac{8}{3}-1 = 5/3.$$

L.H.S = R.H.S

i.e $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy = 5/3$.

13 a) (i) If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$.

Solution: $w = u+iv$ is an analytic function of z.

Then by C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Also $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{\partial}{\partial x}(u+iv)$$

$$= \frac{\partial w}{\partial x}$$

$$\therefore \frac{dw}{dz} = \frac{\partial w}{\partial x}$$

Also $\frac{dw}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] = -i\frac{\partial}{\partial y}(u+iv) = (-i)\frac{\partial w}{\partial y}$

8 $\therefore \frac{dw}{dz} = \frac{\partial w}{\partial x} = (-i)\frac{\partial w}{\partial y}$

(ii) Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Find also its conjugate. 5

Given $u = \frac{1}{2}\log(x^2+y^2)$.

To prove $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Consider $u = \frac{1}{2}\log(x^2+y^2)$

Differentiating u w.r.to x and y partially, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

$\therefore$ u is harmonic

To find the harmonic conjugate

Let $v(x,y)$ be the conjugate harmonic. Then $w = u+iv$ is analytic.

By C-R equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

We have $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = -\frac{y}{x^2+y^2}dx \quad \text{(By exact differential)}$$

Integrating, we get

$$v = \tan^{-1}\left(\frac{y}{x}\right) + C \text{ where } C \text{ is a constant.}$$

13 b) (i) If $f(z)$ is an analytic function prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

If $z = x+iy$ then $\bar{z} = x-iy$

$\therefore x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}(1/2) + \partial/\partial y\left(\frac{1}{2i}\right) = \frac{\partial}{\partial x}(1/2) - \frac{\partial}{\partial y}(i/2)$$

$$\therefore \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Also $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \partial/\partial y(-1/2i)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\partial/\partial y\right)$$

$$\therefore \frac{\partial^2}{\partial z\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \partial^2/\partial y^2\right)$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}.$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\partial\bar{z}}\left\{|f(z)|^2\right\}$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left\{f(z)\overline{f(z)}\right\} \qquad \overline{f(z)} = f(\bar{z})$$

$$= 4\frac{\partial}{\partial z}\left\{f(z)\,f'(\bar{z})\right\}$$

$$= 4\left\{f'(\bar{z})f'(z)\right\}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\underline{|f(z)|^2} = 4\,|f'(z)|^2.$$

ii) Find the image of $|z|=2$ under the mapping 1. $w = z+3+2i$ 2. $w=3\bar{z}$

1. $w = z+3+2i \Rightarrow u = x+3,\ v = y+2.$

10 $x = u-3,\ y = v-2.$

$|z| = 2 \Rightarrow x^2+y^2 = 4 \Rightarrow (u-3)^2 + (v-2)^2 = 4.$

$\therefore |z|=2$, maps on to a circle with centre $(3,2)$ and radius $= 2$

the W-plan.

2. $u^2+v^2=36$

$u^2+v^2=6^2$

It is a circle with centre at origin and radius $=6$ in the wplane.

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14) a) i) Expand $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ as Laurent's series in the region $2<|z|<3$.

Consider $\frac{z^2-1}{(z+2)(z+3)} = A+\frac{B}{z+2}+\frac{C}{z+3}$

$z^2-1 = A(z+2)(z+3)+B(z+3)+C(z+2)$

put $z=-2$, $\boxed{B=3}$

Put $z=-3$, $\boxed{C=-8}$

Equating Coefficients of z^2, $\boxed{1=A}$

$$\therefore f(z) = 1+\frac{3}{z+2}-\frac{8}{z+3}$$

$2<|z|<3 \Rightarrow 2<|z|$ and $|z|<3$

$\Rightarrow \frac{2}{|z|}<1$ $\qquad \frac{|z|}{3}<1$

$\Rightarrow \left|\frac{2}{z}\right|<1$ $\qquad \Rightarrow |z/3|<1$

$$f(z) = 1+\frac{3}{z+2}-\frac{8}{z+3}$$

$$= 1+\frac{3}{z(1+2/z)}-\frac{8}{3(1+z/3)}$$

$$= 1+\frac{3}{z}(1+2/z)^{-1}-8/3(1+z/3)^{-1}$$

$$f(z) = 1+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\dots\right]-8/3\left[1-(z/3)+(z/3)^2+\dots\right]$$

ii) Evaluate $\int_C \frac{z+1}{(z^2+2z+4)^2}\,dz$ where $C: |z+1+i|=2$ using Cauchy's integral formula.

Given $\int_C \frac{z+1}{(z^2+2z+4)^2}\,dz \quad ; \quad C: |z+1+i|=2.$

Poles are $z = -1 \pm \sqrt{3}$ of order 2.

$z = -1 - i\sqrt{3}$ is inside C

$z = -1 + i\sqrt{3}$ is outside C.

$$\oint_C \frac{\left(\frac{z+1}{(z+1-i\sqrt{3})^2}\right)}{(z+1+i\sqrt{3})^2} dz = \frac{2\pi i f'(a)}{1!}, \quad a = -1 - i\sqrt{3}$$

$$= 2\pi i \left(\tfrac{1}{6}\right)$$

$$= \frac{\pi i}{3}$$

14) b) (i) Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$ using contour integration

Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Consider $\int_C f(z) dz$ where C is the closed contour consisting of Γ, semi-large circle of radius R and the real axis from $-R$ to R.

Then $\int_C f(z) dz = \int_\Gamma f(z) dz + \int_{-R}^{R} f(x) dx$ — ①

Now $z f(z) = \frac{z^3 - z^2 + 2z}{z^4 + 10z^2 + 9} \to 0$ as $z \to \infty$

$\therefore \lim_{z \to \infty} z f(z) = 0$.

Hence from ①, $\int_{-\infty}^{\infty} f(x) dx = \int_C f(z) dz$

By using residue theorem, $\int_C f(z) dz = 2\pi i \sum R$

$\therefore \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \text{Res} f(z)$.

The poles of $f(x)$ are given by $z^4 + 10z^2 + 9 = 0 \Rightarrow (z^2+9)(z^2+1) = 0$.

$\Rightarrow z = \pm i, \; z = \pm 3i$.

$\therefore$ The poles $z = 3i$, $z = i$ lies in the upper half of the z-plane

$$\left(\text{Res} f(z)\right)_{z=i} = \lim_{z \to i} (z - i) f(z).$$

$$\left(\operatorname{Res} f(z)\right)_{z=i} = \lim_{z\to i}\left[\cancel{(z-i)} \frac{z^2-z+2}{(z+i)\cancel{(z-i)}(z^2+9)}\right] = \frac{1-i}{8(2i)} = \frac{1-i}{16i}$$

$$\left(\operatorname{Res} f(z)\right)_{z=3i} = \lim_{z\to 3i} (z-3i)f(z) = \lim_{z\to 3i}\left[\cancel{(z-3i)} \frac{z^2-z+2}{\cancel{(z-3i)}(z+3i)(z^2+1)}\right]$$

$$= \lim_{z\to 3i} \frac{z^2-z+2}{(z+3i)(z^2+1)} = \frac{-7-3i}{(6i)(-8)} = \frac{7+3i}{48i}$$

$$\therefore \sum \operatorname{Res} f(z) = \frac{1-i}{16i} + \frac{7+3i}{48i} = \frac{10}{48i} = \frac{5}{24i}$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\left(\frac{5}{24i}\right) = \frac{5\pi}{12}$$

(ii) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ using contour integration.

Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ put $z = e^{i\theta} \Rightarrow dz = e^{i\theta}\cdot i\,d\theta \Rightarrow dz = zi\,d\theta$

$d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore I = \int_C \frac{dz}{iz\left[2+\frac{z^2+1}{2z}\right]} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1} = \frac{2}{i}\int_C f(z)dz \text{ where } f(z) = \frac{1}{z^2+4z+1} \quad \text{--- (1)}$$

To find Residues:

The poles of $f(z)$ are given by $z^2+4z+1=0$.

$$z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3}.$$

The pole $z=-2+\sqrt{3}$ lies inside c and the pole $z=-2-\sqrt{3}$ lies outside c and it is simple pole.

$$\therefore \left(\operatorname{Res} f(z)\right)_{z=-2+\sqrt{3}} = \lim_{z\to(-2+\sqrt{3})}\left[z-(-2+\sqrt{3})\right]f(z)$$

$$= \lim_{z\to(-2+\sqrt{3})}\left[\frac{(z-(-2+\sqrt{3}))}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))}\right] = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}}$$

$$\sum R = \frac{1}{2\sqrt{3}}$$

$\therefore$ By Cauchy's Residue theorem, $\int_C f(z)dz = 2\pi i \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}}$

$\therefore$ (1) becomes, $I = \frac{2}{\cancel{i}}\frac{\pi \cancel{i}}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

15(a) i) Find the Laplace transform of $f(t) = \begin{cases} t, & \text{for } 0<t<a \\ 2a-t, & \text{for } a<t<2a. \end{cases}$

$f(t+2a)=f(t)$.

The function $f(t)$ has a period $2a$.

$$L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left(\frac{te^{-st}}{-s}\right)_0^a + \frac{1}{s}\int_0^a e^{-st}dt\right] + \left[\left(\frac{(2a-t)e^{-st}}{-s}\right)_a^{2a} - \frac{1}{s}\int_a^{2a} e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{1}{s^2}\left(e^{-st}\right)_0^a\right] + \left[\frac{ae^{-as}}{s} + \frac{1}{s^2}\left(e^{-2as} - e^{-as}\right)\right]$$

$$= \frac{1}{(1-e^{-2as})}\left[\frac{-ae^{-as}}{s} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + \frac{ae^{-as}}{s} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2e^{-as}}{s^2} + \frac{e^{-2as}}{s^2}\right] = \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{s^2}\cdot\frac{(1-e^{-as})^2}{(1-e^{-as})(1+e^{-as})} = \frac{1}{s^2}\,\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

(ii) Using Convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$.

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}\, e^{-b(t-u)}\,du = \int_0^t e^{-au}\cdot e^{-bt}\cdot e^{bu}\,du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\,du = e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \frac{e^{-bt}}{-(a-b)}\left[e^{-(a-b)t} - e^0\right] = \frac{e^{-bt}}{-(a-b)}\left[e^{-at}\cdot e^{bt} - 1\right]$$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{e^{-at} - e^{-bt}}{b-a}$$

15) b) i) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$.

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \quad \text{—} \; (1).$$

$$f(t) = \cos at - \cos bt.$$

$$F(s) = L[f(t)] = L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}.$$

$$\therefore \int_s^\infty F(s)\,ds = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\log \frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}\right]_s^\infty$$

$$= \log\left[\frac{\cancel{s}\sqrt{1+a^2/s^2}}{\cancel{s}\sqrt{1+b^2/s^2}}\right]_s^\infty$$

$$= \log 1 - \log \frac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

ii) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x=0$ and $\frac{dx}{dt} = 5$ for $t=0$ using Laplace transform method.

Given $x'' - 3x' + 2x = 2$, $x(0)=0$, $x'(0)=5$. $\;(1)$

Taking Laplace Transform on bothsides of (1),

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = L[2].$$

$$= 2.$$

$$\left[s^2L[x(t)] - s\cancel{x(0)}^{0} - x'(0)\right] - 3\left[sL[x(t)] - \cancel{x(0)}^{0}\right] + 2L[x(t)] = \frac{2}{s}.$$

$$L[x(t)](s^2 - 3s + 2) - 5 = \frac{2}{s}$$

$$\bar{X}(s)\,[s^2 - 3s + 2] = \frac{2}{s} + 5.$$

$$\bar{X}(s) = \frac{5s+2}{s} \times \frac{1}{s^2-3s+2}$$

$$\bar{X}(s) = \frac{5s+2}{s(s^2-3s+2)} \; ; \; \bar{X}(s) = \frac{5s+2}{s(s-1)(s-2)}$$

$$x(t) = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right]$$

Consider $\frac{5s+2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$5s+2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

Put $s=0$,

$$2 = A(-1)(-2) \Rightarrow 2 = 2A \Rightarrow \boxed{A=1}$$

Put $s=1$,

$$5+2 = A(0) + B1(1-2) + C(0)$$

$$7 = B(-1) \Rightarrow \boxed{B=-7}$$

Put $s=2$,

$$10+2 = A(0) + B(0) + C2(2-1)$$

$$12 = 2C \Rightarrow \boxed{C=6}$$

$$\therefore x(t) = L^{-1}\left[\frac{5s+2}{s(s-1)(s-2)}\right] = L^{-1}\left[\frac{1}{s} + \frac{7}{s-1} - \frac{6}{s-2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{7}{s-1}\right] - L^{-1}\left[\frac{6}{s-2}\right]$$

$$\therefore x(t) = 1 + 7e^{t} - 6e^{2t}$$